Shifting numerical monoids

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Dec 5, 2016

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A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

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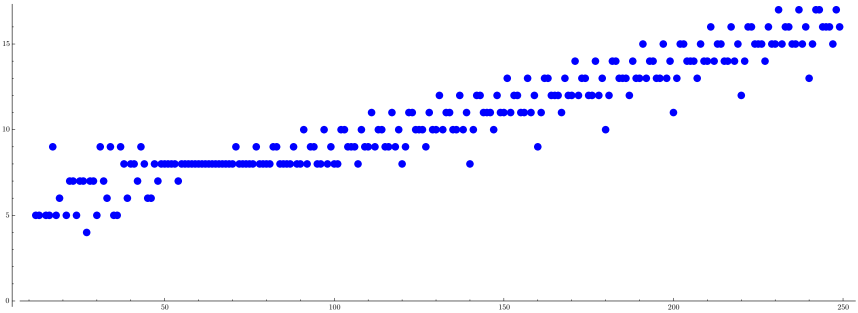

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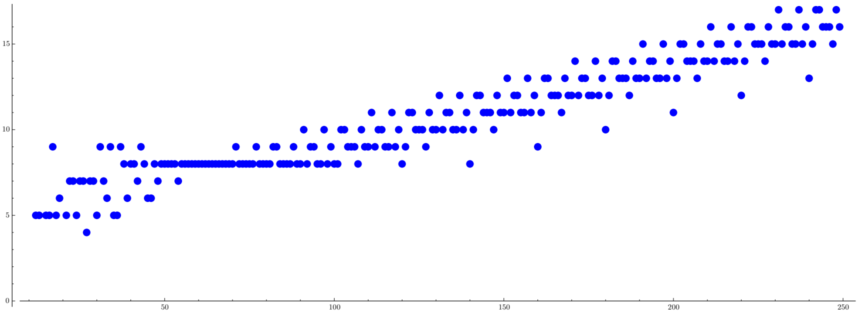
$\mathsf{c}(M_n)$ is periodic-linear (quasilinear) for $n \geq 126$.



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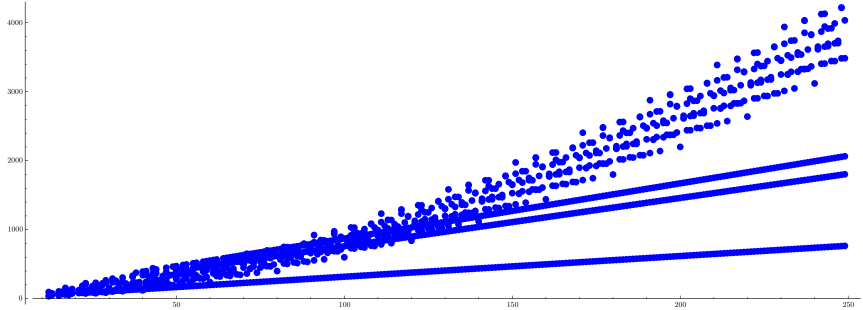

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Underlying cause: minimal presentations!

Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

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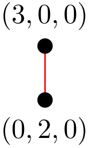

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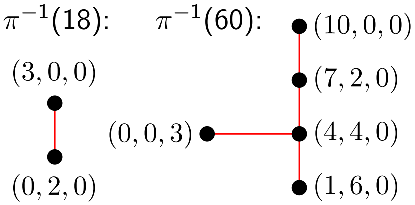

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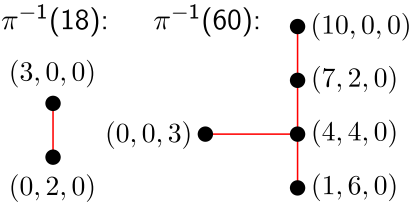

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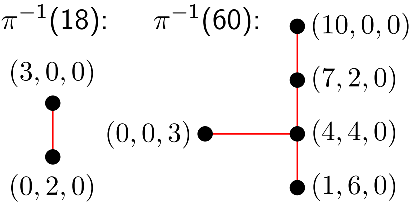

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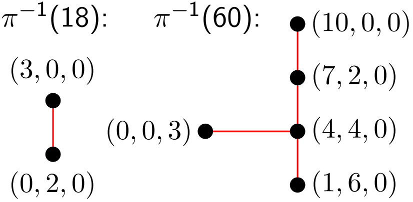

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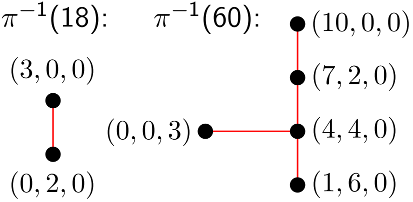

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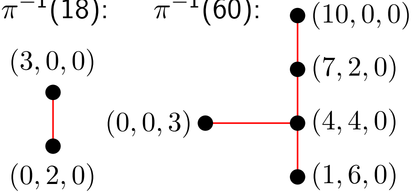

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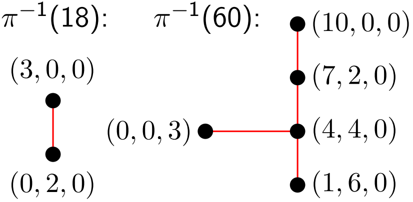

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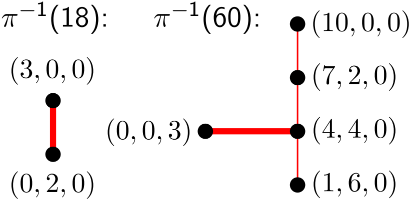

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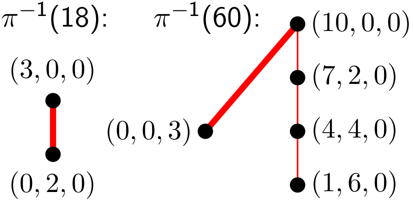

All minimal presentations:

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Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

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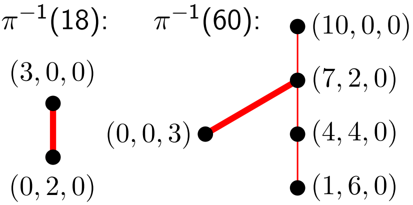

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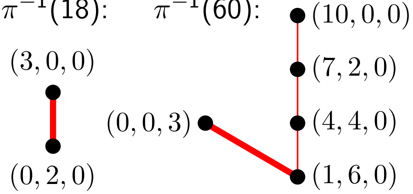

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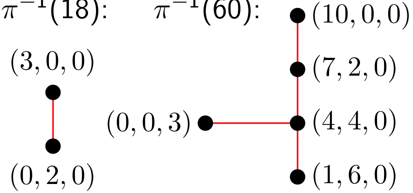

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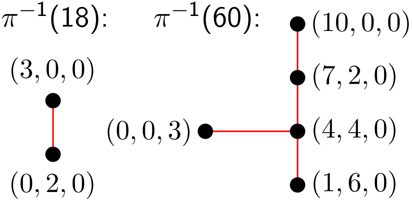

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$$\beta_0(I_S) = \{18, 60\}$$

Intuition: "sufficiently shifted" monoids

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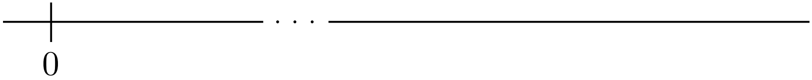

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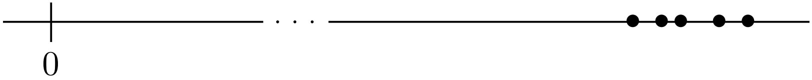

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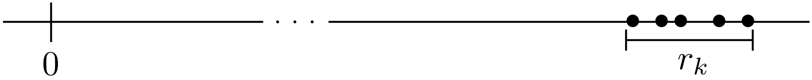

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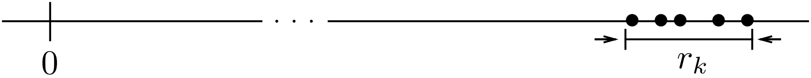

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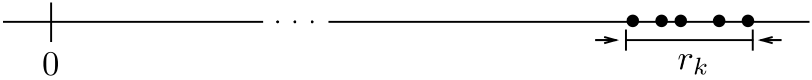

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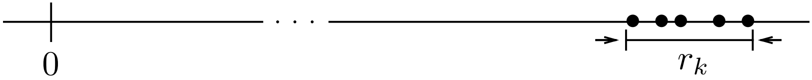

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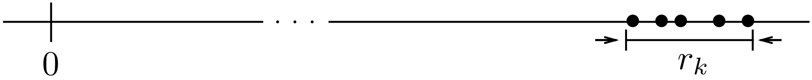

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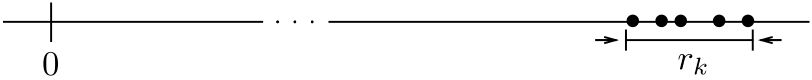

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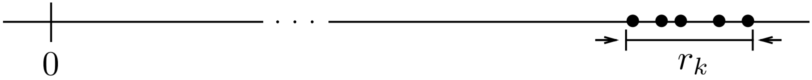

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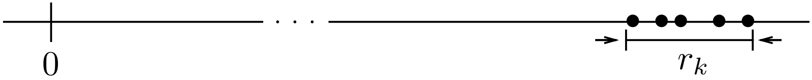

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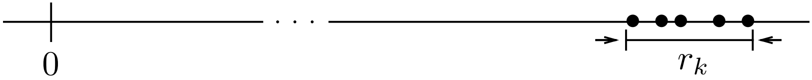

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$$\text{mostly } a_k \quad \longleftrightarrow \quad \text{mostly } b_0$$

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DON'T PANIC!

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M_{450}:

$$\left\{ \begin{array}{l} ((\ 0, 0, 8, 0), (3, 2, 0,\ \ 3)), ((\ 0, 1, 6, 0), (4, 0, 0,\ \ 3)), ((\ 0, 3, 0, 0), (1, 0, 2,\ \ 0)), \\ ((20, 5, 0, 0), (0, 0, 0, 24)), ((25, 1, 0, 0), (0, 0, 4, 21)), ((26, 0, 0, 0), (0, 2, 2, 21)) \end{array} \right\}$$

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M_{470}:

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M_{490}:

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- Φ_n is well-defined.

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- Φ_n is well-defined.

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- Only missing link: transitivity.

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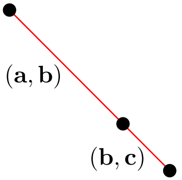

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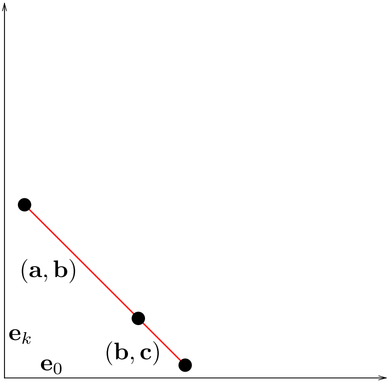

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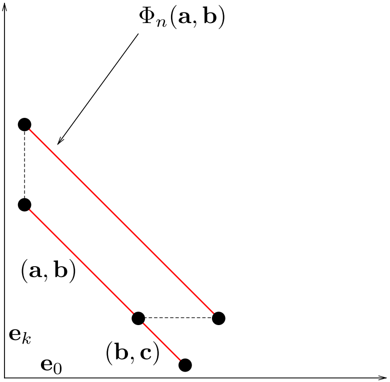

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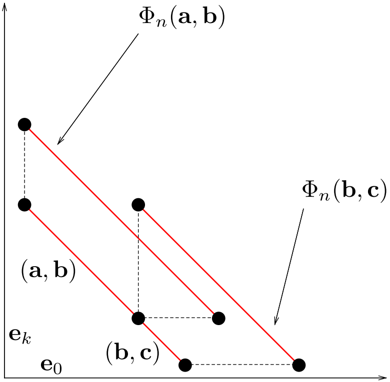

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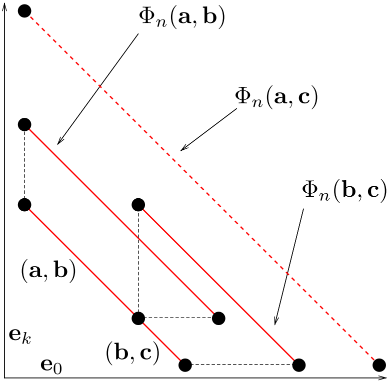

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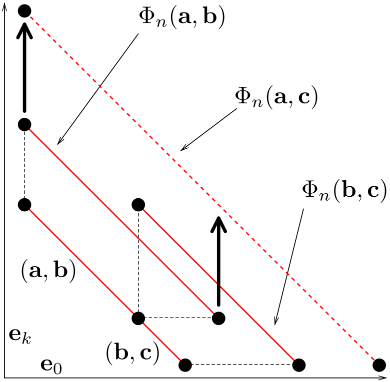

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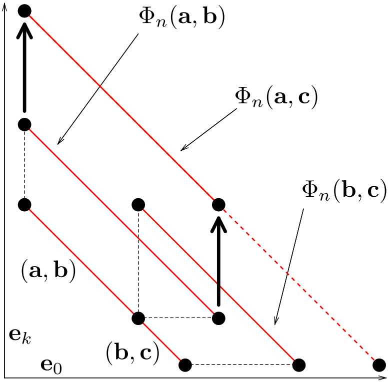

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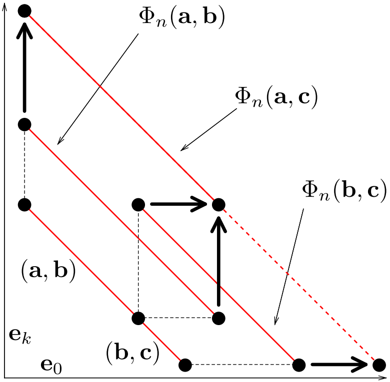

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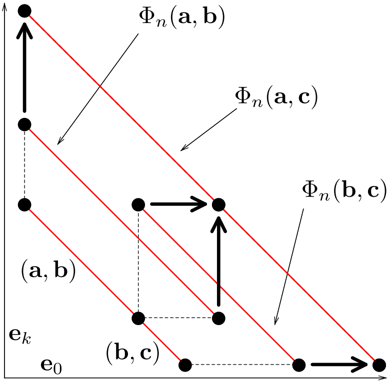

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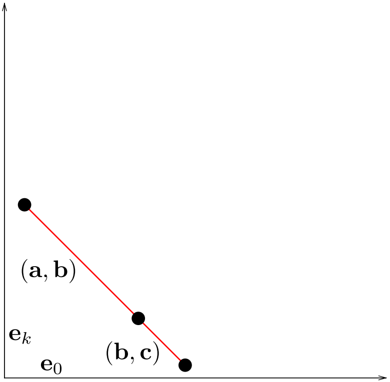

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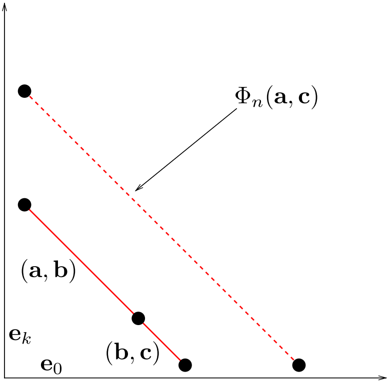

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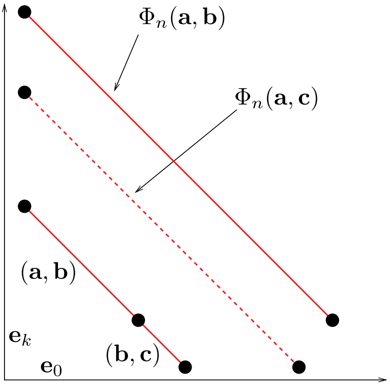

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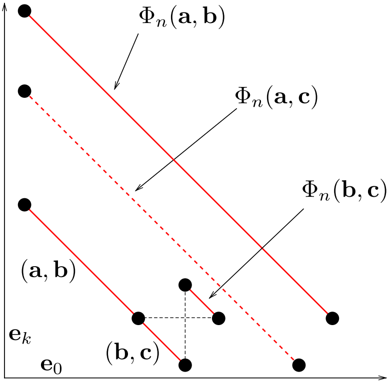

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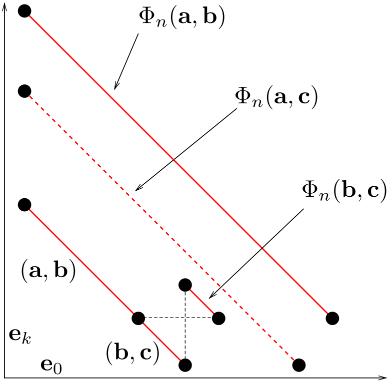
$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big| |\mathbf{a}| - |\mathbf{a}'| \big|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{c}| < |\mathbf{b}|.$$

Need: *monotone* chains are sufficient for transitive closure.



The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

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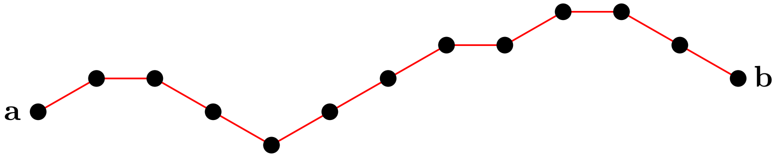

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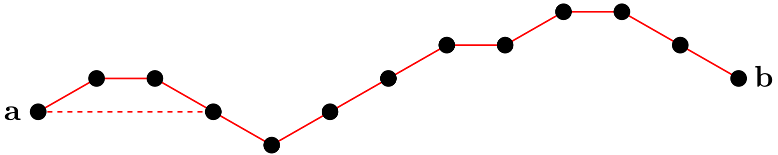

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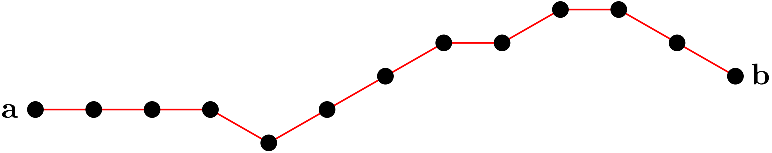

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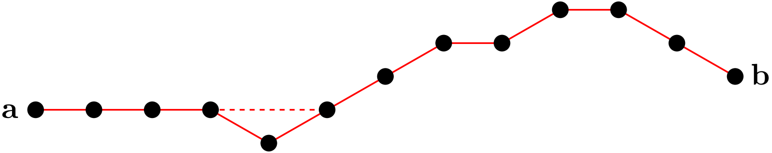

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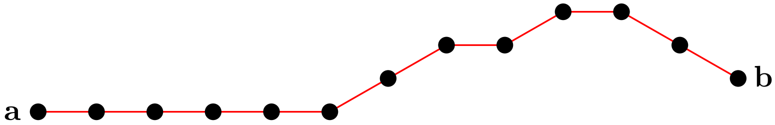

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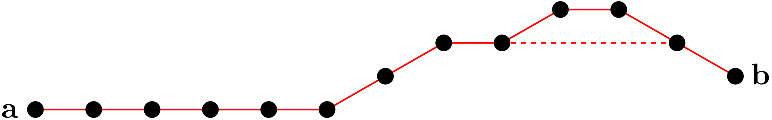

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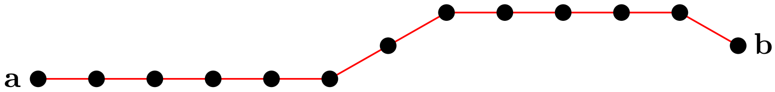

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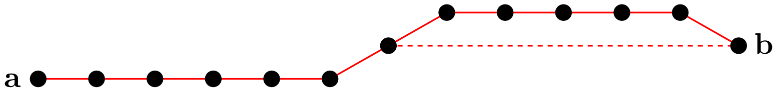

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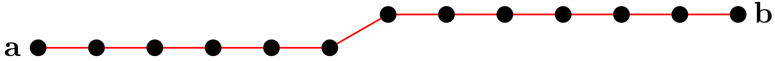

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  Graded degrees for $\beta_0(M_n)$ are $\pi_n(\mathbf{a})$ for each $(\mathbf{a}, \mathbf{a}') \in \rho$,
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Application: computing minimal presentations

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$S = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \ldots, n + r_k \rangle$

$$n = 1234 \qquad r_3 = 1254 - 1234 = 20$$

Verify $n > r_k^2$: $1234 > 400$ ✓

$\langle 414, 420, 423, 434 \rangle$:

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$((0,3,0,0),(1,0,2,0)),$
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n	M_n	Min. Pres. Runtime
50	$\langle 50, 56, 59, 70 \rangle$	1 ms
200	$\langle 200, 206, 209, 220 \rangle$	40 ms
400	$\langle 400, 406, 409, 420 \rangle$	210 ms
1000	$\langle 1000, 1006, 1009, 1020 \rangle$	3 sec
3000	$\langle 3000, 3006, 3009, 3020 \rangle$	2 min
5000	$\langle 5000, 5006, 5009, 5020 \rangle$	18 min
10000	$\langle 10000, 10006, 10009, 10020 \rangle$	4.2 hr

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GAP Numerical Semigroups Package, available at

`http://www.gap-system.org/Packages/numericalsgps.html`.

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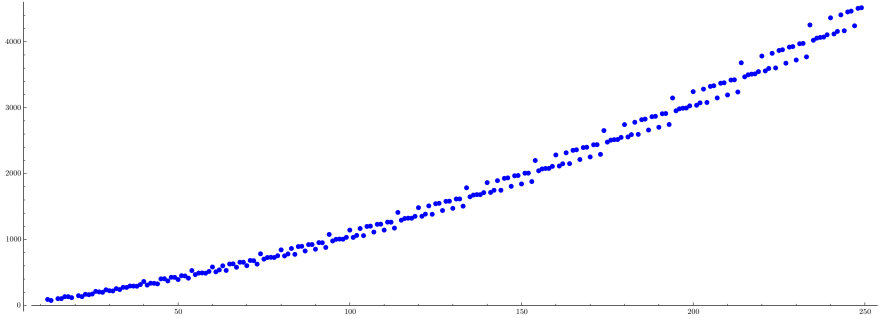

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