Shifting numerical monoids

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A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

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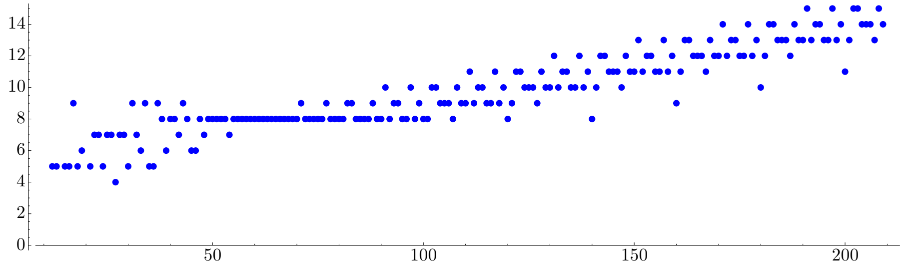

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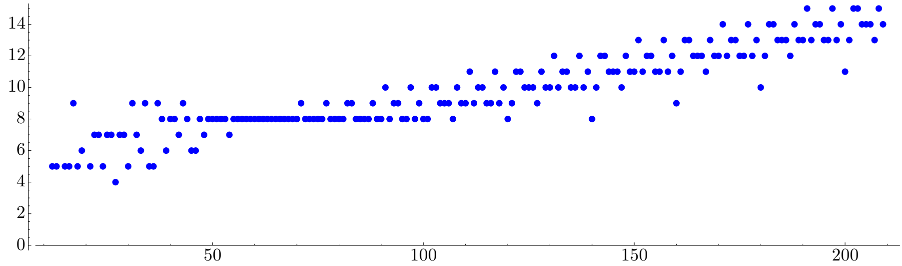
$\mathsf{c}(M_n)$ is periodic-linear (quasilinear) for $n \geq 126$.



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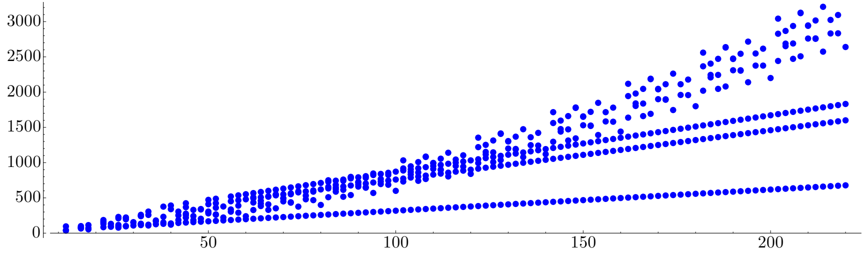

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Underlying cause: minimal presentations!

Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

$$n = a_1 r_1 + \cdots + a_k r_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$$

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that is closed under *translation*.

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Kernel congruences and minimal presentations

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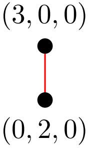

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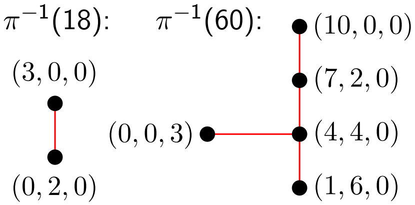

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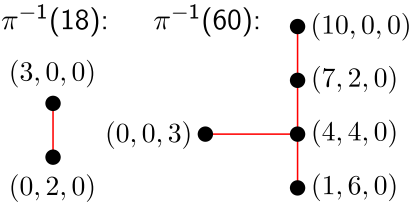

$$\begin{aligned} ((7,2,0),(4,4,0)) = {} & ((3,0,0),(0,2,0)) \\ & + ((4,2,0),(4,2,0)) \end{aligned}$$

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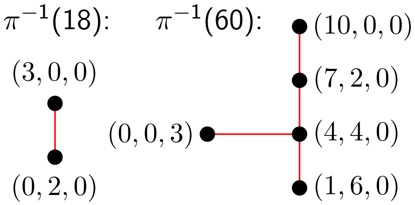

$$((7,2,0),(4,4,0)) = ((3,0,0),(0,2,0)) + ((4,2,0),(4,2,0))$$

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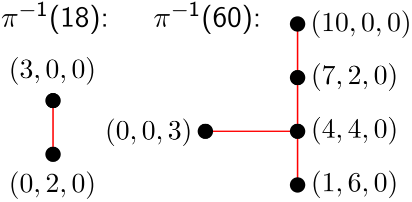

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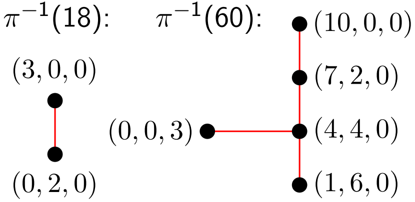

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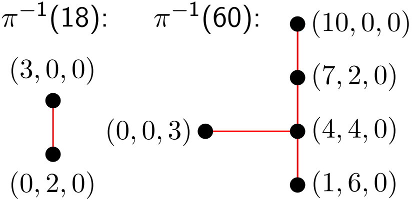

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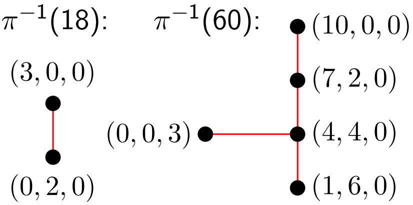

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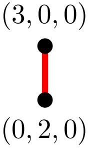

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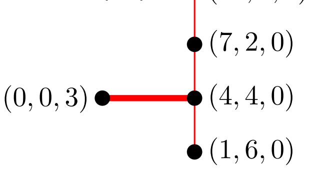

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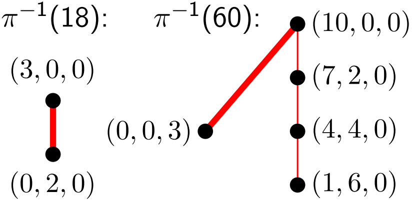

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Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

$$n = a_1 r_1 + \cdots + a_k r_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$$

$$\begin{array}{rcl} \pi : \mathbb{N}^k & \longrightarrow & \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} & \longmapsto & a_1 r_1 + \cdots + a_k r_k \end{array}$$

Definition

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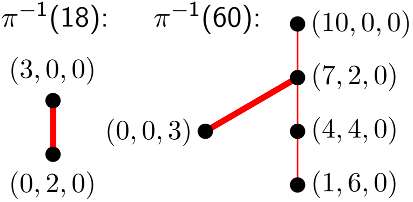
$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3,0,0),(0,2,0)),((4,4,0),(0,0,3))\} \subset \ker \pi$



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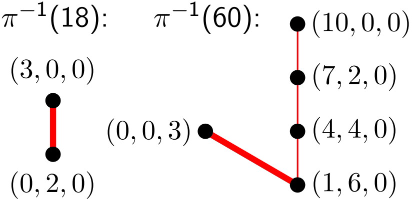

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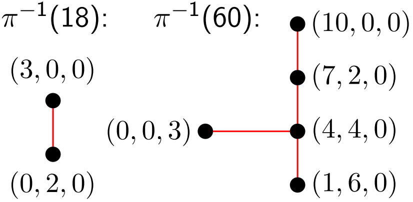

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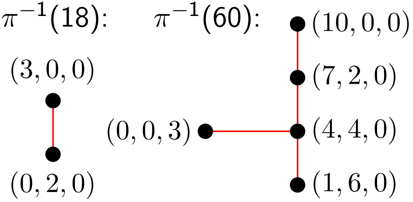

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$$\beta_0(I_S) = \{18, 60\}$$

Intuition: "sufficiently shifted" monoids

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

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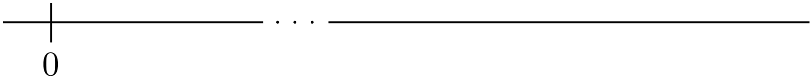

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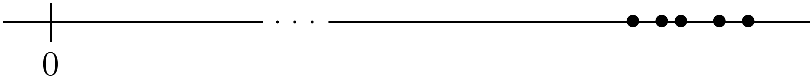

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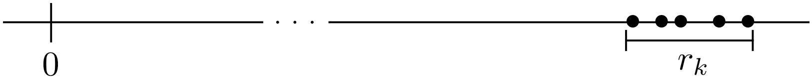

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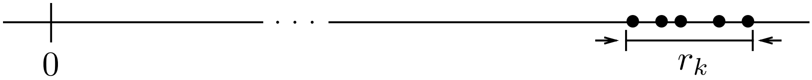

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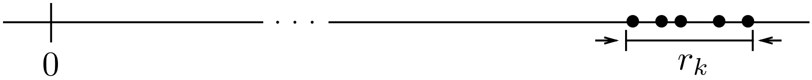

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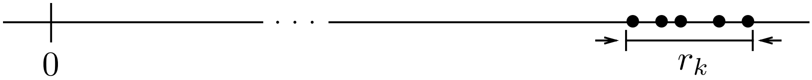

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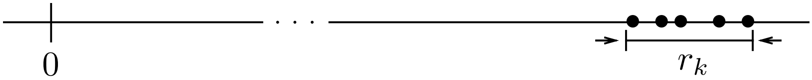

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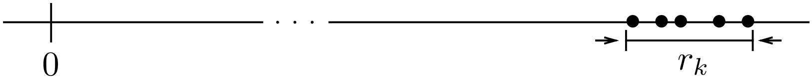

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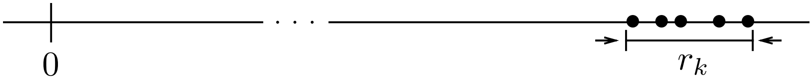

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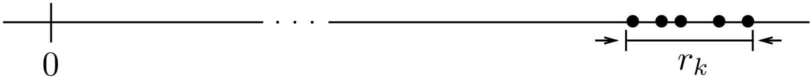

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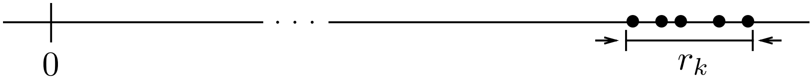

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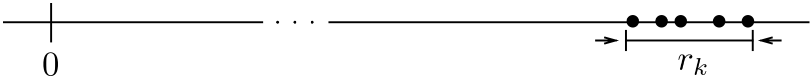

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In $M_n = \langle n, n+6, n+9, n+20 \rangle$ with $n = 450$:

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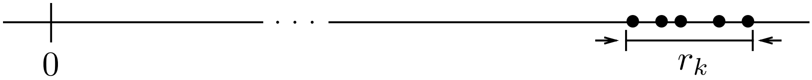

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In $M_n = \langle n, n+6, n+9, n+20 \rangle$ with $n = 450$:

$$\begin{array}{rcll} 3(n+6) & = & n + 2(n+9) & \text{is cheap} \\ 4(n+9) + 21(n+20) & = & 25n + (n+6) & \text{is costly} \end{array}$$

The shifting map

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DON'T PANIC!

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Sneak peek for $M_n = \langle n, n + 6, n + 9, n + 20 \rangle$ and $n \gg 0$:

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M_{450}:

$$\left\{ \begin{array}{l} ((\ 0,0,8,0),(3,2,0,\ \ 3)),((\ 0,1,6,0),(4,0,0,\ \ 3)),((\ 0,3,0,0),(1,0,2,\ \ 0)), \\ ((20,5,0,0),(0,0,0,24)),((25,1,0,0),(0,0,4,21)),((26,0,0,0),(0,2,2,21)) \end{array} \right\}$$

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M_{470}:

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where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Sneak peek for $M_n = \langle n, n+6, n+9, n+20 \rangle$ and $n \gg 0$:

M_{450}:

$$\left\{ \begin{array}{l} ((\ \,0,0,8,0),(3,2,0,\ \,3)),((\ \,0,1,6,0),(4,0,0,\ \,3)),((\ \,0,3,0,0),(1,0,2,\ \,0)), \\ ((20,5,0,0),(0,0,0,24)),((25,1,0,0),(0,0,4,21)),((26,0,0,0),(0,2,2,21)) \end{array} \right\}$$

M_{470}:

$$\left\{ \begin{array}{l} ((\ \,0,0,8,0),(3,2,0,\ \,3)),((\ \,0,1,6,0),(4,0,0,\ \,3)),((\ \,0,3,0,0),(1,0,2,\ \,0)), \\ ((21,5,0,0),(0,0,0,25)),((26,1,0,0),(0,0,4,22)),((27,0,0,0),(0,2,2,22)) \end{array} \right\}$$

M_{490}:

$$\left\{ \begin{array}{l} ((\ \,0,0,8,0),(3,2,0,\ \,3)),((\ \,0,1,6,0),(4,0,0,\ \,3)),((\ \,0,3,0,0),(1,0,2,\ \,0)), \\ ((22,5,0,0),(0,0,0,26)),((27,1,0,0),(0,0,4,23)),((28,0,0,0),(0,2,2,23)) \end{array} \right\}$$

The shifting map

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- Φ_n is well-defined.

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- Φ_n preserves reflexive and symmetric closure.

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- Only missing link: transitivity.

Monotone chains

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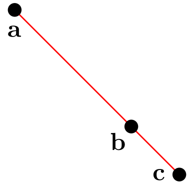

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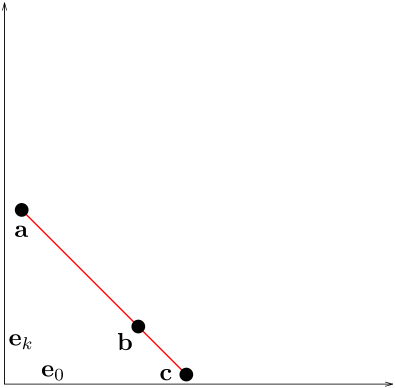

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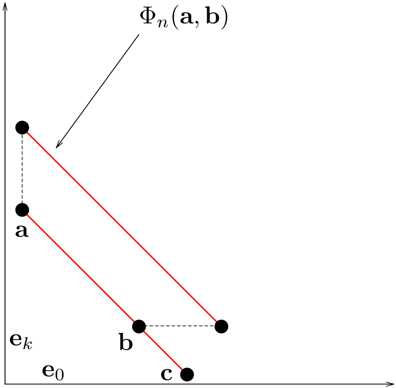

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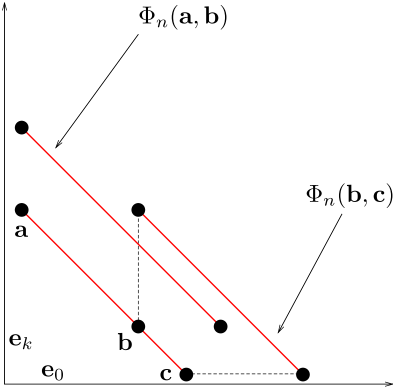

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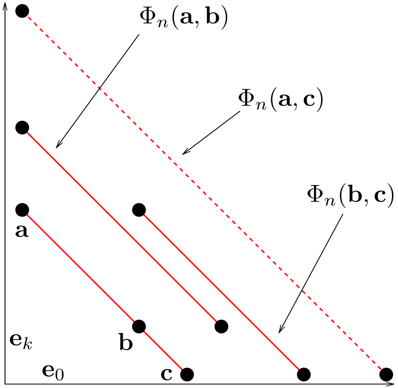

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If $|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|$:

translate $\Phi_n(\mathbf{a}, \mathbf{b})$ and $\Phi_n(\mathbf{b}, \mathbf{c})$
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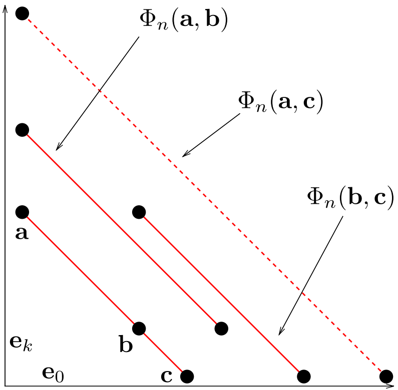

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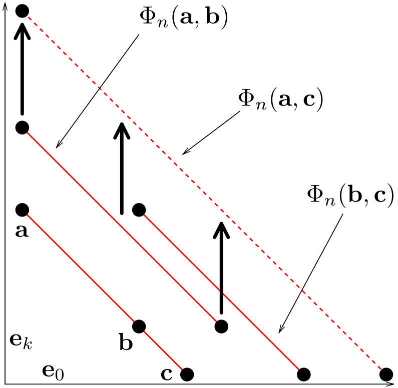

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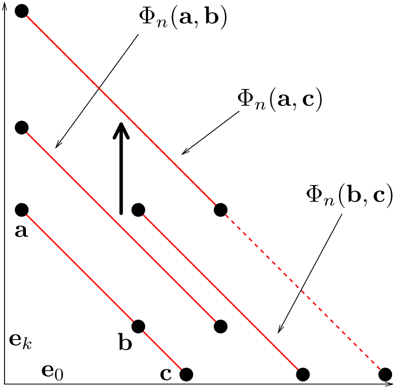

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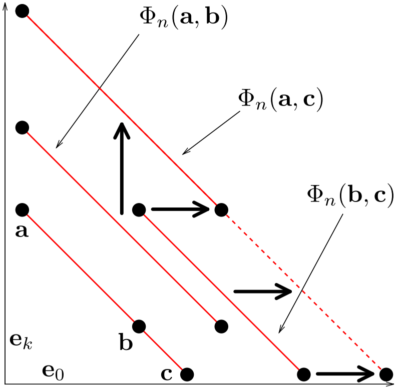

Monotone chains

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

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Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}), (\mathbf{a}, \mathbf{c}) \in \ker \pi_n$.

If $|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|$:

translate $\Phi_n(\mathbf{a}, \mathbf{b})$ and $\Phi_n(\mathbf{b}, \mathbf{c})$
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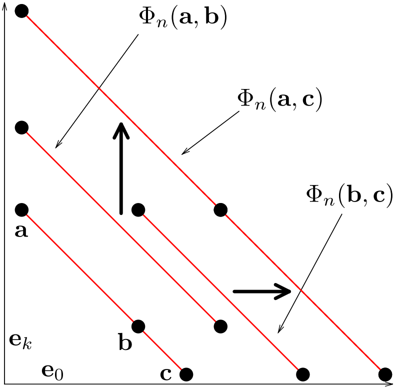

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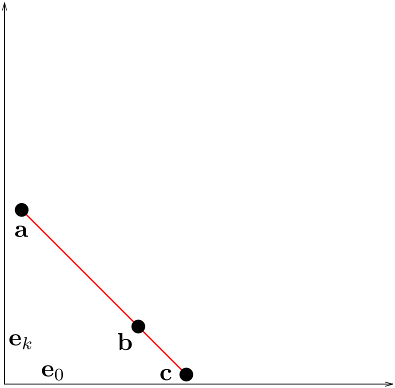

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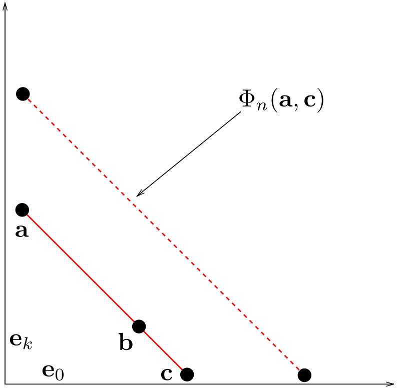

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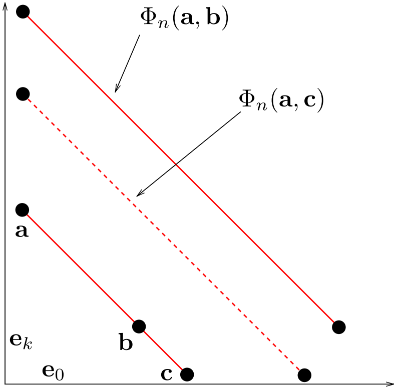

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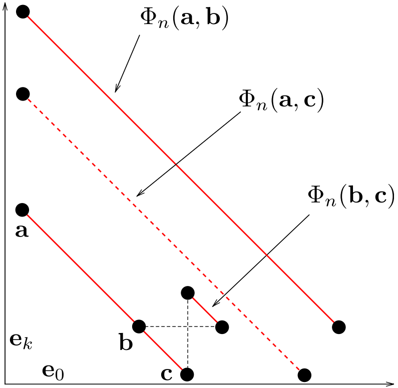

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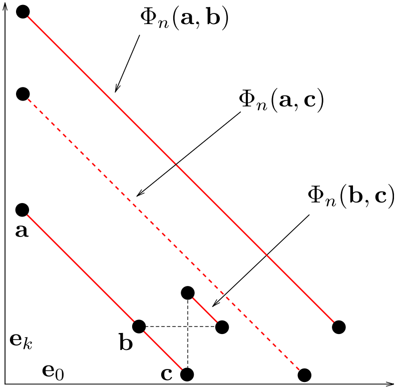

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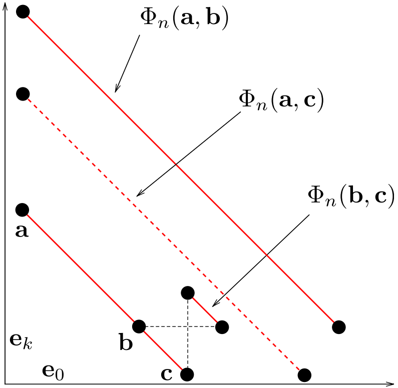
Need: *monotone* chains for Φ_n to preserve transitive closure.



The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

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If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$ (costly), then $a_0 > 0$ and $a_k' > 0$.

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Consequences:

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  Graded degrees for $\beta_0(M_n)$ are $\pi_n(\mathbf{a})$ for each $(\mathbf{a}, \mathbf{a}') \in \rho$

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Application: computing minimal presentations

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$S = \langle 1234, 1240, 1243, 1254 \rangle$

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$\langle 414, 420, 423, 434 \rangle$:

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$\langle 414, 420, 423, 434 \rangle$:

$((0,0,8,0),(3,2,0,3))$,
$((0,1,6,0),(4,0,0,3))$,
$((0,3,0,0),(1,0,2,0))$,
$((21,1,0,0),(0,0,0,21))$,
$((25,0,0,0),(0,0,6,18))$

Application: computing minimal presentations

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.

$$\{\textit{minimal presentations of } M_n\} \quad \leftrightsquigarrow \quad \{\textit{minimal presentations of } M_{n+r_k}\}$$
$$\rho \subset \ker \pi_n \quad \longmapsto \quad \Phi_n(\rho) \subset \ker \pi_{n+r_k}$$

$S = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \ldots, n + r_k \rangle$

$$n = 1234 \qquad r_3 = 1254 - 1234 = 20$$

Verify $n > r_k^2$: $1234 > 400$ ✓

$\langle 414, 420, 423, 434 \rangle$:		$\langle 1234, 1240, 1243, 1254 \rangle$:
$((0,0,8,0),(3,2,0,3))$,		$((0,0,8,0),(3,2,0,3))$,
$((0,1,6,0),(4,0,0,3))$,	$\rightsquigarrow$	$((0,1,6,0),(4,0,0,3))$,
$((0,3,0,0),(1,0,2,0))$,		$((0,3,0,0),(1,0,2,0))$,
$((21,1,0,0),(0,0,0,21))$,		$((62,1,0,0),(0,0,0,62))$,
$((25,0,0,0),(0,0,6,18))$		$((66,0,0,0),(0,0,6,59))$

Application: computing minimal presentations

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For any $n > r_k^2$, *the image* $\Phi_n(\ker \pi_n)$ *generates* $\ker \pi_{n+r_k}$ *as a congruence.*

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$$\rho \subset \ker \pi_n \quad \longmapsto \quad \Phi_n(\rho) \subset \ker \pi_{n+r_k}$$

n	M_n	Min. Pres. Runtime
50	$\langle 50, 56, 59, 70 \rangle$	1 ms
200	$\langle 200, 206, 209, 220 \rangle$	40 ms
400	$\langle 400, 406, 409, 420 \rangle$	210 ms
1000	$\langle 1000, 1006, 1009, 1020 \rangle$	3 sec
3000	$\langle 3000, 3006, 3009, 3020 \rangle$	2 min
5000	$\langle 5000, 5006, 5009, 5020 \rangle$	18 min
10000	$\langle 10000, 10006, 10009, 10020 \rangle$	4.2 hr

Application: computing minimal presentations

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GAP Numerical Semigroups Package, available at

`http://www.gap-system.org/Packages/numericalsgps.html`.

Future shifty work

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Frobenius number: $F(S) = \max(\mathbb{N} \setminus S)$.

Example

If $S = \langle 6, 9, 20 \rangle$, then $F(S) = 43$ since

$$\mathbb{N} \setminus S = \{1, 2, 3, 4, 5, 7, 8, 10, 11, 13, \ldots, 31, 34, 37, 43\}.$$

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Sneak peek for $F(\langle n, n+6, n+9, n+20 \rangle)$:

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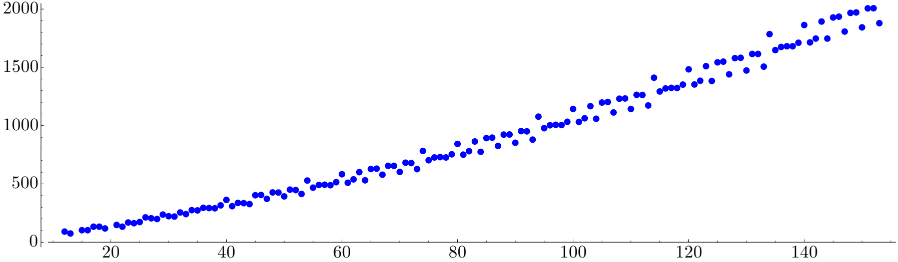
Sneak peek for $F(\langle n, n+6, n+9, n+20 \rangle)$:



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Thanks!