# Numerical semigroups in Sage

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$$32 = 15 + 17 \qquad\qquad 56 = 2 \cdot 17 + 22$$

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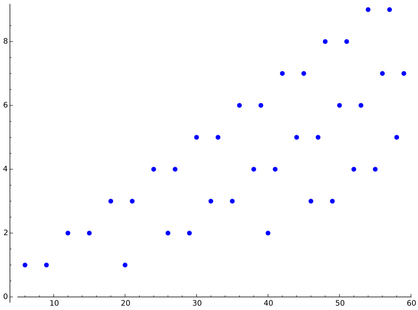
- $\omega$-primality:

$$\omega(n) \in \mathbb{Z}_{\ge 1}$$

Invariant behavior for large numerical monoid elements:

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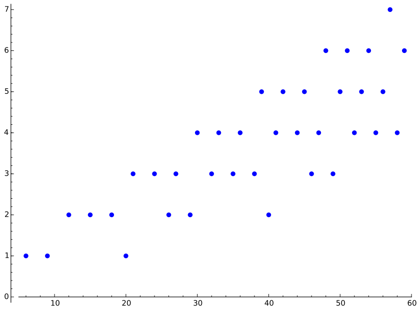


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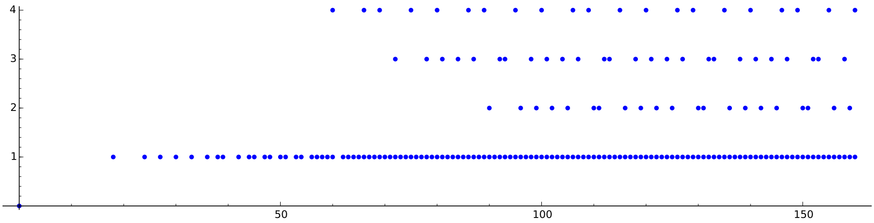


$$\Delta : \langle 6, 9, 20 \rangle \to 2^{\mathbb{N}}$$

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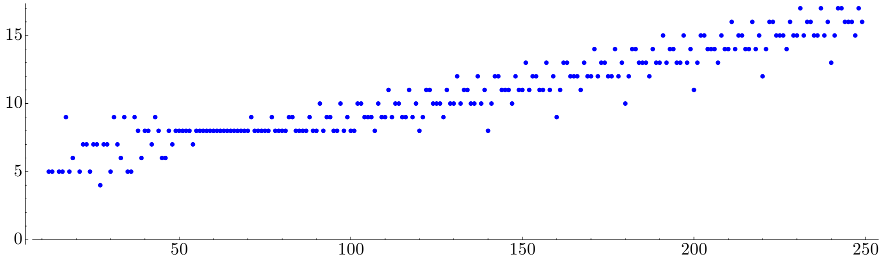


$$n \longmapsto \mathsf{c}(\langle n, n+6, n+9, n+20 \rangle)$$

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Let's see it in action!

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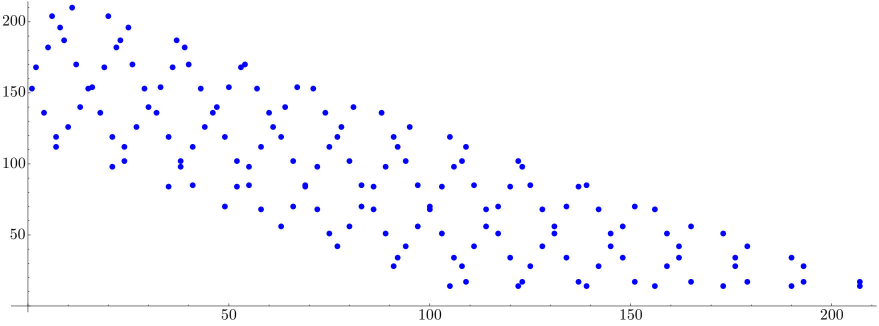


$S = \langle 14, 17 \rangle$

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