Invariants of non-unique factorization

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Factorial domains

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1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, and
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The point: it's nontrivial.

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$$\begin{array}{rcl} (R, +, \cdot) & \rightsquigarrow & (R \setminus \{0\}, \cdot) \\ (\mathbb{C}[M], +, \cdot) & \rightsquigarrow & (M, \cdot) \end{array}$$

Interesting monoids

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An *arithmetical congruence monoid* is a **multiplicative** submonoid

$$M_{a,b} = \{n : n \equiv a \bmod b\} \subset \mathbb{Z}_{>0}$$

for $a, b > 0$ with $a^2 \equiv a \bmod b$.

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  $= (3^2) \cdot (7^2) = (3 \cdot 7) \cdot (3 \cdot 7)$.

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- This is (almost) the best we could hope for.

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- $\rho(n) \leq 20/6$ for all $n \in S$.

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- Elasticity of $M_{4,6}$ is *not accepted*.

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$$\mathsf{Z}_S(n) = \left\{ \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k : n = a_1 n_1 + \cdots + a_k n_k \right\}$$

denotes the *set of factorizations* of n.

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is the set of solutions to a *knapsack problem*.

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Motivation: discrete optimization

Let $A = [n_1 \ n_2 \ \cdots \ n_k]$.

$$\begin{aligned} \mathsf{Z}_S(n) &= \left\{\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k : n = a_1 n_1 + \cdots + a_k n_k\right\} \\ &= \{\mathbf{a} \in \mathbb{N}^k : n = A \cdot \mathbf{a}\} \end{aligned}$$

is the set of solutions to a *knapsack problem*.

Motivation: additive combinatorics

A brief aside for motivation...

Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$.

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Let $F(S) = \max(\mathbb{N} \setminus S)$ denote the *Frobenius number* of S.

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$$F(\langle n_1, n_2 \rangle) = n_1 n_2 - n_1 - n_2$$

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$$\begin{aligned} F(\langle n_1, n_2 \rangle) &= n_1 n_2 - n_1 - n_2 \\ F(\langle 6, 9, 20 \rangle) &= 43 \end{aligned}$$

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Frobenius coin-exchange problem: find the largest unchangeable value with coins $n_1, \ldots, n_k$.

The set of factorizations: a complete invariant

Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$,

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Definition

The *set of factorizations* of S is the set

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Example

$$\mathcal{Z}(\langle 6, 9, 20 \rangle) = \left\{ \right.$$

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Example

$$\mathcal{Z}(\langle 6, 9, 20 \rangle) = \left\{ \begin{array}{l} \{(1,0,0)\}, \{(0,1,0)\}, \{(0,0,1)\}, \\ \\ \\ \end{array} \right.$$

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Observations

- $\mathcal{Z}(S)$ encapsulates all relations between $n_1, \ldots, n_k$.

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Observations

- $\mathcal{Z}(S)$ encapsulates all relations between $n_1, \ldots, n_k$.
- $\mathcal{Z}(S)$ determines S up to isomorphism.

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Definition

The *set of factorizations* of S is the set

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Observations

- $\mathcal{Z}(S)$ encapsulates all relations between $n_1, \ldots, n_k$.
- $\mathcal{Z}(S)$ determines S up to isomorphism.
- $\mathcal{Z}(S)$ is **enormous**.

The set of factorizations: a complete invariant

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Observations

- $\mathcal{Z}(S)$ encapsulates all relations between $n_1, \ldots, n_k$.
- $\mathcal{Z}(S)$ determines S up to isomorphism.
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Example

In $McN = \langle 6, 9, 20 \rangle$, $\mathsf{Z}(60)$ uniquely determines $\mathcal{Z}(McN)$.

A step in the right direction: length sets

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Definition

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$,

$$\mathsf{L}(n) \quad = \quad \{a_1 + \cdots + a_k : (a_1, \ldots, a_k) \in \mathsf{Z}(n)\}$$

denotes the *length set* of n.

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Example

$McN = \langle 6, 9, 20 \rangle$:

$$\mathsf{Z}(60) = \{(0,0,3), (1,6,0), (4,4,0), (7,2,0), (10,0,0)\}$$

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Example

$McN = \langle 6, 9, 20 \rangle$:

$$\begin{aligned} \mathsf{Z}(60) &= \{(0,0,3), (1,6,0), (4,4,0), (7,2,0), (10,0,0)\} \\ \mathsf{L}(60) &= \{3, 7, 8, 9, 10\} \end{aligned}$$

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Example

$McN = \langle 6, 9, 20 \rangle$:

$$\begin{aligned} \mathsf{Z}(60) &= \{(0,0,3),(1,6,0),(4,4,0),(7,2,0),(10,0,0)\} \\ \mathsf{L}(60) &= \{3,7,8,9,10\} \end{aligned}$$

$S = \langle 7, 10, 13, 16 \rangle$:

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$S = \langle 7, 10, 13, 16 \rangle$:

$$\mathsf{Z}(60) = \{(0,6,0,0), (1,4,1,0), (2,2,2,0), (3,0,3,0), \ldots\}$$

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$S = \langle 7, 10, 13, 16 \rangle$:

$$\begin{aligned} \mathsf{Z}(60) &= \{(0,6,0,0), (1,4,1,0), (2,2,2,0), (3,0,3,0), \ldots\} \\ \mathsf{L}(60) &= \{6\} \end{aligned}$$

Maximum and minimum factorization length

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$,

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For $n \in S$, let

$$\begin{aligned} \mathsf{M}(n) &= \max \mathsf{L}(n) \qquad \text{and} \\ \mathsf{m}(n) &= \min \mathsf{L}(n) \end{aligned}$$

denote the *maximum* and *minimum* factorization lengths of n.

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denote the *maximum* and *minimum* factorization lengths of n.

Example

$S = \langle 6, 9, 20 \rangle$:

$$\begin{aligned} \mathsf{Z}(60) &= \{(0,0,3), (1,6,0), (4,4,0), (7,2,0), (10,0,0)\} \\ \mathsf{M}(60) &= 10 \\ \mathsf{m}(60) &= 3 \end{aligned}$$

Maximum and minimum factorization length

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$,

$$\mathsf{M}(n) = \max\ \mathsf{L}(n) \qquad\qquad \mathsf{m}(n) = \min\ \mathsf{L}(n)$$

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Observations

Maximum and minimum factorization length

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Observations

- Max length factorization: lots of small irreducibles

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- Min length factorization: lots of large irreducibles

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Example

$S = \langle 6, 9, 20 \rangle$:

$$\mathsf{M}(40) = 2 \quad \text{and} \quad \mathsf{Z}(40) = \{(0,0,2)\}$$

Maximum and minimum factorization length

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$S = \langle 5, 16, 17, 18, 19 \rangle$:

Maximum and minimum factorization length

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$,

$$\mathsf{M}(n) = \max\ \mathsf{L}(n) \qquad\qquad \mathsf{m}(n) = \min\ \mathsf{L}(n)$$

Observations

- Max length factorization: lots of small irreducibles
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Example

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$$\mathsf{M}(40) = 2 \quad \text{and} \quad \mathsf{Z}(40) = \{(0,0,2)\}$$

$S = \langle 5, 16, 17, 18, 19 \rangle$:

$$\mathsf{m}(82) = 5 \quad \text{and} \quad \mathsf{Z}(82) = \{(0,3,2,0,0), \ldots\}$$

Maximum and minimum factorization length

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$,

$$\mathsf{M}(n) = \max \mathsf{L}(n) \qquad\qquad \mathsf{m}(n) = \min \mathsf{L}(n)$$

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$S = \langle 6, 9, 20 \rangle$:

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$S = \langle 5, 16, 17, 18, 19 \rangle$:

$$\begin{aligned} \mathsf{m}(82) &= 5 \quad \text{and} \quad \mathsf{Z}(82) = \{(0,3,2,0,0), \ldots\} \\ \mathsf{m}(462) &= 25 \quad \text{and} \quad \mathsf{Z}(462) = \{(0,3,2,0,20), \ldots\} \end{aligned}$$

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Theorem (Barron–O–Pelayo, 2014)

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n > n_k(n_{k-1} - 1)$,

$$\begin{aligned} \mathsf{M}(n + n_1) &= 1 + \mathsf{M}(n) \\ \mathsf{m}(n + n_k) &= 1 + \mathsf{m}(n) \end{aligned}$$

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Equivalently, $\mathsf{M}(n)$, $\mathsf{m}(n)$ are eventually quasilinear:

$$\begin{aligned} \mathsf{M}(n) &= \tfrac{1}{n_1} n + a_0(n) \\ \mathsf{m}(n) &= \tfrac{1}{n_k} n + b_0(n) \end{aligned}$$

for periodic functions $a_0(n)$, $b_0(n)$.

Maximum and minimum factorization length

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Interpretation:

Factorizations are chaotic for small monoid elements,

Maximum and minimum factorization length

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Theorem (Barron–O–Pelayo, 2014)

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Equivalently, $\mathsf{M}(n)$, $\mathsf{m}(n)$ *are eventually quasilinear:*

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for periodic functions $a_0(n)$, $b_0(n)$.

Interpretation:

Factorizations are chaotic for small monoid elements,

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Maximum and minimum factorization length

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n > n_k(n_{k-1} - 1)$,

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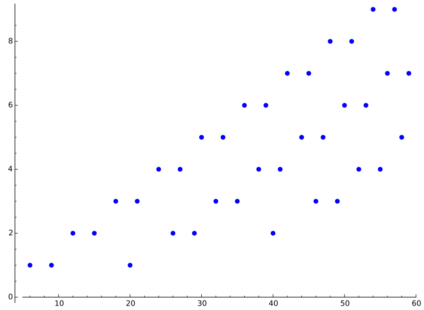
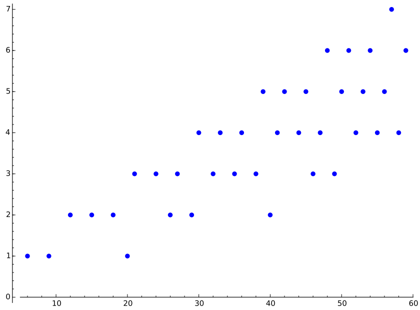


$\mathsf{M}(n) : S \to \mathbb{N}$



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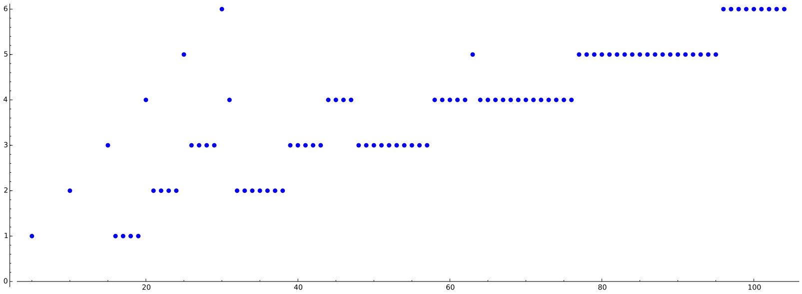

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Return to elasticity

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$$\rho(n) \;=\; \mathsf{M}(n)/\mathsf{m}(n)$$

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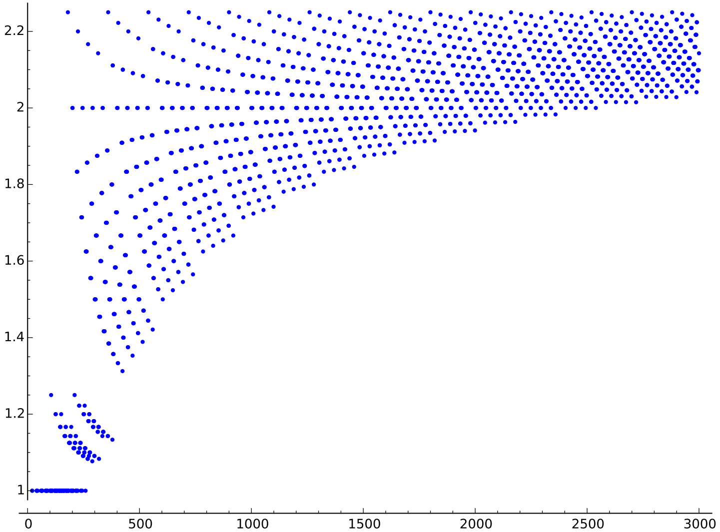
More generally:

$$\rho(m) = n_k/n_1 \quad \Leftrightarrow \quad n_1, n_k \text{ divide } m$$

Return to elasticity

$S = \langle 20, 21, 45 \rangle$:



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Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$. *For* $n \gg 0$,

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Aside from finitely many values, the set $R(S)$ equals a union of finitely many monotone increasing sequences, each approaching $\rho(S) = n_k/n_1$.

$$\rho(n + n_1 n_k) = \frac{\mathsf{M}(n) + n_k}{\mathsf{m}(n) + n_1}$$

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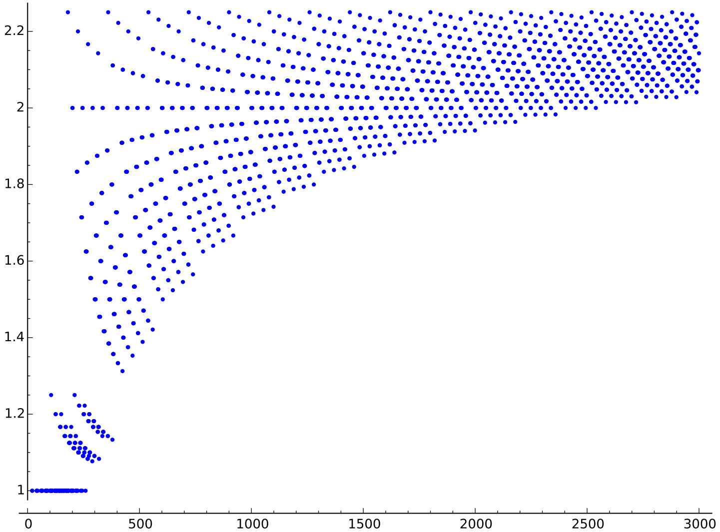

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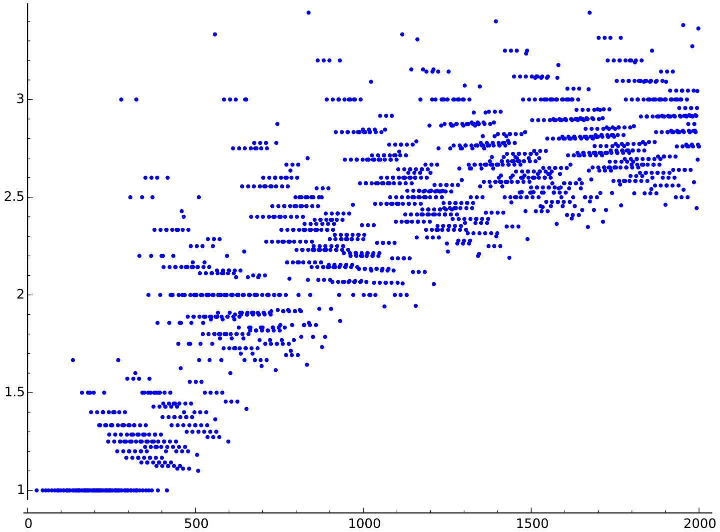
$S = \langle 27, 45, 62, 93 \rangle$:



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$$\mathsf{Z}(60) = \{(0,0,3), (1,6,0), (4,4,0), (7,2,0), (10,0,0)\}$$

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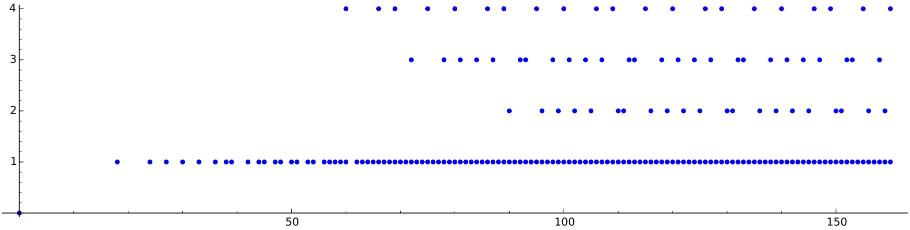

$$(n, i) \rightsquigarrow i \in \Delta(n)$$

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Theorem (Chapman–Hoyer–Kaplan, 2000)

$S = \langle n_1, \ldots, n_k \rangle$. *For* $n \geq 2kn_2n_k^2$,

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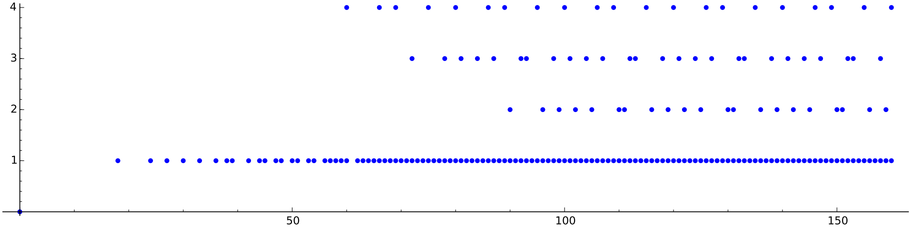

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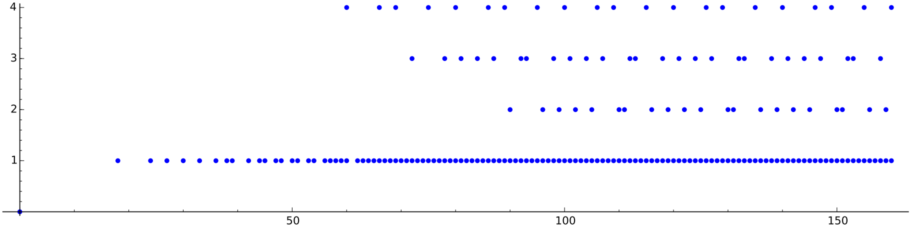

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$S = \langle 4, 6, 7 \rangle \subset \mathbb{N}$, $\mathbf{a} = (3, 1, 1), \mathbf{b} = (1, 0, 3) \in \mathsf{Z}_S(25)$.

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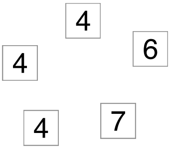

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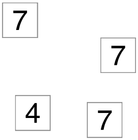

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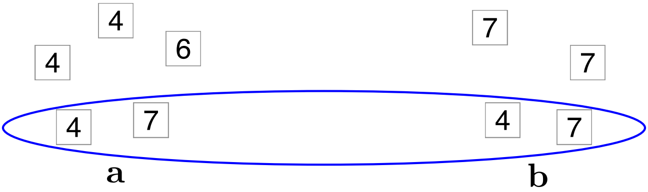

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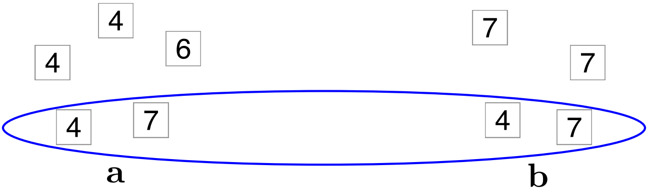

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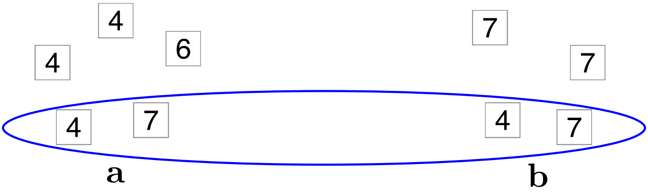

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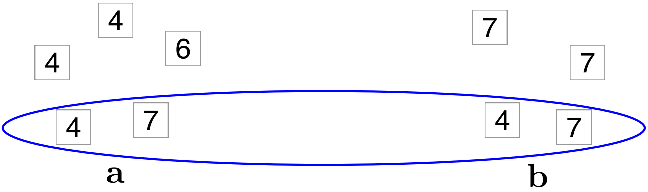

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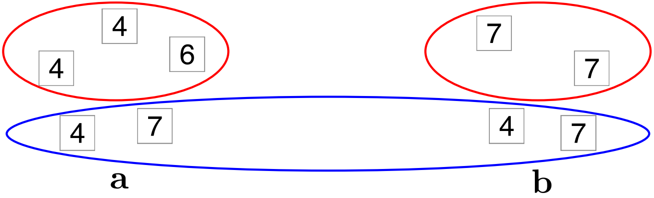

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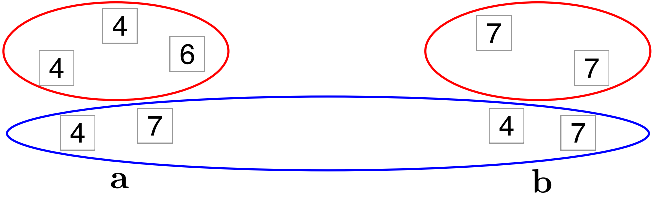

The catenary degree: definition by a big example

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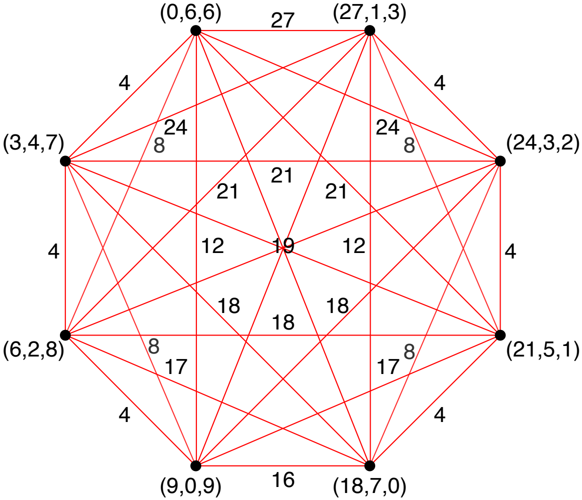

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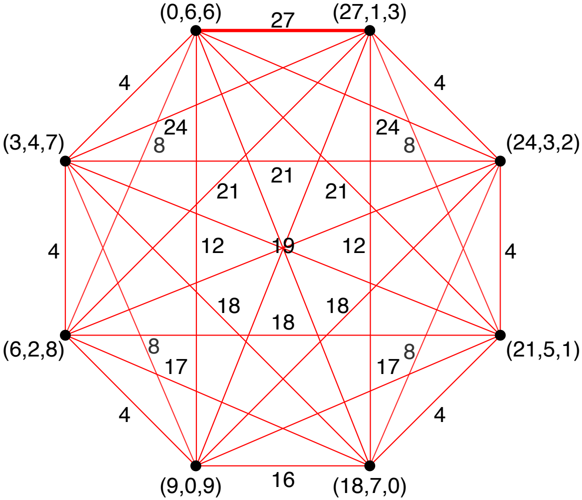

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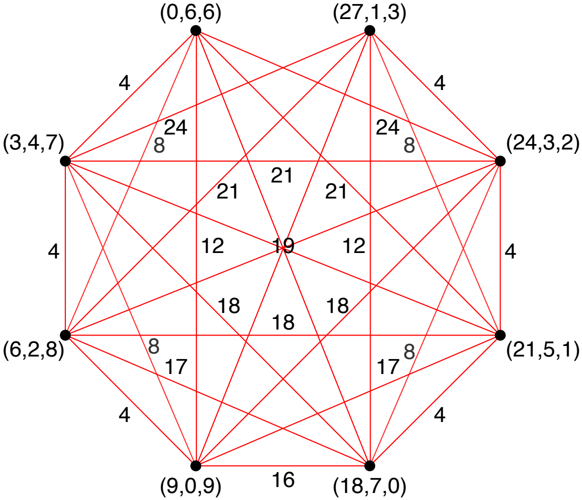

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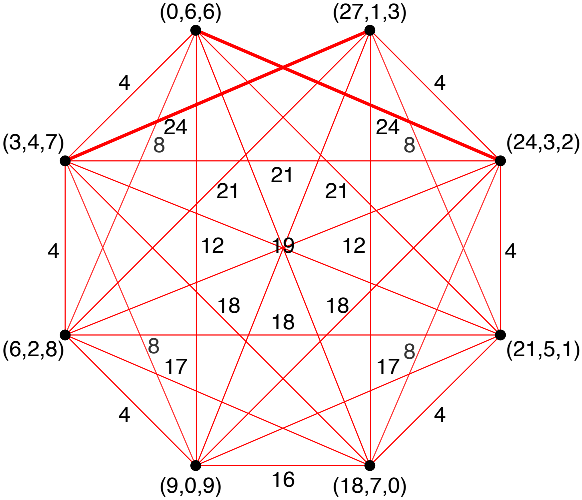

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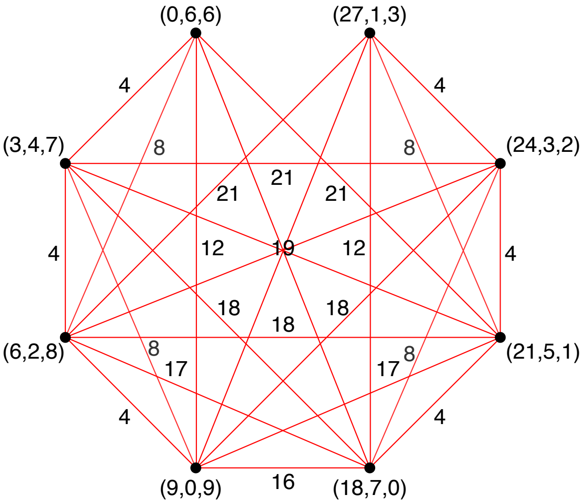

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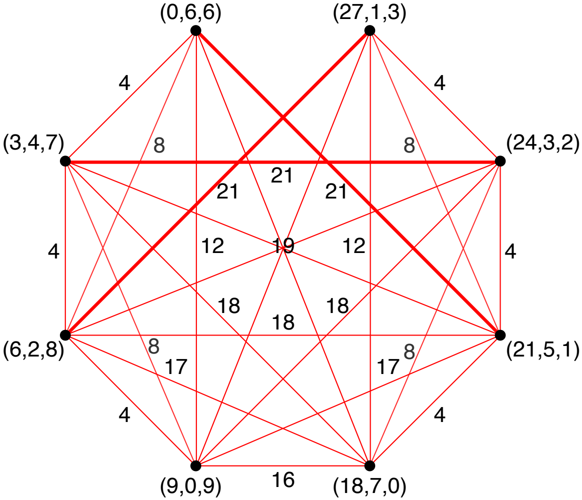

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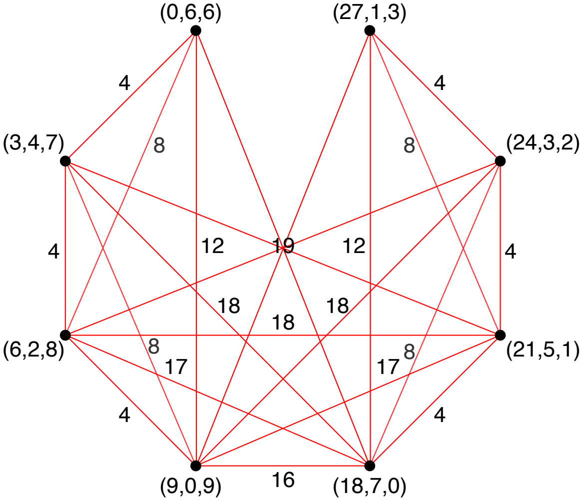

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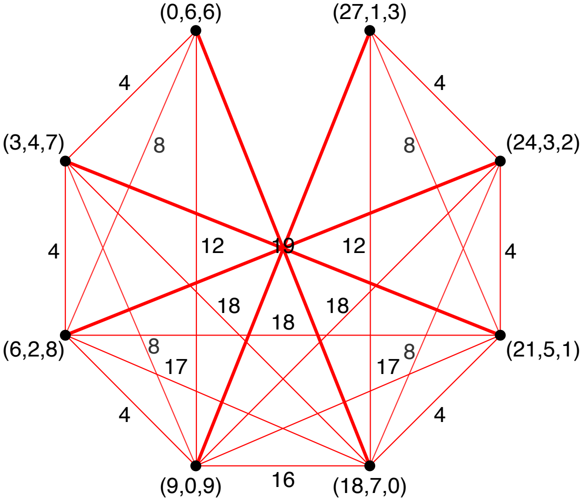

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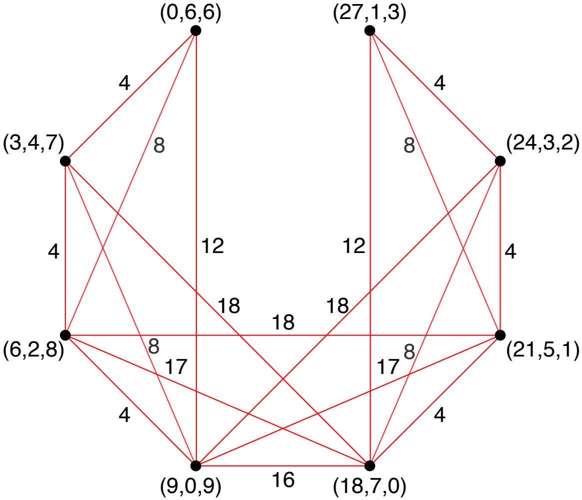

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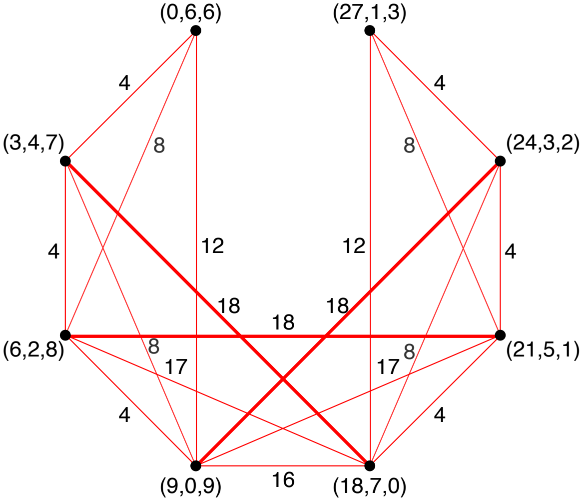

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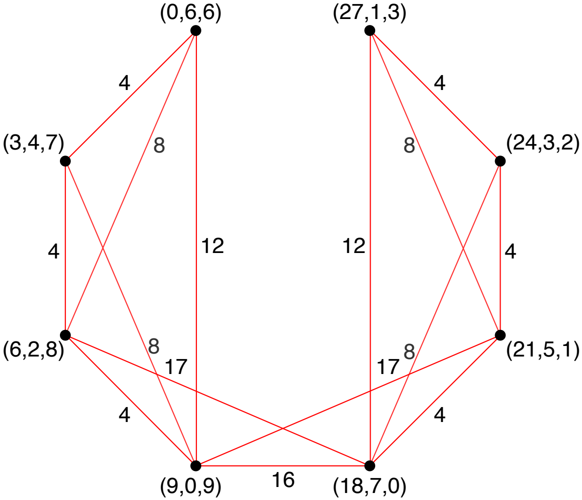

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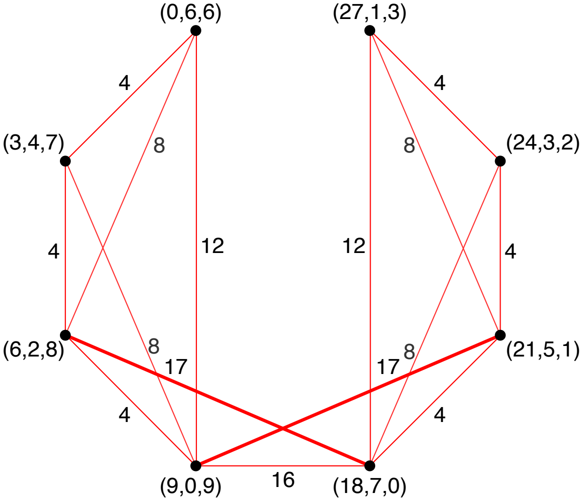

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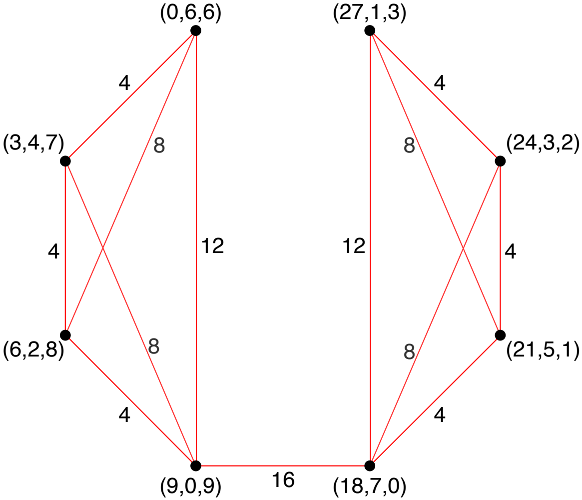

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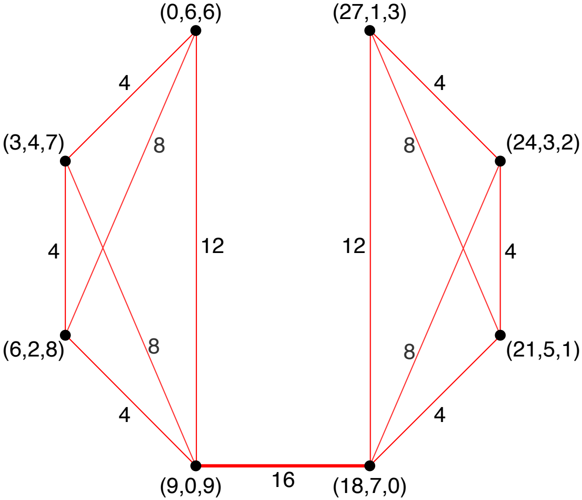

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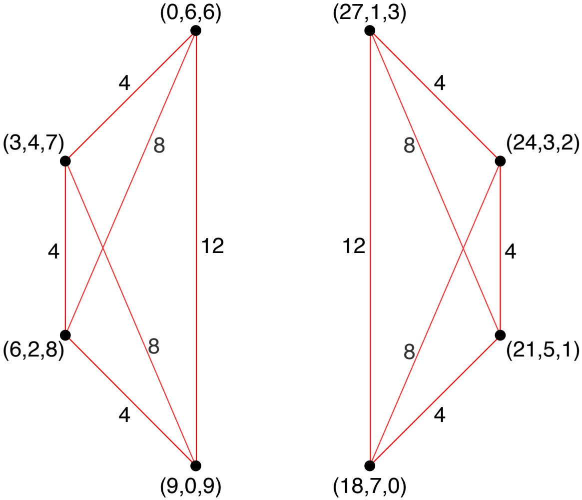

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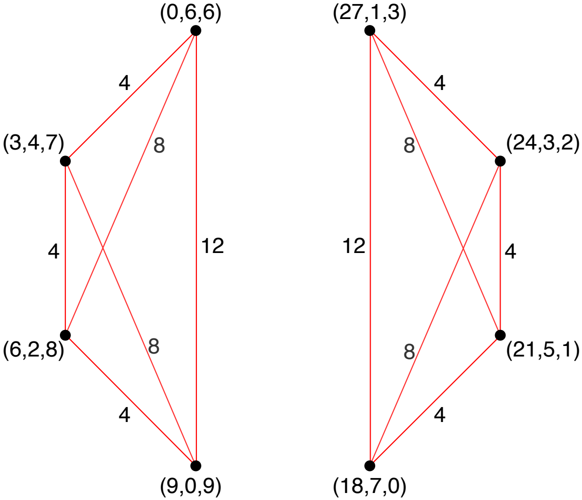
$S = \langle 11, 36, 39 \rangle, n = 450,\ c(n) = 16$



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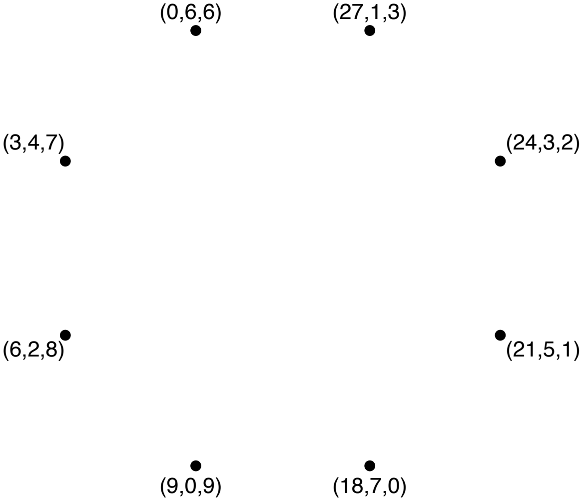

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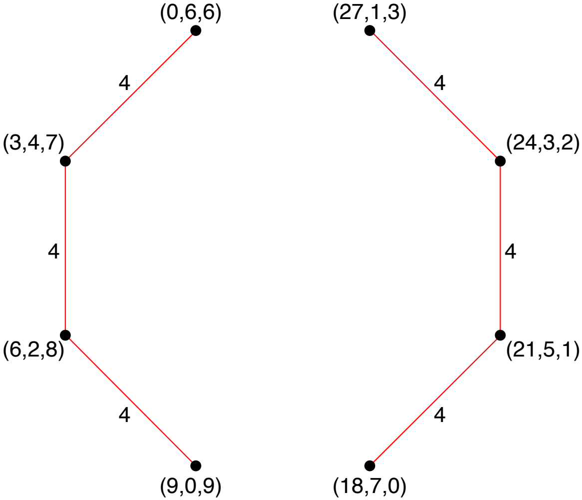

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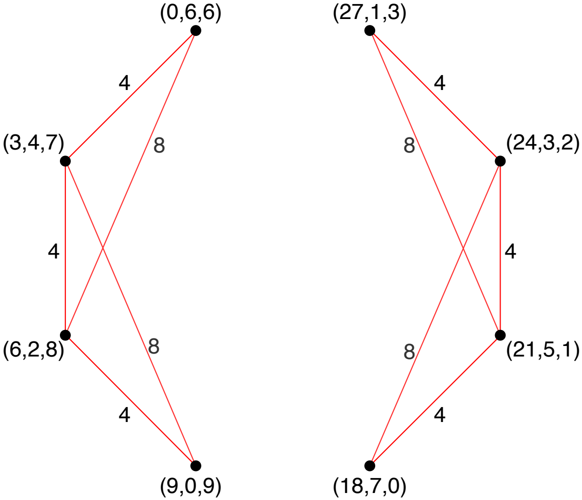

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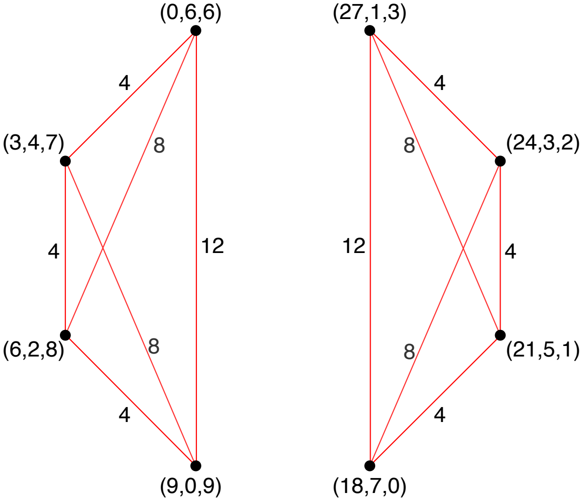

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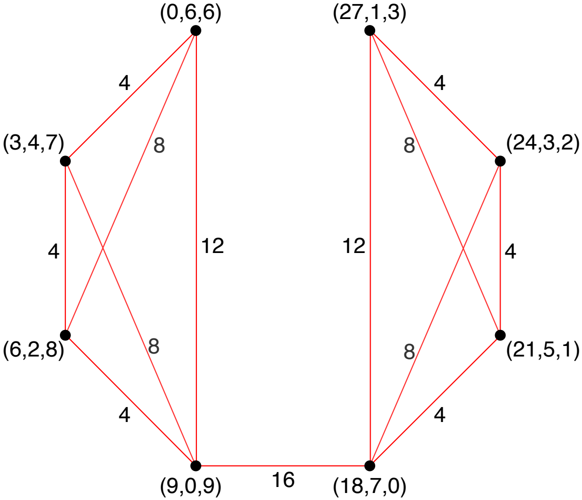

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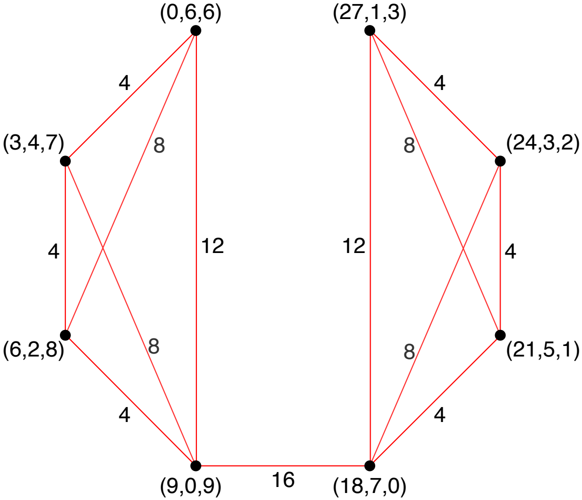

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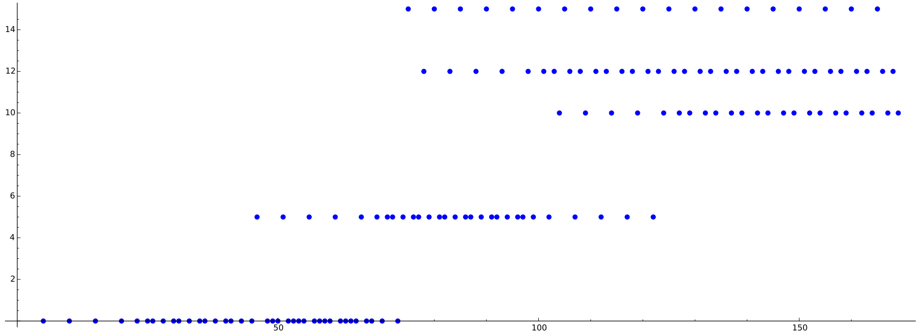

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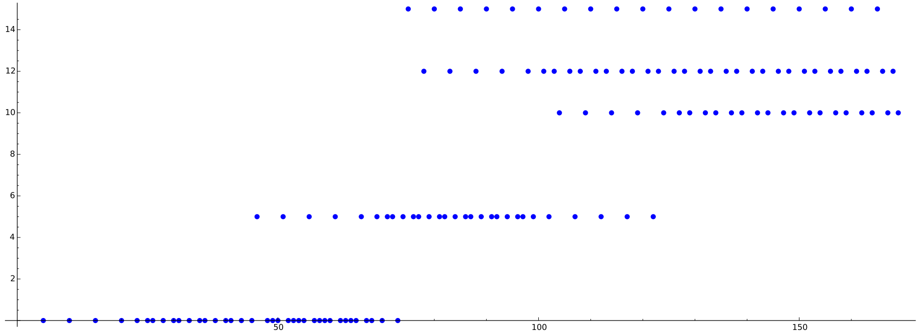

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Theorem (Chapman-Corrales-Miller-Miller-Patel, 2014)

Let $S = \langle n_1, \ldots, n_k \rangle$. *For* $n \gg 0$,

$$\mathsf{c}(n) = \mathsf{c}(n + n_1 \cdots n_k).$$

The ω-primality invariant

Fix an integral domain R and nonzero, nonunit $x \in R$.

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Define $\omega(x) = m$ minimal such that

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Fact

R is factorial if and only if every irreducible element of R is prime. Moreover, $\omega(p_1 \cdots p_r) = r$ for any primes $p_1, \ldots, p_r \in R$.

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- "Prime element of S" is different from "prime integer"

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Theorem (O–Pelayo, 2013)

Let $S = \langle n_1, \ldots, n_k \rangle \subset \mathbb{N}$*. For* $n \gg 0$*,*

$$\omega(n) = \frac{1}{n_1} n + a_0(n)$$

for some n_1*-periodic function* $a_0(n)$*. Equivalently,*

$$\omega(n + n_1) = 1 + \omega(n)$$

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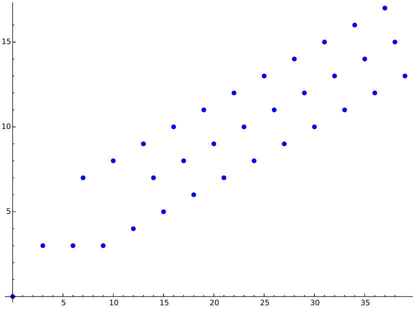
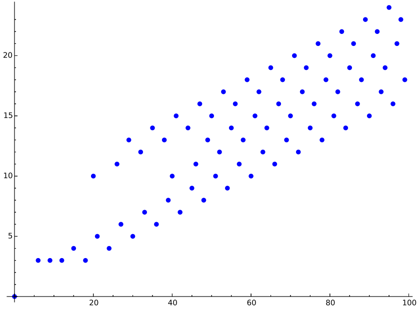


$S = \langle 3, 7 \rangle$



$S = \langle 6, 9, 20 \rangle$

Bringing it all together

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- $\omega(n)$ is eventually quasilinear
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Answer: Hilbert functions!

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The values of $\mathsf{M}(n)$, $\mathsf{m}(n)$, $\omega(n)$, $\Delta(n)$, *and* $\mathsf{c}(n)$ *over any numerical monoid are each determined by Hilbert functions.*

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Hilbert's Theorem $\Rightarrow$ $\mathrm{M}(n)$ is quasilinear.

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Thanks!