

Minimal presentations of shifted numerical monoids

Christopher O'Neill

University of California Davis

coneill@math.ucdavis.edu

Joint with Rebecca Conaway*, Felix Gotti, Jesse Horton*,
Roberto Pelayo, Mesa Williams*, and Brian Wissman

* = undergraduate student

Sep 26, 2016

Definition

A *numerical monoid* S is an **additive** submonoid of \mathbb{N} with $|\mathbb{N} \setminus S| < \infty$.

Definition

A *numerical monoid* S is an **additive** submonoid of \mathbb{N} with $|\mathbb{N} \setminus S| < \infty$.

Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \dots\}$.

Definition

A *numerical monoid* S is an **additive** submonoid of \mathbb{N} with $|\mathbb{N} \setminus S| < \infty$.

Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \dots\}$. “McNugget Monoid”

Definition

A *numerical monoid* S is an **additive** submonoid of \mathbb{N} with $|\mathbb{N} \setminus S| < \infty$.

Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \dots\}$. “McNugget Monoid”

Factorizations:

$$60 =$$

Definition

A *numerical monoid* S is an **additive** submonoid of \mathbb{N} with $|\mathbb{N} \setminus S| < \infty$.

Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \dots\}$. “McNugget Monoid”

Factorizations:

$$60 = 7(6) + 2(9)$$

Definition

A *numerical monoid* S is an **additive** submonoid of \mathbb{N} with $|\mathbb{N} \setminus S| < \infty$.

Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \dots\}$. “McNugget Monoid”

Factorizations:

$$\begin{aligned} 60 &= 7(6) + 2(9) \\ &= \qquad\qquad\qquad 3(20) \end{aligned}$$

Definition

A *numerical monoid* S is an **additive** submonoid of \mathbb{N} with $|\mathbb{N} \setminus S| < \infty$.

Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \dots\}$. “McNugget Monoid”

Factorizations:

$$\begin{array}{rclcl} 60 & = & 7(6) + 2(9) & \rightsquigarrow & (7, 2, 0) \\ & = & 3(20) & \rightsquigarrow & (0, 0, 3) \end{array}$$

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \dots, r_k \rangle$.

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \dots, r_k \rangle$.

$$\pi : \mathbb{N}^k \longrightarrow \langle r_1, \dots, r_k \rangle$$

$$\mathbf{a} \longmapsto a_1 r_1 + \dots + a_k r_k$$

denotes the *factorization homomorphism* of S .

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \dots, r_k \rangle$.

$$\begin{aligned}\pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k\end{aligned}$$

denotes the *factorization homomorphism* of S . In particular,

$$\pi^{-1}(n) = \{ \mathbf{a} \in \mathbb{N}^k : n = a_1 r_1 + \dots + a_k r_k \}$$

is the *set of factorizations* of $n \in S$.

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \dots, r_k \rangle$.

$$\begin{aligned}\pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k\end{aligned}$$

denotes the *factorization homomorphism* of S . In particular,

$$\pi^{-1}(n) = \{ \mathbf{a} \in \mathbb{N}^k : n = a_1 r_1 + \dots + a_k r_k \}$$

is the *set of factorizations* of $n \in S$.

$$|\mathbf{a}| = a_1 + \dots + a_k$$

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \dots, r_k \rangle$.

$$\begin{aligned}\pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k\end{aligned}$$

denotes the *factorization homomorphism* of S . In particular,

$$\pi^{-1}(n) = \{ \mathbf{a} \in \mathbb{N}^k : n = a_1 r_1 + \dots + a_k r_k \}$$

is the *set of factorizations* of $n \in S$.

$$|\mathbf{a}| = a_1 + \dots + a_k \text{ (length of } \mathbf{a} \text{)}$$

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \dots, r_k \rangle$.

$$\begin{aligned}\pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k\end{aligned}$$

denotes the *factorization homomorphism* of S . In particular,

$$\pi^{-1}(n) = \{ \mathbf{a} \in \mathbb{N}^k : n = a_1 r_1 + \dots + a_k r_k \}$$

is the *set of factorizations* of $n \in S$.

$$|\mathbf{a}| = a_1 + \dots + a_k \text{ (length of } \mathbf{a} \text{)}$$

Example

$$S = \langle 6, 9, 20 \rangle:$$

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \dots, r_k \rangle$.

$$\begin{aligned}\pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k\end{aligned}$$

denotes the *factorization homomorphism* of S . In particular,

$$\pi^{-1}(n) = \{ \mathbf{a} \in \mathbb{N}^k : n = a_1 r_1 + \dots + a_k r_k \}$$

is the *set of factorizations* of $n \in S$.

$$|\mathbf{a}| = a_1 + \dots + a_k \text{ (length of } \mathbf{a} \text{)}$$

Example

$S = \langle 6, 9, 20 \rangle$:

$$\pi^{-1}(60) = \{(10, 0, 0), (7, 2, 0), (4, 4, 0), (1, 6, 0), (0, 0, 3)\}$$

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \dots, r_k \rangle$.

$$\begin{aligned}\pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k\end{aligned}$$

denotes the *factorization homomorphism* of S . In particular,

$$\pi^{-1}(n) = \{ \mathbf{a} \in \mathbb{N}^k : n = a_1 r_1 + \dots + a_k r_k \}$$

is the *set of factorizations* of $n \in S$.

$$|\mathbf{a}| = a_1 + \dots + a_k \text{ (length of } \mathbf{a} \text{)}$$

Example

$S = \langle 6, 9, 20 \rangle$:

$$\pi^{-1}(60) = \{(10, 0, 0), (7, 2, 0), (4, 4, 0), (1, 6, 0), (0, 0, 3)\}$$

Possible factorization lengths for $n = 60$: 3, 7, 8, 9, 10.

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \dots, r_k \rangle$.

$$\begin{aligned}\pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k\end{aligned}$$

denotes the *factorization homomorphism* of S . In particular,

$$\pi^{-1}(n) = \{ \mathbf{a} \in \mathbb{N}^k : n = a_1 r_1 + \dots + a_k r_k \}$$

is the *set of factorizations* of $n \in S$.

$$|\mathbf{a}| = a_1 + \dots + a_k \text{ (length of } \mathbf{a} \text{)}$$

Example

$S = \langle 6, 9, 20 \rangle$:

$$\pi^{-1}(60) = \{(10, 0, 0), (7, 2, 0), (4, 4, 0), (1, 6, 0), (0, 0, 3)\}$$

Possible factorization lengths for $n = 60$: 3, 7, 8, 9, 10.

$$\pi^{-1}(1000001) =$$

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \dots, r_k \rangle$.

$$\begin{aligned}\pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k\end{aligned}$$

denotes the *factorization homomorphism* of S . In particular,

$$\pi^{-1}(n) = \{ \mathbf{a} \in \mathbb{N}^k : n = a_1 r_1 + \dots + a_k r_k \}$$

is the *set of factorizations* of $n \in S$.

$$|\mathbf{a}| = a_1 + \dots + a_k \text{ (length of } \mathbf{a} \text{)}$$

Example

$S = \langle 6, 9, 20 \rangle$:

$$\pi^{-1}(60) = \{(10, 0, 0), (7, 2, 0), (4, 4, 0), (1, 6, 0), (0, 0, 3)\}$$

Possible factorization lengths for $n = 60$: 3, 7, 8, 9, 10.

$$\pi^{-1}(1000001) = \{ \underbrace{\hspace{10em}}_{\text{shortest}}, \dots, \underbrace{\hspace{10em}}_{\text{longest}} \}$$

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \dots, r_k \rangle$.

$$\begin{aligned}\pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k\end{aligned}$$

denotes the *factorization homomorphism* of S . In particular,

$$\pi^{-1}(n) = \{ \mathbf{a} \in \mathbb{N}^k : n = a_1 r_1 + \dots + a_k r_k \}$$

is the *set of factorizations* of $n \in S$.

$$|\mathbf{a}| = a_1 + \dots + a_k \text{ (length of } \mathbf{a} \text{)}$$

Example

$S = \langle 6, 9, 20 \rangle$:

$$\pi^{-1}(60) = \{ (10, 0, 0), (7, 2, 0), (4, 4, 0), (1, 6, 0), (0, 0, 3) \}$$

Possible factorization lengths for $n = 60$: 3, 7, 8, 9, 10.

$$\pi^{-1}(1000001) = \{ \underbrace{(2, 1, 49999)}_{\text{shortest}}, \dots, \underbrace{\hspace{10em}}_{\text{longest}} \}$$

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \dots, r_k \rangle$.

$$\begin{aligned}\pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k\end{aligned}$$

denotes the *factorization homomorphism* of S . In particular,

$$\pi^{-1}(n) = \{ \mathbf{a} \in \mathbb{N}^k : n = a_1 r_1 + \dots + a_k r_k \}$$

is the *set of factorizations* of $n \in S$.

$$|\mathbf{a}| = a_1 + \dots + a_k \text{ (length of } \mathbf{a} \text{)}$$

Example

$S = \langle 6, 9, 20 \rangle$:

$$\pi^{-1}(60) = \{ (10, 0, 0), (7, 2, 0), (4, 4, 0), (1, 6, 0), (0, 0, 3) \}$$

Possible factorization lengths for $n = 60$: 3, 7, 8, 9, 10.

$$\pi^{-1}(1000001) = \{ \underbrace{(2, 1, 49999)}_{\text{shortest}}, \dots, \underbrace{(166662, 1, 1)}_{\text{longest}} \}$$

To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

Catenary degree $c(M_n)$: measures spread of factorizations in M_n .

To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

Catenary degree $c(M_n)$: measures spread of factorizations in M_n .

$$M_n = \langle n, n + 6, n + 9, n + 20 \rangle:$$

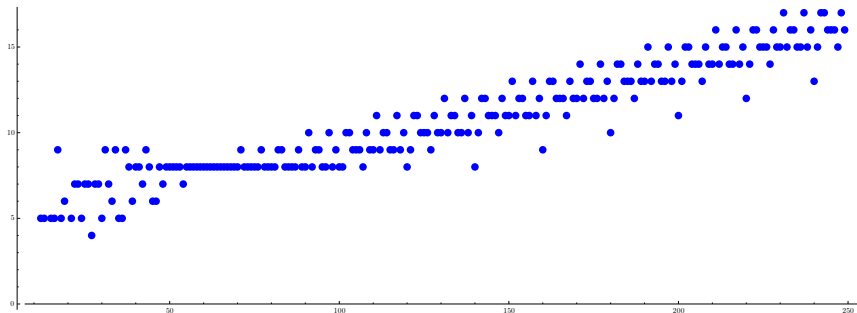
To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

Catenary degree $c(M_n)$: measures spread of factorizations in M_n .

$M_n = \langle n, n + 6, n + 9, n + 20 \rangle$:



To shift a numerical monoid...

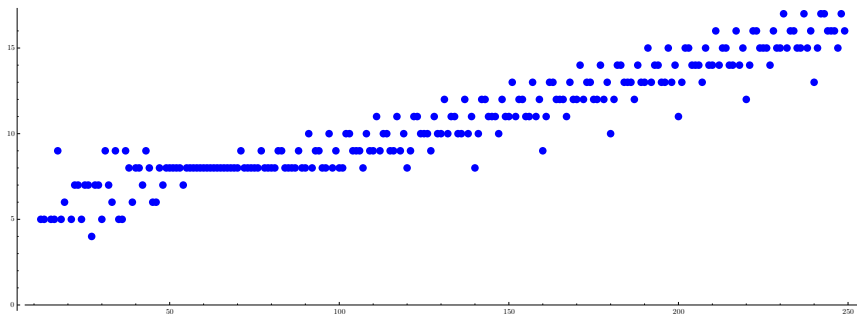
Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

Catenary degree $c(M_n)$: measures spread of factorizations in M_n .

$M_n = \langle n, n + 6, n + 9, n + 20 \rangle$:

$c(M_n)$ is periodic-linear (quasilinear) for $n \geq 126$.



To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

Delta set $\Delta(M_n)$: successive factorization length differences in M_n .

To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

Delta set $\Delta(M_n)$: successive factorization length differences in M_n .

Theorem (Chapman-Kaplan-Lemburg-Niles-Zlogar, 2014)

The delta set $\Delta(M_n)$ is singleton for $n \gg 0$.

To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

Delta set $\Delta(M_n)$: successive factorization length differences in M_n .

Theorem (Chapman-Kaplan-Lemburg-Niles-Zlogar, 2014)

The delta set $\Delta(M_n)$ is singleton for $n \gg 0$.

$M_n = \langle n, n + 6, n + 9, n + 20 \rangle$:

To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

Delta set $\Delta(M_n)$: successive factorization length differences in M_n .

Theorem (Chapman-Kaplan-Lemburg-Niles-Zlogar, 2014)

The delta set $\Delta(M_n)$ is singleton for $n \gg 0$.

$M_n = \langle n, n + 6, n + 9, n + 20 \rangle$:

$$\Delta(M_n) = \{1\} \text{ for all } n \geq 48$$

To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

Betti numbers $\beta_i(M_n)$: Betti numbers of the defining toric ideal I_{M_n} .

To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

Betti numbers $\beta_i(M_n)$: Betti numbers of the defining toric ideal I_{M_n} .

Theorem (Vu, 2014)

The Betti numbers of M_n are eventually r_k -periodic in n .

To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

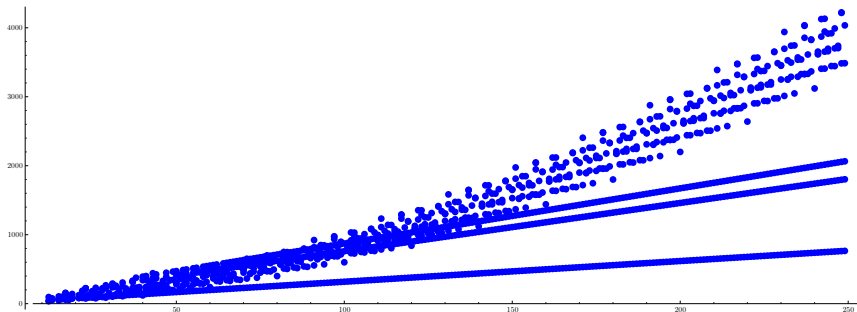
$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

Betti numbers $\beta_i(M_n)$: Betti numbers of the defining toric ideal I_{M_n} .

Theorem (Vu, 2014)

The Betti numbers of M_n are eventually r_k -periodic in n .

$M_n = \langle n, n + 6, n + 9, n + 20 \rangle$: Graded degrees for $\beta_0(M_n)$



To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

Observations:

To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

Observations:

- Known: the Betti numbers $n \mapsto \beta_i(M_n)$ are eventually r_k -periodic.

To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

Observations:

- Known: the Betti numbers $n \mapsto \beta_i(M_n)$ are eventually r_k -periodic.
- Known: the function $n \mapsto \Delta(M_n)$ is eventually singleton.

To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

Observations:

- Known: the Betti numbers $n \mapsto \beta_i(M_n)$ are eventually r_k -periodic.
- Known: the function $n \mapsto \Delta(M_n)$ is eventually singleton.
- Observed: the function $n \mapsto c(M_n)$ is eventually r_k -quasilinear.

To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

Observations:

- Known: the Betti numbers $n \mapsto \beta_i(M_n)$ are eventually r_k -periodic.
- Known: the function $n \mapsto \Delta(M_n)$ is eventually singleton.
- Observed: the function $n \mapsto c(M_n)$ is eventually r_k -quasilinear.

Underlying cause:

To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

Observations:

- Known: the Betti numbers $n \mapsto \beta_i(M_n)$ are eventually r_k -periodic.
- Known: the function $n \mapsto \Delta(M_n)$ is eventually singleton.
- Observed: the function $n \mapsto c(M_n)$ is eventually r_k -quasilinear.

Underlying cause: minimal presentations!

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \Leftrightarrow \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k \end{aligned}$$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k \end{aligned}$$

Definition

The *kernel* $\ker \pi$ is the relation \sim on \mathbb{N}^k with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b})$$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k \end{aligned}$$

Definition

The *kernel* $\ker \pi$ is the relation \sim on \mathbb{N}^k with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b})$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\begin{aligned} \mathbf{a} &\sim \mathbf{a} \\ \mathbf{a} \sim \mathbf{b} &\Rightarrow \mathbf{b} \sim \mathbf{a} \\ \mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} &\Rightarrow \mathbf{a} \sim \mathbf{c} \end{aligned}$$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k \end{aligned}$$

Definition

The *kernel* $\ker \pi$ is the relation \sim on \mathbb{N}^k with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b})$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

that is closed under *translation*.

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k \end{aligned}$$

Definition

The *kernel* $\ker \pi$ is the relation \sim on \mathbb{N}^k with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b})$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

that is closed under *translation*.

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

Monomial map:

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k \end{aligned}$$

Definition

The *kernel* $\ker \pi$ is the relation \sim on \mathbb{N}^k with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b})$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

that is closed under *translation*.

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

Monomial map:

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k \end{aligned}$$

$$\begin{aligned} \varphi : \mathbb{k}[x_1, \dots, x_k] &\longrightarrow \mathbb{k}[y] \\ x_i &\longmapsto y^{r_i} \end{aligned}$$

Definition

The *kernel* $\ker \pi$ is the relation \sim on \mathbb{N}^k with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b})$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

that is closed under *translation*.

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

Monomial map:

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k \end{aligned}$$

$$\begin{aligned} \varphi : \mathbb{k}[x_1, \dots, x_k] &\longrightarrow \mathbb{k}[y] \\ x_i &\longmapsto y^{r_i} \end{aligned}$$

Definition

The *kernel* $\ker \pi$ is the relation \sim on \mathbb{N}^k with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b}) \quad x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S = \ker \varphi$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

that is closed under *translation*.

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

Monomial map:

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k \end{aligned}$$

$$\begin{aligned} \varphi : \mathbb{k}[x_1, \dots, x_k] &\longrightarrow \mathbb{k}[y] \\ x_i &\longmapsto y^{r_i} \end{aligned}$$

Definition

The *kernel* $\ker \pi$ is the relation \sim on \mathbb{N}^k with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b}) \quad x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S = \ker \varphi$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a} \quad x^{\mathbf{a}} - x^{\mathbf{a}} = 0 \in I_S$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

that is closed under *translation*.

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

Monomial map:

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k \end{aligned}$$

$$\begin{aligned} \varphi : \mathbb{k}[x_1, \dots, x_k] &\longrightarrow \mathbb{k}[y] \\ x_i &\longmapsto y^{r_i} \end{aligned}$$

Definition

The *kernel* $\ker \pi$ is the relation \sim on \mathbb{N}^k with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b}) \quad x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S = \ker \varphi$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

that is closed under *translation*.

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

$$\begin{aligned} x^{\mathbf{a}} - x^{\mathbf{a}} &= 0 \in I_S \\ x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S &\Rightarrow x^{\mathbf{b}} - x^{\mathbf{a}} \in I_S \end{aligned}$$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

Monomial map:

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k \end{aligned}$$

$$\begin{aligned} \varphi : \mathbb{k}[x_1, \dots, x_k] &\longrightarrow \mathbb{k}[y] \\ x_i &\longmapsto y^{r_i} \end{aligned}$$

Definition

The *kernel* $\ker \pi$ is the relation \sim on \mathbb{N}^k with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b}) \quad x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S = \ker \varphi$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

that is closed under *translation*.

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

$$x^{\mathbf{a}} - x^{\mathbf{a}} = 0 \in I_S$$

$$x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S \Rightarrow x^{\mathbf{b}} - x^{\mathbf{a}} \in I_S$$

$$(x^{\mathbf{a}} - x^{\mathbf{b}}) + (x^{\mathbf{b}} - x^{\mathbf{c}}) = x^{\mathbf{a}} - x^{\mathbf{c}}$$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

Monomial map:

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \dots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \dots + a_k r_k \end{aligned}$$

$$\begin{aligned} \varphi : \mathbb{k}[x_1, \dots, x_k] &\longrightarrow \mathbb{k}[y] \\ x_i &\longmapsto y^{r_i} \end{aligned}$$

Definition

The *kernel* $\ker \pi$ is the relation \sim on \mathbb{N}^k with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b}) \quad x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S = \ker \varphi$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

that is closed under *translation*.

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

$$x^{\mathbf{a}} - x^{\mathbf{a}} = 0 \in I_S$$

$$x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S \Rightarrow x^{\mathbf{b}} - x^{\mathbf{a}} \in I_S$$

$$(x^{\mathbf{a}} - x^{\mathbf{b}}) + (x^{\mathbf{b}} - x^{\mathbf{c}}) = x^{\mathbf{a}} - x^{\mathbf{c}}$$

$$x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S \Rightarrow x^{\mathbf{c}}(x^{\mathbf{a}} - x^{\mathbf{b}}) \in I_S$$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

$$\pi : \mathbb{N}^k \longrightarrow \langle r_1, \dots, r_k \rangle$$

$$\mathbf{a} \longmapsto a_1 r_1 + \dots + a_k r_k$$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

$$\pi : \mathbb{N}^k \longrightarrow \langle r_1, \dots, r_k \rangle$$

$$\mathbf{a} \longmapsto a_1 r_1 + \dots + a_k r_k$$

Definition

A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

$$\pi : \mathbb{N}^k \longrightarrow \langle r_1, \dots, r_k \rangle$$

$$\mathbf{a} \longmapsto a_1 r_1 + \dots + a_k r_k$$

Definition

A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$:

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

$$\pi : \mathbb{N}^k \longrightarrow \langle r_1, \dots, r_k \rangle$$

$$\mathbf{a} \longmapsto a_1 r_1 + \dots + a_k r_k$$

Definition

A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$$S = \langle 6, 9, 20 \rangle: \rho = \{((3, 0, 0), (0, 2, 0)), ((4, 4, 0), (0, 0, 3))\} \subset \ker \pi$$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

$$\pi : \mathbb{N}^k \longrightarrow \langle r_1, \dots, r_k \rangle$$

$$\mathbf{a} \longmapsto a_1 r_1 + \dots + a_k r_k$$

Definition

A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$$S = \langle 6, 9, 20 \rangle: \rho = \{((3, 0, 0), (0, 2, 0)), ((4, 4, 0), (0, 0, 3))\} \subset \ker \pi$$

$\pi^{-1}(18)$:

$(3, 0, 0)$



$(0, 2, 0)$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

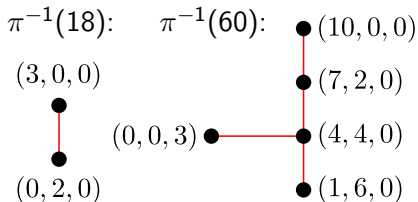
$$\pi : \mathbb{N}^k \longrightarrow \langle r_1, \dots, r_k \rangle$$

$$\mathbf{a} \longmapsto a_1 r_1 + \dots + a_k r_k$$

Definition

A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3, 0, 0), (0, 2, 0)), ((4, 4, 0), (0, 0, 3))\} \subset \ker \pi$



Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

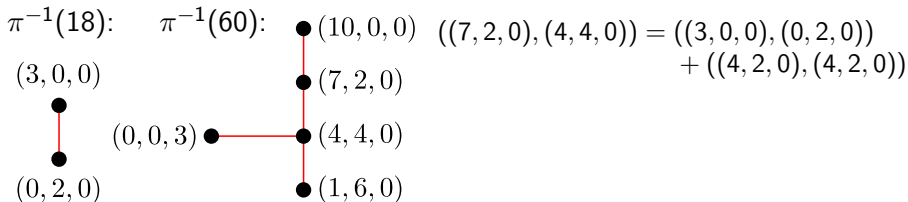
$$\pi : \mathbb{N}^k \longrightarrow \langle r_1, \dots, r_k \rangle$$

$$\mathbf{a} \longmapsto a_1 r_1 + \dots + a_k r_k$$

Definition

A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3, 0, 0), (0, 2, 0)), ((4, 4, 0), (0, 0, 3))\} \subset \ker \pi$



Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

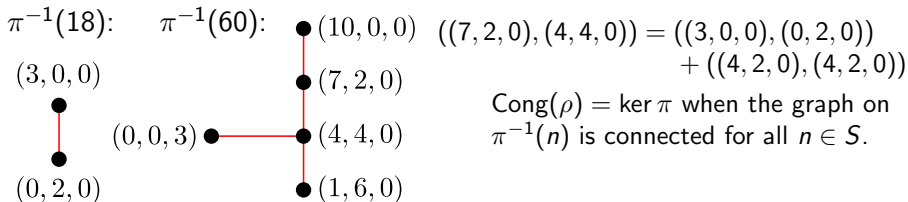
$$\pi : \mathbb{N}^k \longrightarrow \langle r_1, \dots, r_k \rangle$$

$$\mathbf{a} \longmapsto a_1 r_1 + \dots + a_k r_k$$

Definition

A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3, 0, 0), (0, 2, 0)), ((4, 4, 0), (0, 0, 3))\} \subset \ker \pi$



Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

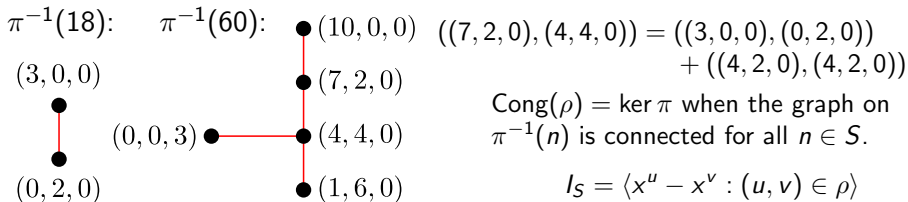
$$\pi : \mathbb{N}^k \longrightarrow \langle r_1, \dots, r_k \rangle$$

$$\mathbf{a} \longmapsto a_1 r_1 + \dots + a_k r_k$$

Definition

A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3, 0, 0), (0, 2, 0)), ((4, 4, 0), (0, 0, 3))\} \subset \ker \pi$



Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

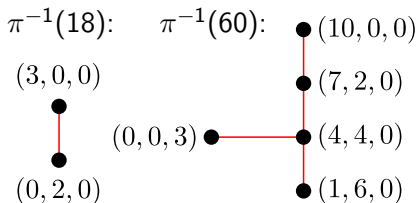
$$\pi : \mathbb{N}^k \longrightarrow \langle r_1, \dots, r_k \rangle$$

$$\mathbf{a} \longmapsto a_1 r_1 + \dots + a_k r_k$$

Definition

A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3, 0, 0), (0, 2, 0)), ((4, 4, 0), (0, 0, 3))\} \subset \ker \pi$



Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

$$\pi : \mathbb{N}^k \longrightarrow \langle r_1, \dots, r_k \rangle$$

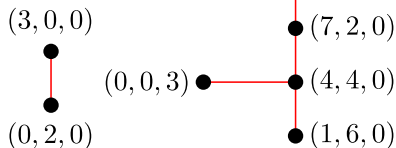
$$\mathbf{a} \longmapsto a_1 r_1 + \dots + a_k r_k$$

Definition

A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3, 0, 0), (0, 2, 0)), ((4, 4, 0), (0, 0, 3))\} \subset \ker \pi$

$\pi^{-1}(18)$: $\pi^{-1}(60)$: $\bullet (10, 0, 0)$ All minimal presentations:



Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

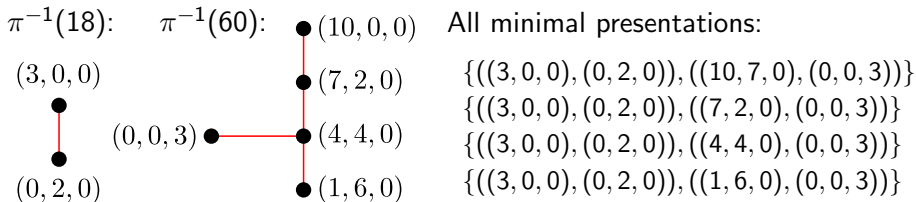
$$\pi : \mathbb{N}^k \longrightarrow \langle r_1, \dots, r_k \rangle$$

$$\mathbf{a} \longmapsto a_1 r_1 + \dots + a_k r_k$$

Definition

A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3, 0, 0), (0, 2, 0)), ((4, 4, 0), (0, 0, 3))\} \subset \ker \pi$



Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

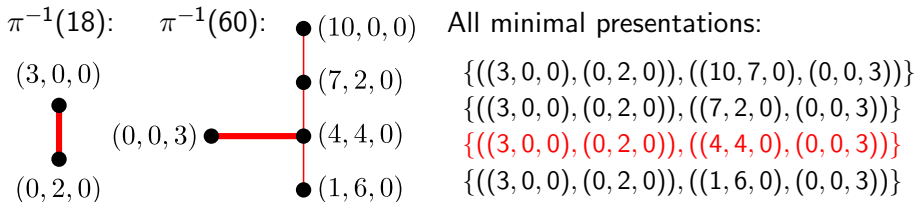
$$\pi : \mathbb{N}^k \longrightarrow \langle r_1, \dots, r_k \rangle$$

$$\mathbf{a} \longmapsto a_1 r_1 + \dots + a_k r_k$$

Definition

A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3, 0, 0), (0, 2, 0)), ((4, 4, 0), (0, 0, 3))\} \subset \ker \pi$



Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

$$\pi : \mathbb{N}^k \longrightarrow \langle r_1, \dots, r_k \rangle$$

$$\mathbf{a} \longmapsto a_1 r_1 + \dots + a_k r_k$$

Definition

A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3, 0, 0), (0, 2, 0)), ((4, 4, 0), (0, 0, 3))\} \subset \ker \pi$

$\pi^{-1}(18)$: $\pi^{-1}(60)$:

$(3, 0, 0)$



$(0, 0, 3)$



$(0, 2, 0)$

$(10, 0, 0)$

$(7, 2, 0)$

$(4, 4, 0)$

$(1, 6, 0)$

All minimal presentations:

$\{((3, 0, 0), (0, 2, 0)), ((10, 7, 0), (0, 0, 3))\}$

$\{((3, 0, 0), (0, 2, 0)), ((7, 2, 0), (0, 0, 3))\}$

$\{((3, 0, 0), (0, 2, 0)), ((4, 4, 0), (0, 0, 3))\}$

$\{((3, 0, 0), (0, 2, 0)), ((1, 6, 0), (0, 0, 3))\}$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

$$\pi : \mathbb{N}^k \longrightarrow \langle r_1, \dots, r_k \rangle$$

$$\mathbf{a} \longmapsto a_1 r_1 + \dots + a_k r_k$$

Definition

A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3, 0, 0), (0, 2, 0)), ((4, 4, 0), (0, 0, 3))\} \subset \ker \pi$

$\pi^{-1}(18)$: $\pi^{-1}(60)$:

$(3, 0, 0)$



$(0, 0, 3)$

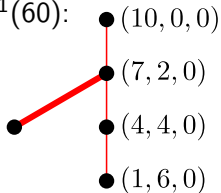
$(0, 2, 0)$

$(10, 0, 0)$

$(7, 2, 0)$

$(4, 4, 0)$

$(1, 6, 0)$



All minimal presentations:

$\{((3, 0, 0), (0, 2, 0)), ((10, 7, 0), (0, 0, 3))\}$

$\{((3, 0, 0), (0, 2, 0)), ((7, 2, 0), (0, 0, 3))\}$

$\{((3, 0, 0), (0, 2, 0)), ((4, 4, 0), (0, 0, 3))\}$

$\{((3, 0, 0), (0, 2, 0)), ((1, 6, 0), (0, 0, 3))\}$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

$$\pi : \mathbb{N}^k \longrightarrow \langle r_1, \dots, r_k \rangle$$

$$\mathbf{a} \longmapsto a_1 r_1 + \dots + a_k r_k$$

Definition

A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3, 0, 0), (0, 2, 0)), ((4, 4, 0), (0, 0, 3))\} \subset \ker \pi$

$\pi^{-1}(18)$: $\pi^{-1}(60)$:

$(3, 0, 0)$



$(0, 2, 0)$

$(0, 0, 3)$



$(10, 0, 0)$

$(7, 2, 0)$

$(4, 4, 0)$

$(1, 6, 0)$

All minimal presentations:

$\{((3, 0, 0), (0, 2, 0)), ((10, 7, 0), (0, 0, 3))\}$

$\{((3, 0, 0), (0, 2, 0)), ((7, 2, 0), (0, 0, 3))\}$

$\{((3, 0, 0), (0, 2, 0)), ((4, 4, 0), (0, 0, 3))\}$

$\{((3, 0, 0), (0, 2, 0)), ((1, 6, 0), (0, 0, 3))\}$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

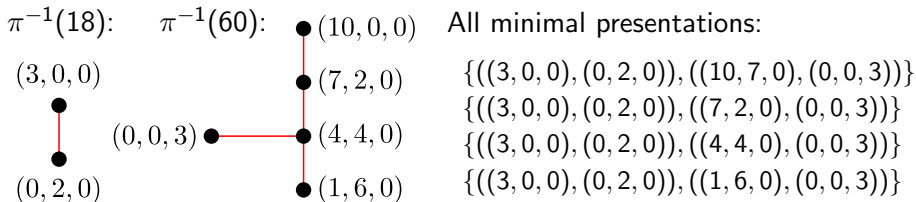
$$\pi : \mathbb{N}^k \longrightarrow \langle r_1, \dots, r_k \rangle$$

$$\mathbf{a} \longmapsto a_1 r_1 + \dots + a_k r_k$$

Definition

A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3, 0, 0), (0, 2, 0)), ((4, 4, 0), (0, 0, 3))\} \subset \ker \pi$



Kernel congruences and minimal presentations

Let $S = \langle r_1, \dots, r_k \rangle$.

$$n = a_1 r_1 + \dots + a_k r_k \quad \iff \quad \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$$

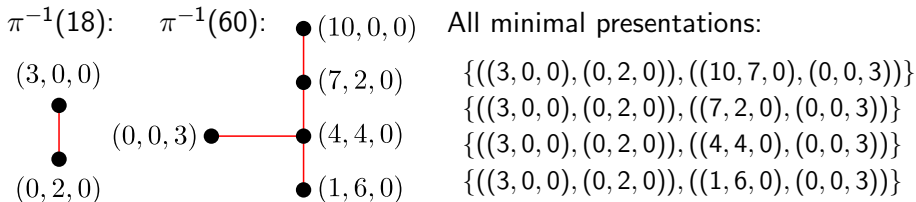
$$\pi : \mathbb{N}^k \longrightarrow \langle r_1, \dots, r_k \rangle$$

$$\mathbf{a} \longmapsto a_1 r_1 + \dots + a_k r_k$$

Definition

A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3, 0, 0), (0, 2, 0)), ((4, 4, 0), (0, 0, 3))\} \subset \ker \pi$



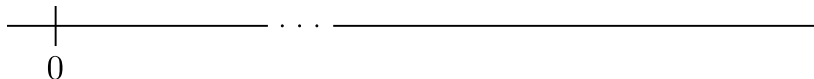
$$\beta_0(I_S) = \{18, 60\}$$

Intuition: “sufficiently shifted” monoids

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

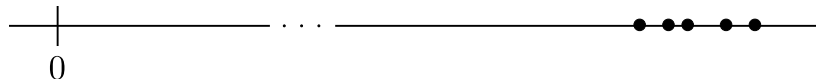
Intuition: “sufficiently shifted” monoids

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$



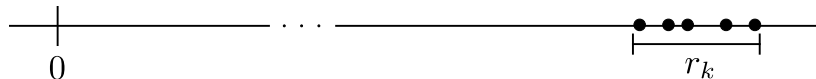
Intuition: “sufficiently shifted” monoids

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$



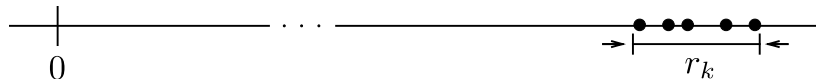
Intuition: “sufficiently shifted” monoids

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$



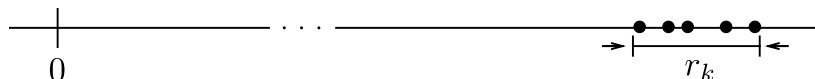
Intuition: “sufficiently shifted” monoids

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$



Intuition: “sufficiently shifted” monoids

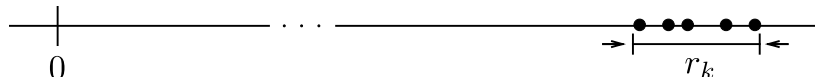
$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$



$$_n + _(n + r_1) + \dots + _(n + r_k)$$

Intuition: “sufficiently shifted” monoids

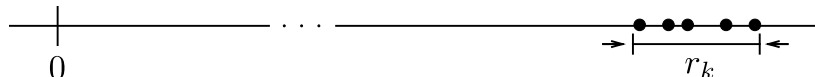
$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$



$$\underline{\quad} n + \underline{\quad} (n + r_1) + \cdots + \underline{\quad} (n + r_k)$$

Intuition: “sufficiently shifted” monoids

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

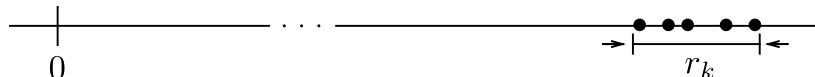


$$\underline{\quad} n + \underline{\quad} (n + r_1) + \cdots + \underline{\quad} (n + r_k)$$

2 types of minimal relations $\mathbf{a} \sim \mathbf{b}$:

Intuition: “sufficiently shifted” monoids

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$



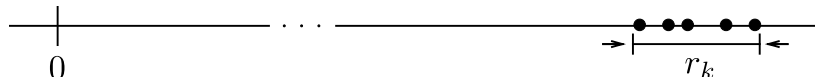
$$_ n + _ (n + r_1) + \dots + _ (n + r_k)$$

2 types of minimal relations $\mathbf{a} \sim \mathbf{b}$:

- Relations among r_1, \dots, r_k (cheap)

Intuition: “sufficiently shifted” monoids

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$



$$_ n + _ (n + r_1) + \dots + _ (n + r_k)$$

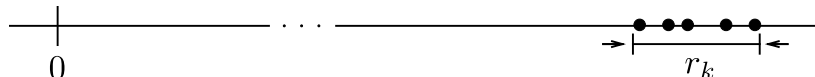
2 types of minimal relations $\mathbf{a} \sim \mathbf{b}$:

- Relations among r_1, \dots, r_k (cheap)

$$|\mathbf{a}| = |\mathbf{b}|$$

Intuition: “sufficiently shifted” monoids

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$



$$_n + _(n + r_1) + \dots + _(n + r_k)$$

2 types of minimal relations $\mathbf{a} \sim \mathbf{b}$:

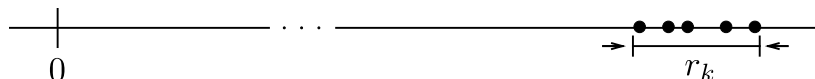
- Relations among r_1, \dots, r_k (cheap)

$$|\mathbf{a}| = |\mathbf{b}|$$

- Relations that change $\#$ copies of n (costly)

Intuition: “sufficiently shifted” monoids

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$



$$_n + _(n + r_1) + \dots + _(n + r_k)$$

2 types of minimal relations $\mathbf{a} \sim \mathbf{b}$:

- Relations among r_1, \dots, r_k (cheap)

$$|\mathbf{a}| = |\mathbf{b}|$$

- Relations that change $\#$ copies of n (costly)

$$\text{mostly } a_k \quad \longleftrightarrow \quad \text{mostly } b_0$$

The shifting map

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

The shifting map

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

The shifting map

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

The shifting map

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

DON'T PANIC!

The shifting map

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

The shifting map

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

Sneak peek for $M_n = \langle n, n + 6, n + 9, n + 20 \rangle$ and $n \gg 0$:

The shifting map

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

Sneak peek for $M_n = \langle n, n + 6, n + 9, n + 20 \rangle$ and $n \gg 0$:

M_{450} :

$$\{ ((0, 0, 8, 0), (3, 2, 0, 3)), ((0, 1, 6, 0), (4, 0, 0, 3)), ((0, 3, 0, 0), (1, 0, 2, 0)), \\ \{ ((20, 5, 0, 0), (0, 0, 0, 24)), ((25, 1, 0, 0), (0, 0, 4, 21)), ((26, 0, 0, 0), (0, 2, 2, 21)) \}$$

The shifting map

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

Sneak peek for $M_n = \langle n, n + 6, n + 9, n + 20 \rangle$ and $n \gg 0$:

M_{450} :

$$\left\{ \begin{array}{l} ((0, 0, 8, 0), (3, 2, 0, 3)), ((0, 1, 6, 0), (4, 0, 0, 3)), ((0, 3, 0, 0), (1, 0, 2, 0)), \\ ((20, 5, 0, 0), (0, 0, 0, 24)), ((25, 1, 0, 0), (0, 0, 4, 21)), ((26, 0, 0, 0), (0, 2, 2, 21)) \end{array} \right\}$$

M_{470} :

$$\left\{ \begin{array}{l} ((0, 0, 8, 0), (3, 2, 0, 3)), ((0, 1, 6, 0), (4, 0, 0, 3)), ((0, 3, 0, 0), (1, 0, 2, 0)), \\ ((21, 5, 0, 0), (0, 0, 0, 25)), ((26, 1, 0, 0), (0, 0, 4, 22)), ((27, 0, 0, 0), (0, 2, 2, 22)) \end{array} \right\}$$

The shifting map

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

Sneak peek for $M_n = \langle n, n + 6, n + 9, n + 20 \rangle$ and $n \gg 0$:

M_{450} :

$$\left\{ \begin{aligned} &((0, 0, 8, 0), (3, 2, 0, 3)), ((0, 1, 6, 0), (4, 0, 0, 3)), ((0, 3, 0, 0), (1, 0, 2, 0)), \\ &((20, 5, 0, 0), (0, 0, 0, 24)), ((25, 1, 0, 0), (0, 0, 4, 21)), ((26, 0, 0, 0), (0, 2, 2, 21)) \end{aligned} \right\}$$

M_{470} :

$$\left\{ \begin{aligned} &((0, 0, 8, 0), (3, 2, 0, 3)), ((0, 1, 6, 0), (4, 0, 0, 3)), ((0, 3, 0, 0), (1, 0, 2, 0)), \\ &((21, 5, 0, 0), (0, 0, 0, 25)), ((26, 1, 0, 0), (0, 0, 4, 22)), ((27, 0, 0, 0), (0, 2, 2, 22)) \end{aligned} \right\}$$

M_{490} :

$$\left\{ \begin{aligned} &((0, 0, 8, 0), (3, 2, 0, 3)), ((0, 1, 6, 0), (4, 0, 0, 3)), ((0, 3, 0, 0), (1, 0, 2, 0)), \\ &((22, 5, 0, 0), (0, 0, 0, 26)), ((27, 1, 0, 0), (0, 0, 4, 23)), ((28, 0, 0, 0), (0, 2, 2, 23)) \end{aligned} \right\}$$

The shifting map

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

The shifting map

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

- Φ_n is well-defined.

The shifting map

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

- Φ_n is well-defined.

$$\begin{aligned} \pi_n(\mathbf{a}) &= a_0 n + \sum_{i=1}^k a_i (n + r_i) = |\mathbf{a}| n + \sum_{i=1}^k a_i r_i \\ \pi_{n+r_k}(\mathbf{a}) &= \qquad \qquad \qquad = |\mathbf{a}| n + |\mathbf{a}| r_k + \sum_{i=1}^k a_i r_i \end{aligned}$$

The shifting map

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

- Φ_n is well-defined.

$$\begin{aligned} \pi_n(\mathbf{a}) &= a_0 n + \sum_{i=1}^k a_i (n + r_i) = |\mathbf{a}| n + \sum_{i=1}^k a_i r_i \\ \pi_{n+r_k}(\mathbf{a}) &= \qquad \qquad \qquad = |\mathbf{a}| n + |\mathbf{a}| r_k + \sum_{i=1}^k a_i r_i \end{aligned}$$

- Φ_n preserves reflexive and symmetric closure.

The shifting map

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

- Φ_n is well-defined.

$$\begin{aligned} \pi_n(\mathbf{a}) &= a_0 n + \sum_{i=1}^k a_i (n + r_i) = |\mathbf{a}|n + \sum_{i=1}^k a_i r_i \\ \pi_{n+r_k}(\mathbf{a}) &= \qquad \qquad \qquad = |\mathbf{a}|n + |\mathbf{a}|r_k + \sum_{i=1}^k a_i r_i \end{aligned}$$

- Φ_n preserves reflexive and symmetric closure.
- Φ_n preserves translation closure.

The shifting map

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

- Φ_n is well-defined.

$$\begin{aligned} \pi_n(\mathbf{a}) &= a_0 n + \sum_{i=1}^k a_i (n + r_i) = |\mathbf{a}| n + \sum_{i=1}^k a_i r_i \\ \pi_{n+r_k}(\mathbf{a}) &= \phantom{\pi_n(\mathbf{a})} = |\mathbf{a}| n + |\mathbf{a}| r_k + \sum_{i=1}^k a_i r_i \end{aligned}$$

- Φ_n preserves reflexive and symmetric closure.
- Φ_n preserves translation closure.

$$\Phi_n((\mathbf{a}, \mathbf{a}') + (\mathbf{b}, \mathbf{b})) = \Phi_n(\mathbf{a}, \mathbf{a}') + (\mathbf{b}, \mathbf{b})$$

The shifting map

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

- Φ_n is well-defined.

$$\begin{aligned} \pi_n(\mathbf{a}) &= a_0 n + \sum_{i=1}^k a_i (n + r_i) = |\mathbf{a}| n + \sum_{i=1}^k a_i r_i \\ \pi_{n+r_k}(\mathbf{a}) &= \phantom{\pi_n(\mathbf{a})} = |\mathbf{a}| n + |\mathbf{a}| r_k + \sum_{i=1}^k a_i r_i \end{aligned}$$

- Φ_n preserves reflexive and symmetric closure.
- Φ_n preserves translation closure.

$$\Phi_n((\mathbf{a}, \mathbf{a}') + (\mathbf{b}, \mathbf{b})) = \Phi_n(\mathbf{a}, \mathbf{a}') + (\mathbf{b}, \mathbf{b})$$

- Only missing link: transitivity.

Monotone chains

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

Monotone chains

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$

Monotone chains

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

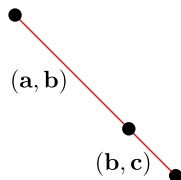
Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$



Monotone chains

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

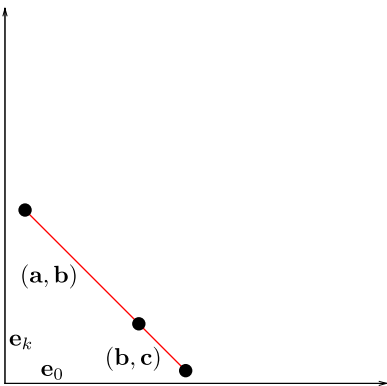
Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$



Monotone chains

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

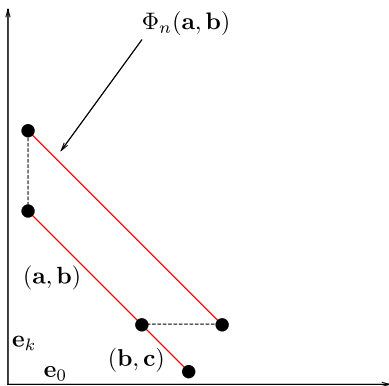
Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$



Monotone chains

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

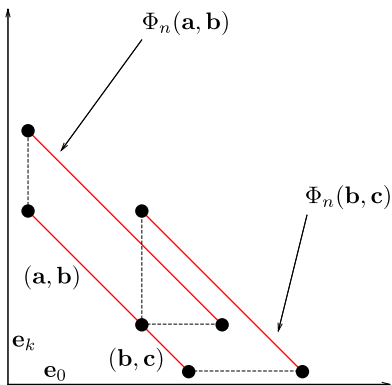
Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$



Monotone chains

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

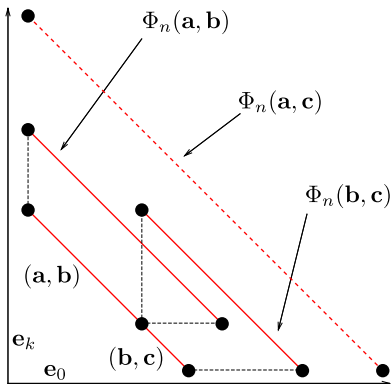
Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$



Monotone chains

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

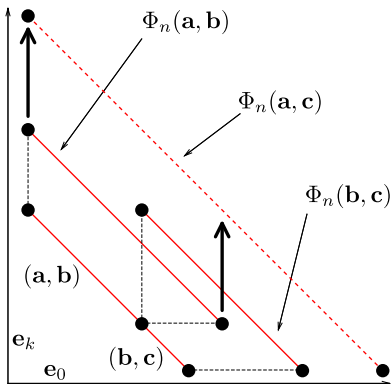
Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$



Monotone chains

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

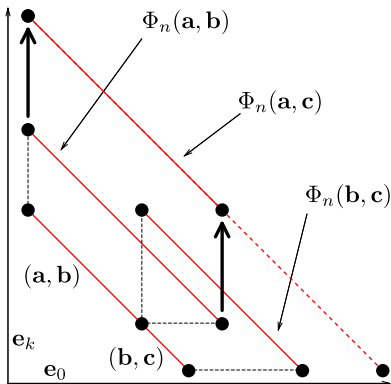
Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$



Monotone chains

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

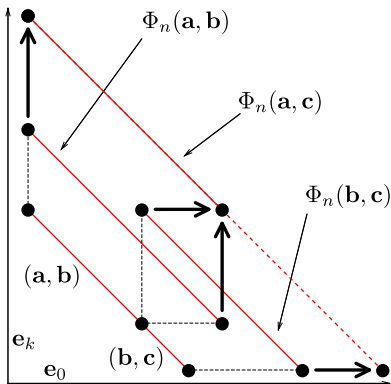
Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$



Monotone chains

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

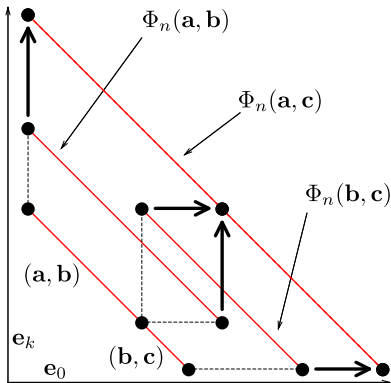
Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$



Monotone chains

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

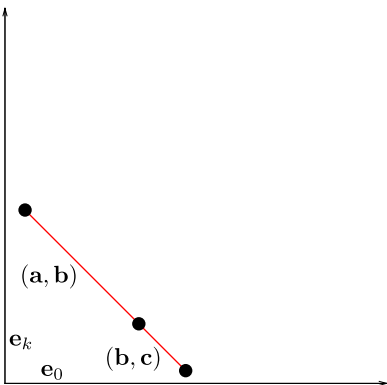
Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{c}| < |\mathbf{b}|.$$



Monotone chains

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

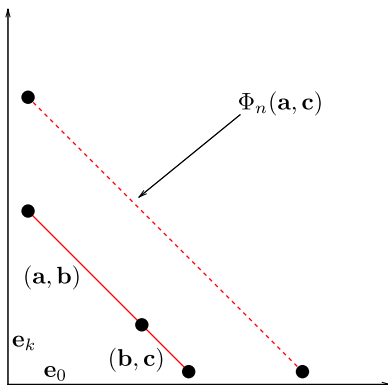
Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \left| |\mathbf{a}| - |\mathbf{a}'| \right|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{c}| < |\mathbf{b}|.$$



Monotone chains

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

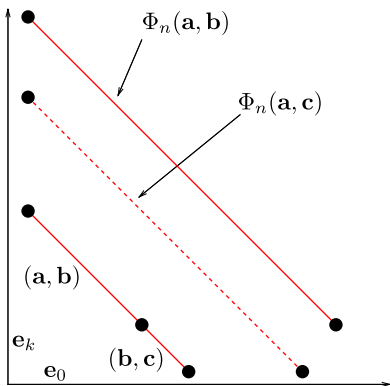
Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \left| |\mathbf{a}| - |\mathbf{a}'| \right|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{c}| < |\mathbf{b}|.$$



Monotone chains

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

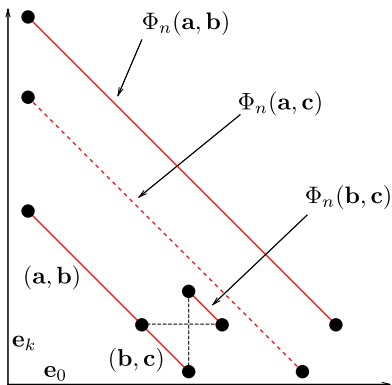
Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \left| |\mathbf{a}| - |\mathbf{a}'| \right|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{c}| < |\mathbf{b}|.$$



Monotone chains

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle, \quad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

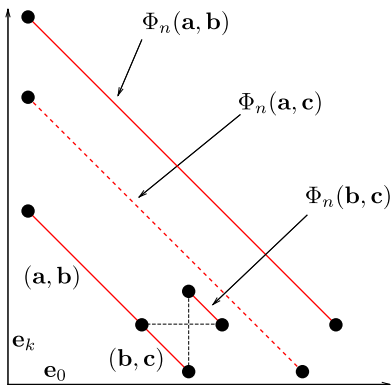
$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{c}| < |\mathbf{b}|.$$

Need: *monotone chains* are sufficient for transitive closure.



The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{array}{ccc} \{\text{minimal presentations of } M_n\} & \longleftrightarrow & \{\text{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

Key Lemma

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$ (costly), then $a_0 > 0$ and $a'_k > 0$.

The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

Key Lemma

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$ (costly), then $a_0 > 0$ and $a'_k > 0$.

The above lemma ensures:

The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

Key Lemma

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$ (costly), then $a_0 > 0$ and $a'_k > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of Φ_n .

The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

Key Lemma

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$ (costly), then $a_0 > 0$ and $a'_k > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of Φ_n .
- Any two factorizations are connected by a monotone ρ -chain.

The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

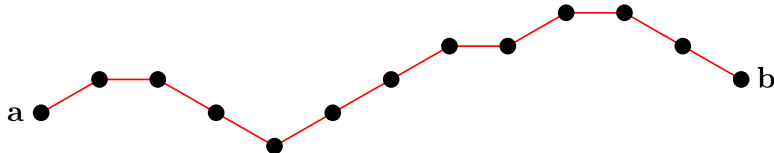
$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

Key Lemma

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$ (costly), then $a_0 > 0$ and $a'_k > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of Φ_n .
- Any two factorizations are connected by a monotone ρ -chain.



The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

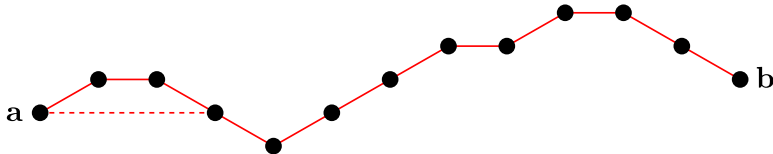
$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

Key Lemma

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$ (costly), then $a_0 > 0$ and $a'_k > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of Φ_n .
- Any two factorizations are connected by a monotone ρ -chain.



The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

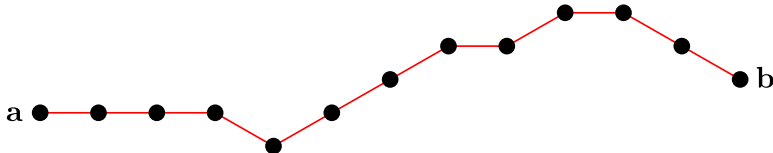
$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

Key Lemma

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$ (costly), then $a_0 > 0$ and $a'_k > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of Φ_n .
- Any two factorizations are connected by a monotone ρ -chain.



The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

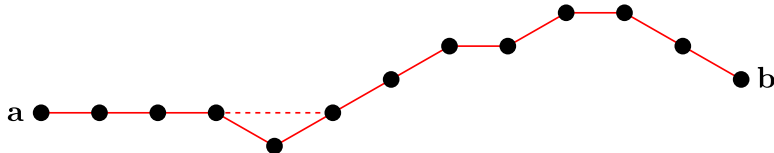
$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

Key Lemma

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$ (costly), then $a_0 > 0$ and $a'_k > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of Φ_n .
- Any two factorizations are connected by a monotone ρ -chain.



The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

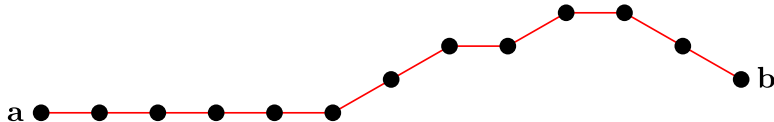
$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

Key Lemma

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$ (costly), then $a_0 > 0$ and $a'_k > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of Φ_n .
- Any two factorizations are connected by a monotone ρ -chain.



The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

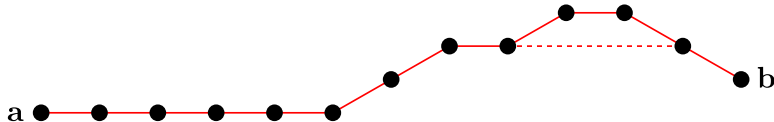
$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

Key Lemma

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$ (costly), then $a_0 > 0$ and $a'_k > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of Φ_n .
- Any two factorizations are connected by a monotone ρ -chain.



The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

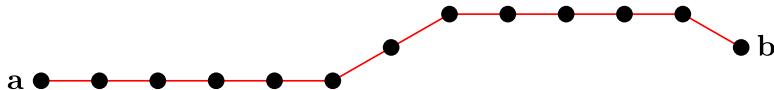
$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

Key Lemma

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$ (costly), then $a_0 > 0$ and $a'_k > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of Φ_n .
- Any two factorizations are connected by a monotone ρ -chain.



The main result

Theorem (Conway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

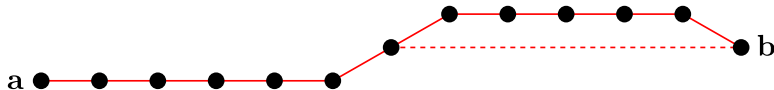
$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

Key Lemma

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$ (costly), then $a_0 > 0$ and $a'_k > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of Φ_n .
- Any two factorizations are connected by a monotone ρ -chain.



The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

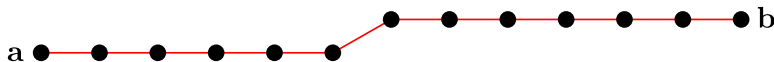
$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

Key Lemma

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$ (costly), then $a_0 > 0$ and $a'_k > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of Φ_n .
- Any two factorizations are connected by a monotone ρ -chain.



The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{array}{ccc} \{\text{minimal presentations of } M_n\} & \longleftrightarrow & \{\text{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{array}{ccc} \{\text{minimal presentations of } M_n\} & \leftrightarrow & \{\text{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \mapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

Consequences:

The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{array}{ccc} \{\text{minimal presentations of } M_n\} & \longleftrightarrow & \{\text{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

Consequences:

- The Betti numbers $n \mapsto \beta_0(M_n)$ are eventually r_k -periodic:
Graded degrees for $\beta_0(M_n)$ are $\pi_n(\mathbf{a})$ for each $(\mathbf{a}, \mathbf{a}') \in \rho$

The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\longleftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\longmapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

Consequences:

- The Betti numbers $n \mapsto \beta_0(M_n)$ are eventually r_k -periodic:
Graded degrees for $\beta_0(M_n)$ are $\pi_n(\mathbf{a})$ for each $(\mathbf{a}, \mathbf{a}') \in \rho$
- The function $n \mapsto \Delta(M_n)$ is eventually singleton:
 $\Delta(M_n) = \{d\}$ when $\|\mathbf{a}\| - \|\mathbf{a}'\| \in \{0, d\}$ for all $(\mathbf{a}, \mathbf{a}') \in \rho$

The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\longleftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\longmapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

Consequences:

- The Betti numbers $n \mapsto \beta_0(M_n)$ are eventually r_k -periodic:
Graded degrees for $\beta_0(M_n)$ are $\pi_n(\mathbf{a})$ for each $(\mathbf{a}, \mathbf{a}') \in \rho$
- The function $n \mapsto \Delta(M_n)$ is eventually singleton:
 $\Delta(M_n) = \{d\}$ when $\|\mathbf{a}\| - \|\mathbf{a}'\| \in \{0, d\}$ for all $(\mathbf{a}, \mathbf{a}') \in \rho$
- The function $n \mapsto c(M_n)$ is eventually r_k -quasilinear:
 $c(M_n)$ is determined by $\{\text{minimal presentations of } M_n\}$

Application: computing minimal presentations

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{array}{ccc} \{\text{minimal presentations of } M_n\} & \leftrightarrow & \{\text{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \mapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

Application: computing minimal presentations

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{array}{ccc} \{\text{minimal presentations of } M_n\} & \longleftrightarrow & \{\text{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

$$S = \langle 1234, 1240, 1243, 1254 \rangle$$

Application: computing minimal presentations

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

$$S = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \dots, n + r_k \rangle$$

Application: computing minimal presentations

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\longleftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\longmapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

$$S = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \dots, n + r_k \rangle$$

$$n = 1234$$

Application: computing minimal presentations

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

$$S = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \dots, n + r_k \rangle$$

$$n = 1234 \quad r_3 = 1254 - 1234 = 20$$

Application: computing minimal presentations

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

$$S = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \dots, n + r_k \rangle$$

$$n = 1234 \quad r_3 = 1254 - 1234 = 20$$

Verify $n > r_k^2$:

Application: computing minimal presentations

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

$$S = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \dots, n + r_k \rangle$$

$$n = 1234 \quad r_3 = 1254 - 1234 = 20$$

Verify $n > r_k^2$: $1234 > 400$

Application: computing minimal presentations

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

$$S = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \dots, n + r_k \rangle$$

$$n = 1234 \quad r_3 = 1254 - 1234 = 20$$

Verify $n > r_k^2$: $1234 > 400$ ✓

Application: computing minimal presentations

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

$$S = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \dots, n + r_k \rangle$$

$$n = 1234 \quad r_3 = 1254 - 1234 = 20$$

Verify $n > r_k^2$: $1234 > 400$ ✓

$\langle 414, 420, 423, 434 \rangle$:

Application: computing minimal presentations

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

$$S = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \dots, n + r_k \rangle$$

$$n = 1234 \quad r_3 = 1254 - 1234 = 20$$

Verify $n > r_k^2$: $1234 > 400$ ✓

$$\langle 414, 420, 423, 434 \rangle :$$

$$((0, 0, 8, 0), (3, 2, 0, 3)),$$

$$((0, 1, 6, 0), (4, 0, 0, 3)),$$

$$((0, 3, 0, 0), (1, 0, 2, 0)),$$

$$((21, 1, 0, 0), (0, 0, 0, 21)),$$

$$((25, 0, 0, 0), (0, 0, 6, 18))$$

Application: computing minimal presentations

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\longleftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\longmapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

$$S = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \dots, n + r_k \rangle$$

$$n = 1234 \quad r_3 = 1254 - 1234 = 20$$

Verify $n > r_k^2$: $1234 > 400$ ✓

$$\begin{aligned} \langle 414, 420, 423, 434 \rangle : & \quad \langle 1234, 1240, 1243, 1254 \rangle : \\ ((0, 0, 8, 0), (3, 2, 0, 3)), & \quad ((0, 0, 8, 0), (3, 2, 0, 3)), \\ ((0, 1, 6, 0), (4, 0, 0, 3)), & \quad ((0, 1, 6, 0), (4, 0, 0, 3)), \\ ((0, 3, 0, 0), (1, 0, 2, 0)), & \quad ((0, 3, 0, 0), (1, 0, 2, 0)), \\ ((21, 1, 0, 0), (0, 0, 0, 21)), & \quad ((62, 1, 0, 0), (0, 0, 0, 62)), \\ ((25, 0, 0, 0), (0, 0, 6, 18)) & \quad ((66, 0, 0, 0), (0, 0, 6, 59)) \end{aligned}$$

Application: computing minimal presentations

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{array}{ccc} \{\text{minimal presentations of } M_n\} & \leftrightarrow & \{\text{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \mapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

Application: computing minimal presentations

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

n	M_n	Min. Pres. Runtime
50	$\langle 50, 56, 59, 70 \rangle$	1 ms
200	$\langle 200, 206, 209, 220 \rangle$	40 ms
400	$\langle 400, 406, 409, 420 \rangle$	210 ms
1000	$\langle 1000, 1006, 1009, 1020 \rangle$	3 sec
3000	$\langle 3000, 3006, 3009, 3020 \rangle$	2 min
5000	$\langle 5000, 5006, 5009, 5020 \rangle$	18 min
10000	$\langle 10000, 10006, 10009, 10020 \rangle$	4.2 hr

Application: computing minimal presentations

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\leftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\mapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

n	M_n	Min. Pres. Runtime
50	$\langle 50, 56, 59, 70 \rangle$	1 ms
200	$\langle 200, 206, 209, 220 \rangle$	40 ms
400	$\langle 400, 406, 409, 420 \rangle$	210 ms
1000	$\langle 1000, 1006, 1009, 1020 \rangle$	3 sec 210 ms
3000	$\langle 3000, 3006, 3009, 3020 \rangle$	2 min 210 ms
5000	$\langle 5000, 5006, 5009, 5020 \rangle$	18 min 210 ms
10000	$\langle 10000, 10006, 10009, 10020 \rangle$	4.2 hr 210 ms

Application: computing minimal presentations

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\begin{aligned} \{ \text{minimal presentations of } M_n \} &\longleftrightarrow \{ \text{minimal presentations of } M_{n+r_k} \} \\ \rho \subset \ker \pi_n &\longmapsto \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{aligned}$$

n	M_n	Min. Pres. Runtime
50	$\langle 50, 56, 59, 70 \rangle$	1 ms
200	$\langle 200, 206, 209, 220 \rangle$	40 ms
400	$\langle 400, 406, 409, 420 \rangle$	210 ms
1000	$\langle 1000, 1006, 1009, 1020 \rangle$	3 sec 210 ms
3000	$\langle 3000, 3006, 3009, 3020 \rangle$	2 min 210 ms
5000	$\langle 5000, 5006, 5009, 5020 \rangle$	18 min 210 ms
10000	$\langle 10000, 10006, 10009, 10020 \rangle$	4.2 hr 210 ms

GAP Numerical Semigroups Package, available at

<http://www.gap-system.org/Packages/numericalsgps.html>.

Future work

Future work

- Improve the bound $n > r_k^2$.

Future work

- Improve the bound $n > r_k^2$.
- Consider other quantities under shifting.

Future work

- Improve the bound $n > r_k^2$.
- Consider other quantities under shifting.

Frobenius number: $F(S) = \max(\mathbb{N} \setminus S)$.

Future work

- Improve the bound $n > r_k^2$.
- Consider other quantities under shifting.

Frobenius number: $F(S) = \max(\mathbb{N} \setminus S)$.

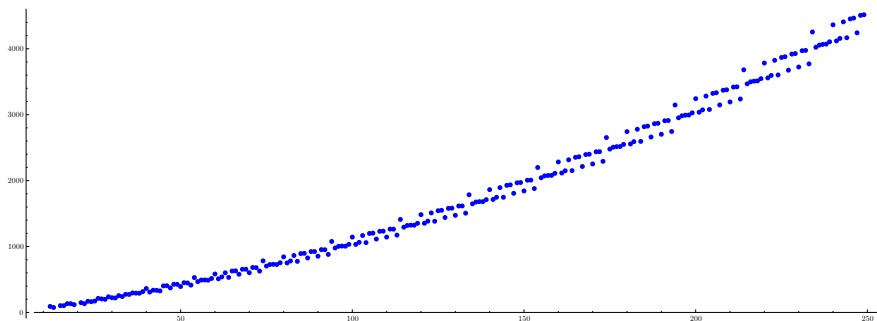
Sneak peek for $F(\langle n, n + 6, n + 9, n + 20 \rangle)$:

Future work





- Improve the bound $n > r_k^2$.
- Consider other quantities under shifting.

Frobenius number: $F(S) = \max(\mathbb{N} \setminus S)$.





Sneak peek for $F(\langle n, n + 6, n + 9, n + 20 \rangle)$:



References

-  S. Chapman, N. Kaplan, T. Lemburg, A. Niles, and C. Zlogar (2014),
Shifts of generators and delta sets of numerical monoids,
Internat. J. Algebra Comput. 24 (2014), no. 5, 655–669.
-  T. Vu (2014),
Periodicity of Betti numbers of monomial curves,
Journal of Algebra 418 (2014) 66–90.
-  R. Conaway, F. Gotti, J. Horton, C. O’Neill, R. Pelayo, M. Williams, and
B. Wissman (2016)
Minimal presentations of shifted numerical monoids.
in preparation.
-  M. Delgado, P. García-Sánchez, and J. Morais,
GAP numerical semigroups package
<http://www.gap-system.org/Packages/numericalsgps.html>.

References

-  S. Chapman, N. Kaplan, T. Lemburg, A. Niles, and C. Zlogar (2014),
Shifts of generators and delta sets of numerical monoids,
Internat. J. Algebra Comput. 24 (2014), no. 5, 655–669.
-  T. Vu (2014),
Periodicity of Betti numbers of monomial curves,
Journal of Algebra 418 (2014) 66–90.
-  R. Conaway, F. Gotti, J. Horton, C. O’Neill, R. Pelayo, M. Williams, and
B. Wissman (2016)
Minimal presentations of shifted numerical monoids.
in preparation.
-  M. Delgado, P. García-Sánchez, and J. Morais,
GAP numerical semigroups package
<http://www.gap-system.org/Packages/numericalsgps.html>.

Thanks!