On the periodicity of irreducible elements in arithmetical congruence monoids

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Joint with Jacob Hartzer (undergraduate)

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Arithmetical congruence monoids (ACMs)

Definition

An *arithmetical congruence monoid* is a **multiplicative** set

$$M_{a,b} = \{a, a+b, a+2b, a+3b, \ldots\} \subset (\mathbb{Z}_{\geq 1}, \cdot)$$

for $0 < a < b$ with $a^2 \equiv a \bmod b$.

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- $441 = 9 \cdot 49 = 21 \cdot 21$
  $= (3^2) \cdot (7^2) = (3 \cdot 7) \cdot (3 \cdot 7)$.

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`ArithmeticalCongruenceMonoid`: a Sage package, available from

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sage: H.Factorizations(47224750041)
[[17, 21, 49, 89, 30333],
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sage: H.IrreduciblesUpToElement(10000001)
sage: H.IsIrreducible(999997)  # immediate
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Periodicity in ACMs

Question [Baginski–Chapman, 2014]

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$M_{1,4}$:	1,	25,	45,	65,	81,	85,	...
$M_{5,20}$:	25,	125,	225,	325,	425,	525,	...
$M_{7,42}$:	49,	343,	637,	931,	1225,	1519,	...
$M_{51,150}$:	2601,	10251,	17901,	25551,	33201,	40401,	...
$M_{25,200}$:	625,	5625,	10625,	15625,	20625,	25625,	...
$M_{341,620}$:	116281,	327701,	539121,	750541,	923521,	961961,	...

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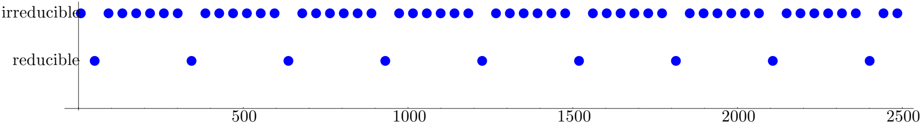
$M_{7,42}$:



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If $a \mid b$ and $a > 1$, then $M_{a,b}$ has periodic irreducible set.

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If $a \mid b$ *and* $a > 1$*, then* $M_{a,b}$ *has periodic irreducible set.*

Example

$M_{5,20} = \{5, 25, 45, 65, 85, 105, 125, 145, 165, 185, 205, 225, 245, \ldots\}$

Reducible elements:

$$
\begin{array}{rcl}
25 = 5 \cdot 5 & 525 = 5 \cdot 105 & 1025 = 5 \cdot 205 \\
125 = 5 \cdot 25 & 625 = 5 \cdot 125 & 1125 = 5 \cdot 225 \\
225 = 5 \cdot 45 & 725 = 5 \cdot 145 & 1225 = 5 \cdot 245 \\
325 = 5 \cdot 65 & 825 = 5 \cdot 165 & 1325 = 5 \cdot 265 \\
425 = 5 \cdot 85 & 925 = 5 \cdot 185 & 1425 = 5 \cdot 285
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For $M_{1,4}$, a sequence of $k = 6$ consecutive reducible elements:

$$20884505 = 5 \cdot 4176901 \qquad 20884517 = 17 \cdot 1228501$$
$$20884509 = 9 \cdot 2320501 \qquad 20884521 = 21 \cdot 994501$$
$$20884513 = 13 \cdot 1606501 \qquad 20884525 = 25 \cdot 835381$$

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For $M_{9,12}$, a sequence of $k = 4$ evenly-spaced reducible elements:

$$31995873 = 21 \cdot 1523613 \qquad 31995945 = 45 \cdot 711021$$
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Idea

Look for (arbitrarily) long sequences of evenly-spaced reducible elements.

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Lemma

Let $g = \gcd(a, b)$. *The elements*

$$g(a + jb) + (a + b - g)\prod_{i=1}^{k}(a + ib) \quad \textit{for} \quad j = 1, \ldots, k$$

are all reducible, with constant difference gb.

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Theorem

$M_{a,b}$ *has periodic irreducible set if and only if* $a \mid b$ *and* $a > 1$.

References

P. Baginski and S. Chapman,
Arithmetic congruence monoids: a survey,
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