Shifting numerical monoids

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A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

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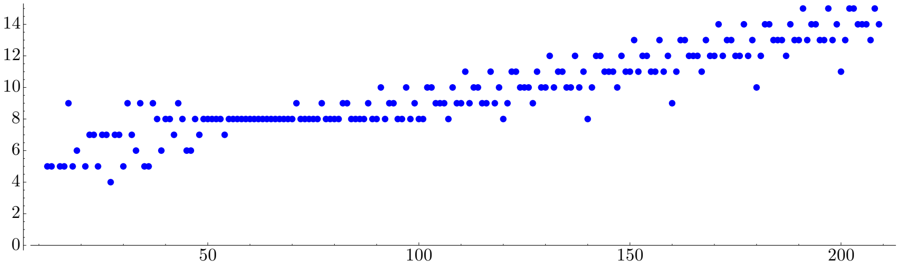

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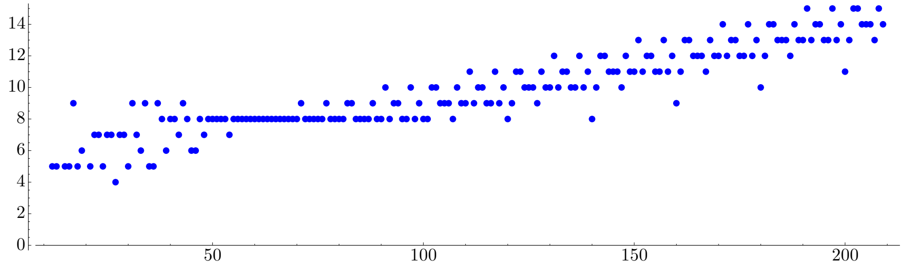
$\mathsf{c}(M_n)$ is periodic-linear (quasilinear) for $n \geq 126$.



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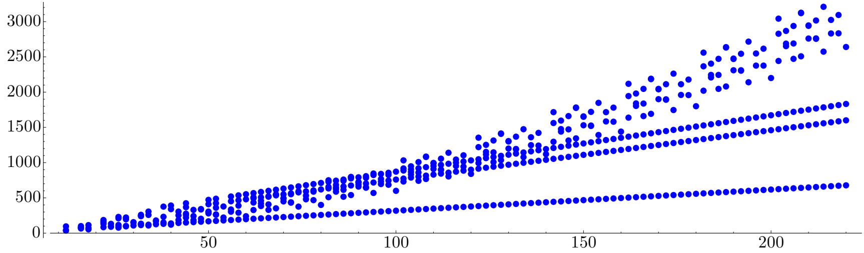

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Underlying cause: minimal presentations!

Kernel congruences and minimal presentations

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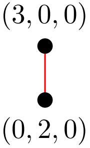

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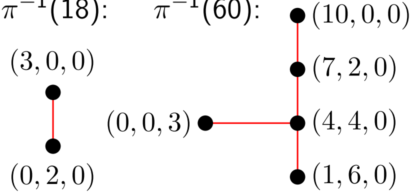

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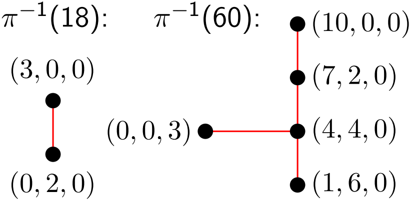

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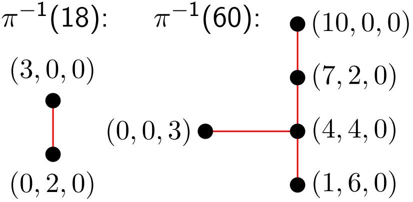

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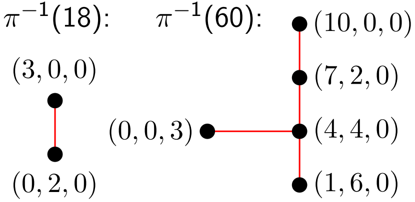

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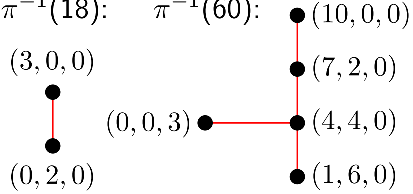

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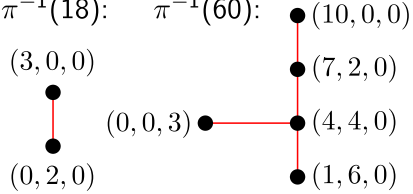

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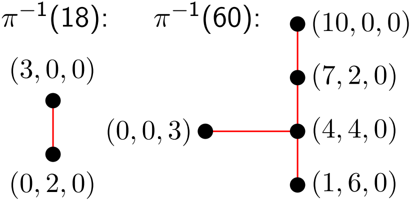

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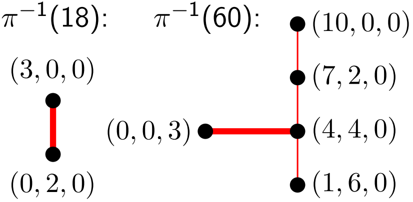

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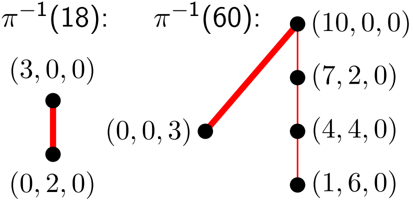

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Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

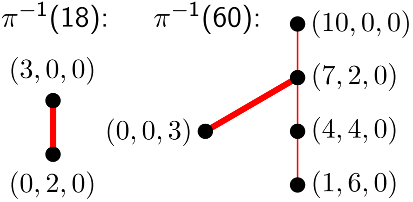
$$n = a_1 r_1 + \cdots + a_k r_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$$

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \cdots + a_k r_k \end{aligned}$$

Definition

A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3,0,0),(0,2,0)), ((4,4,0),(0,0,3))\} \subset \ker \pi$



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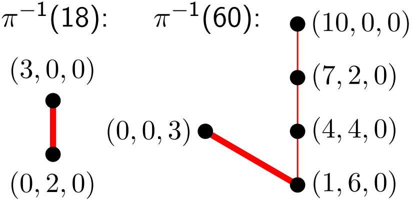

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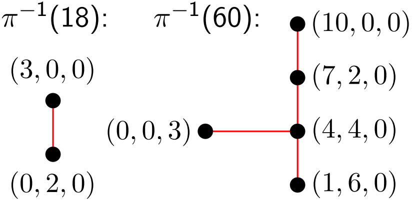

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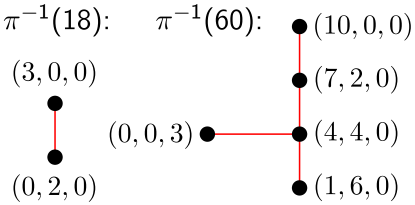

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$$\beta_0(I_S) = \{18, 60\}$$

Kernel congruences and minimal presentations

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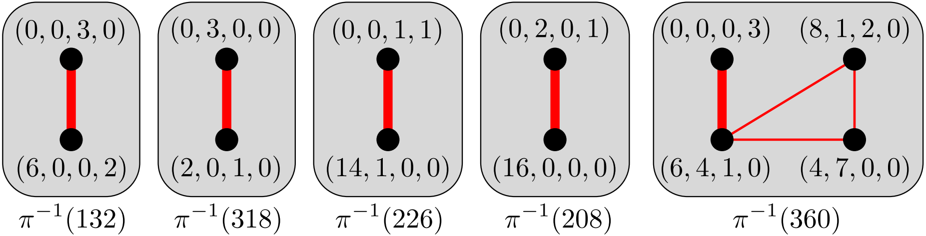

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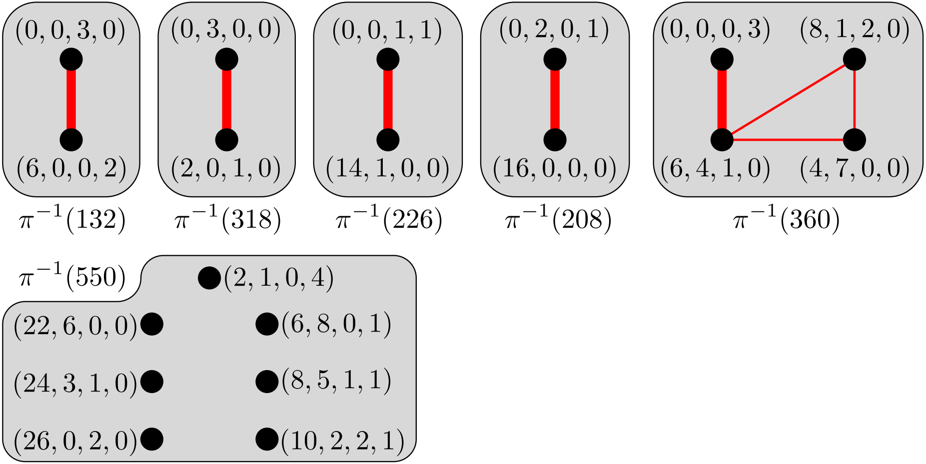

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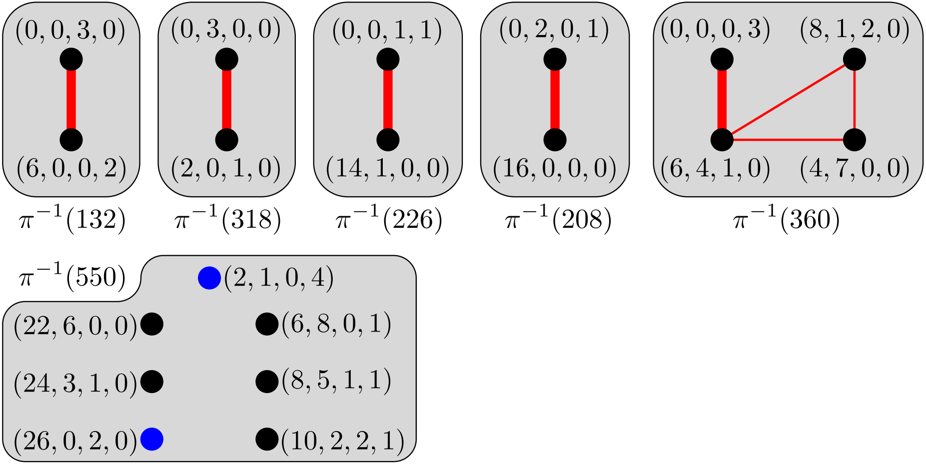

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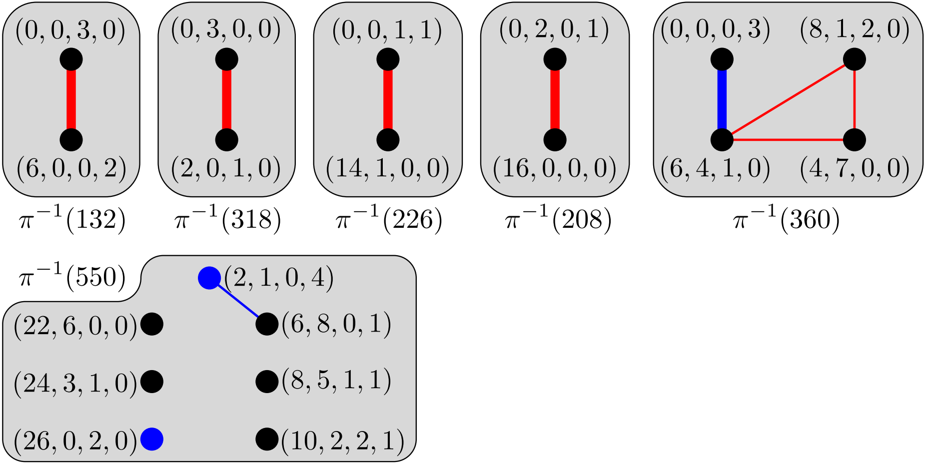

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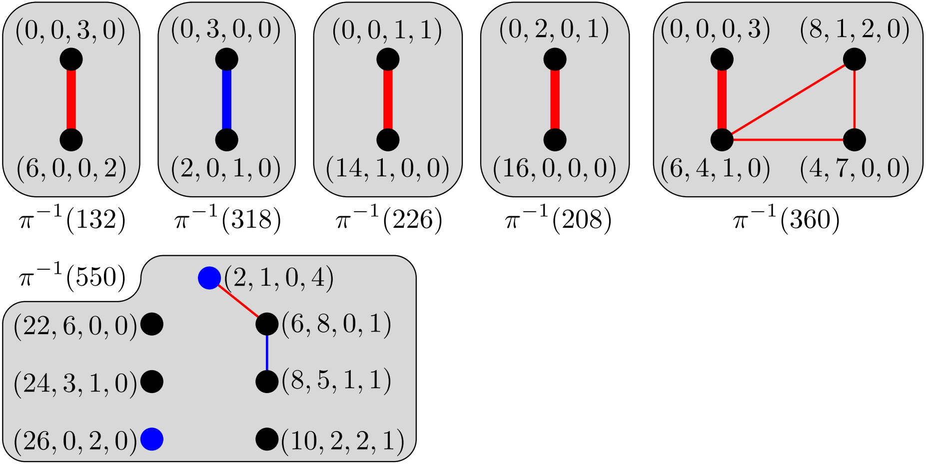

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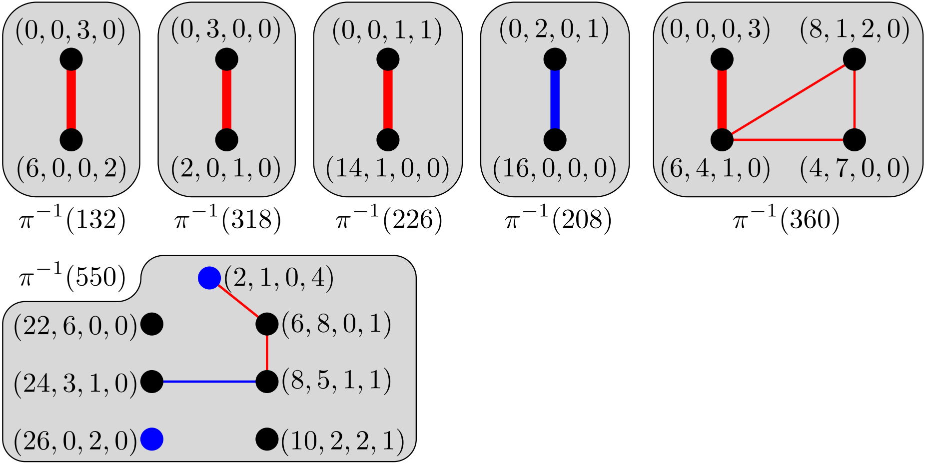

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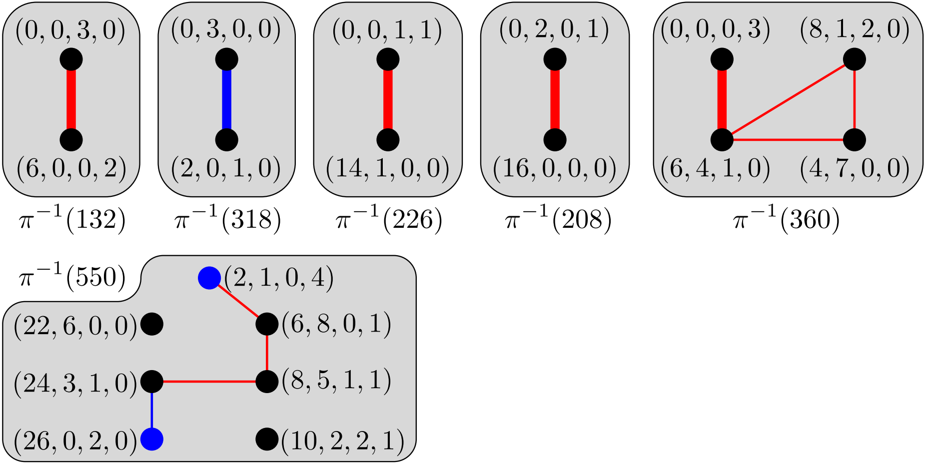

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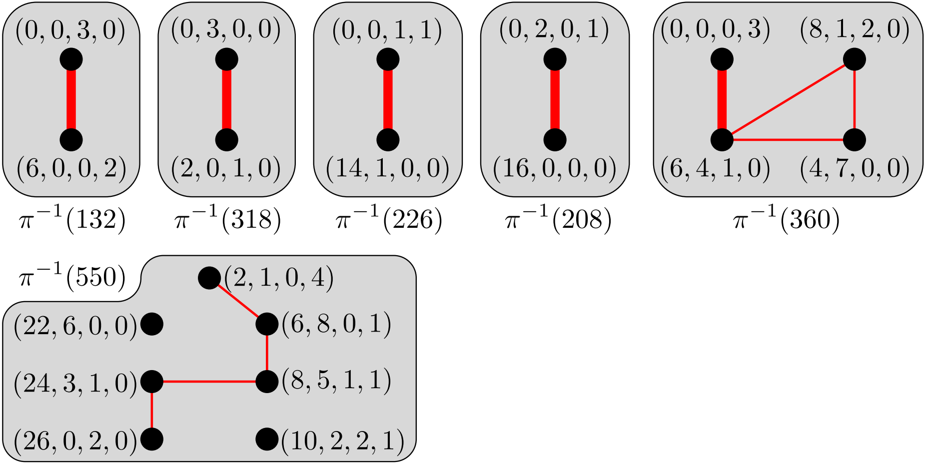

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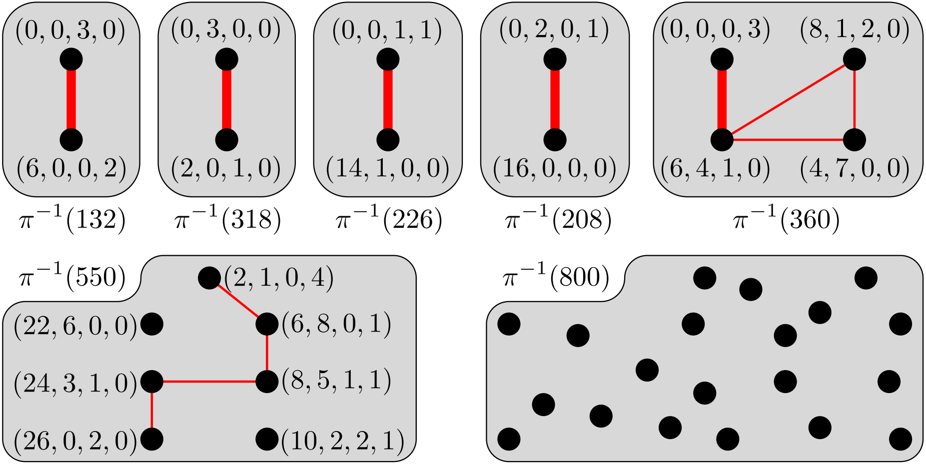

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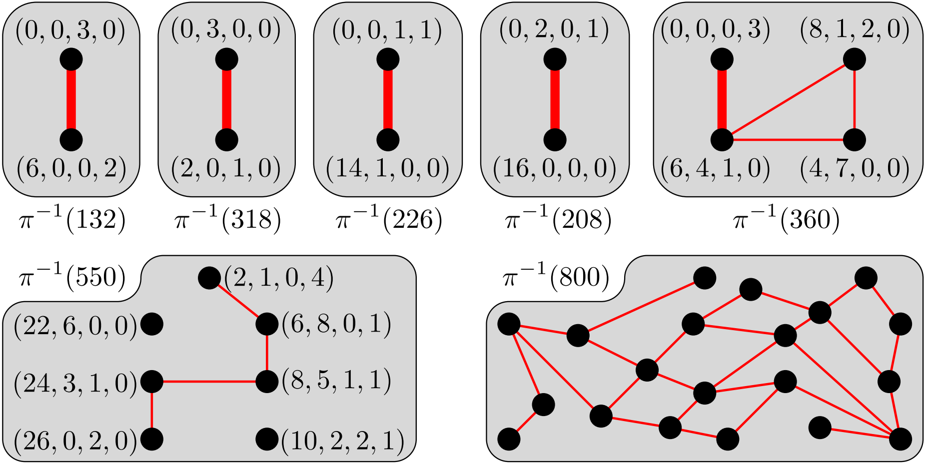

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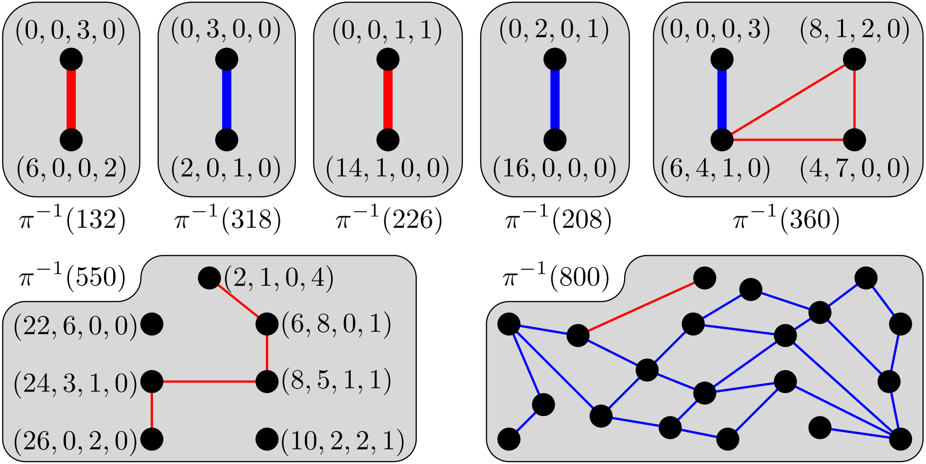

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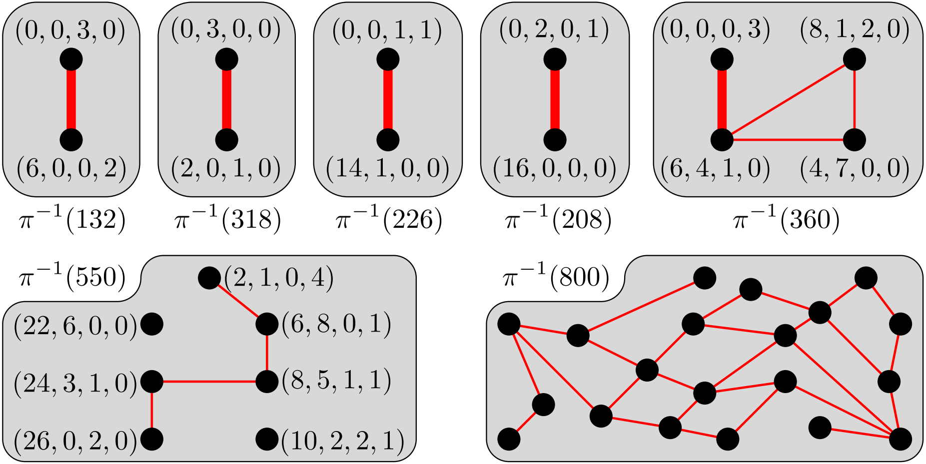

Intuition: "sufficiently shifted" monoids

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

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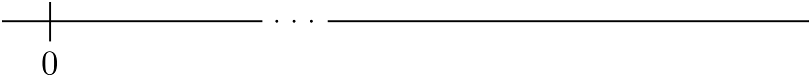

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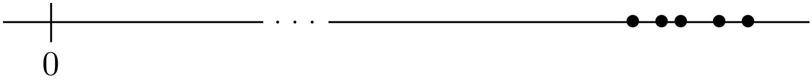

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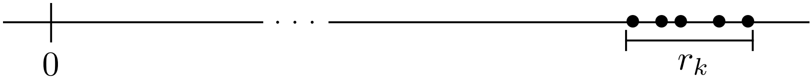

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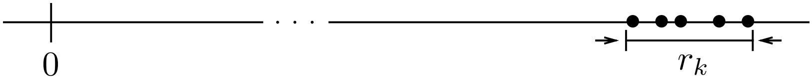

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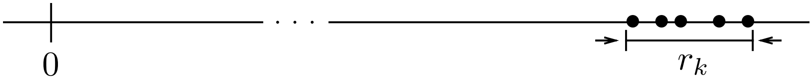

$$a_0 n + a_1(n+r_1) + \cdots + a_k(n+r_k) = b_0 n + b_1(n+r_1) + \cdots + b_k(n+r_k)$$

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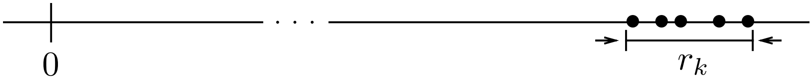

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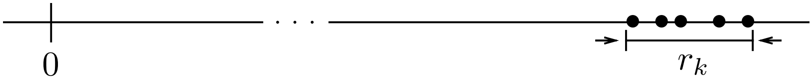

$$\begin{array}{lclclclclclclcl} a_0 n & + & a_1(n+r_1) & + & \cdots & + & a_k(n+r_k) & = & b_0 n & + & b_1(n+r_1) & + & \cdots & + & b_k(n+r_k) \\ |\mathbf{a}| n & + & a_1 r_1 & + & \cdots & + & a_k r_k & = & |\mathbf{b}| n & + & b_1 r_1 & + & \cdots & + & b_k r_k \end{array}$$

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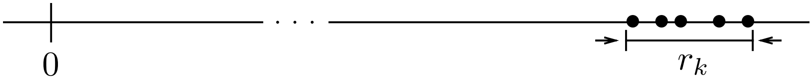

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2 types of minimal relations $\mathbf{a} \sim \mathbf{b}$:

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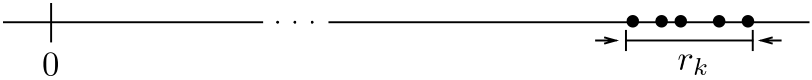

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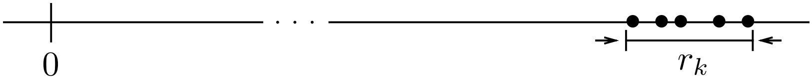

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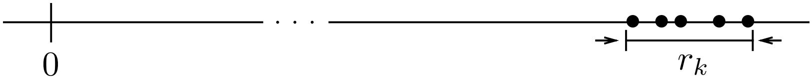

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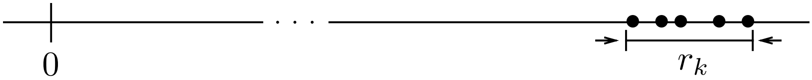

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In $M_n = \langle n, n + 6, n + 9, n + 20 \rangle$ with $n = 450$:

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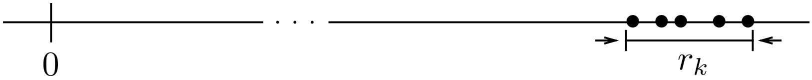

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2 types of minimal relations $\mathbf{a} \sim \mathbf{b}$:

- Relations among $r_1, \ldots, r_k$ (cheap): $|\mathbf{a}| = |\mathbf{b}|$
- Relations that change # copies of n (costly): $|\mathbf{a}| < |\mathbf{b}|$

$$\text{mostly } a_k \longleftrightarrow \text{mostly } b_0$$

In $M_n = \langle n, n+6, n+9, n+20 \rangle$ with $n = 450$:

$$\begin{array}{rcll} 3(n+6) & = & n + 2(n+9) & \text{is cheap} \\ 4(n+9) + 21(n+20) & = & 25n + (n+6) & \text{is costly} \end{array}$$

The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

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where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

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DON'T PANIC!

The shifting map

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Sneak peek for $M_n = \langle n, n + 6, n + 9, n + 20 \rangle$ and $n \gg 0$:

The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

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Sneak peek for $M_n = \langle n, n+6, n+9, n+20 \rangle$ and $n \gg 0$:

M_{450}:

$$\left\{ \begin{array}{l} ((\ 0,0,8,0),(3,2,0,\ 3)),((\ 0,1,6,0),(4,0,0,\ 3)),((\ 0,3,0,0),(1,0,2,\ 0)), \\ ((20,5,0,0),(0,0,0,24)),((25,1,0,0),(0,0,4,21)),((26,0,0,0),(0,2,2,21)) \end{array} \right\}$$

The shifting map

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M_{450}:

$$\left\{ \begin{array}{l} ((0,0,8,0),(3,2,0,3)),((0,1,6,0),(4,0,0,3)),((0,3,0,0),(1,0,2,0)), \\ ((20,5,0,0),(0,0,0,24)),((25,1,0,0),(0,0,4,21)),((26,0,0,0),(0,2,2,21)) \end{array} \right\}$$

M_{470}:

$$\left\{ \begin{array}{l} ((0,0,8,0),(3,2,0,3)),((0,1,6,0),(4,0,0,3)),((0,3,0,0),(1,0,2,0)), \\ ((21,5,0,0),(0,0,0,25)),((26,1,0,0),(0,0,4,22)),((27,0,0,0),(0,2,2,22)) \end{array} \right\}$$

The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ is given by

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where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

Sneak peek for $M_n = \langle n, n+6, n+9, n+20 \rangle$ and $n \gg 0$:

M_{450}:

$$\left\{ \begin{array}{l} ((0,0,8,0),(3,2,0,3)),((0,1,6,0),(4,0,0,3)),((0,3,0,0),(1,0,2,0)), \\ ((20,5,0,0),(0,0,0,24)),((25,1,0,0),(0,0,4,21)),((26,0,0,0),(0,2,2,21)) \end{array} \right\}$$

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M_{490}:

$$\left\{ \begin{array}{l} ((0,0,8,0),(3,2,0,3)),((0,1,6,0),(4,0,0,3)),((0,3,0,0),(1,0,2,0)), \\ ((22,5,0,0),(0,0,0,26)),((27,1,0,0),(0,0,4,23)),((28,0,0,0),(0,2,2,23)) \end{array} \right\}$$

The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ is given by

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where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Congruence properties preserved by Φ_n:

- Reflexive and symmetric closure

The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

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where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

Congruence properties preserved by Φ_n:

- Reflexive and symmetric closure
- Translation closure

The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

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Congruence properties preserved by Φ_n:

- Reflexive and symmetric closure
- Translation closure

$$\Phi_n((\mathbf{a}, \mathbf{a}') + (\mathbf{b}, \mathbf{b})) = \Phi_n(\mathbf{a}, \mathbf{a}') + (\mathbf{b}, \mathbf{b})$$

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Congruence properties preserved by Φ_n:

- Reflexive and symmetric closure
- Translation closure

$$\Phi_n((\mathbf{a}, \mathbf{a}') + (\mathbf{b}, \mathbf{b})) = \Phi_n(\mathbf{a}, \mathbf{a}') + (\mathbf{b}, \mathbf{b})$$

- Only missing link: transitivity

Transitivity before/after shifting

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

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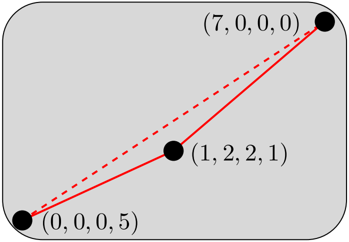
where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

For $M_n = \langle n, n + 6, n + 9, n + 20\rangle$:



$350 \in M_{50}$

Transitivity before/after shifting

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ is given by

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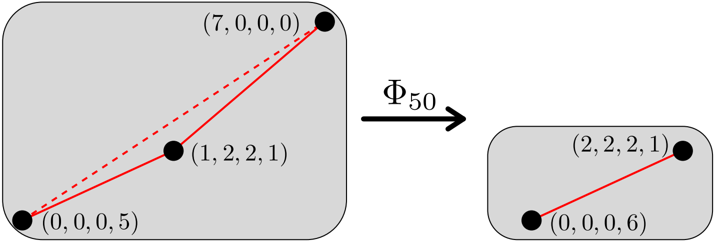

$350 \in M_{50}$ $\qquad$ $540 \in M_{70}$

Transitivity before/after shifting

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

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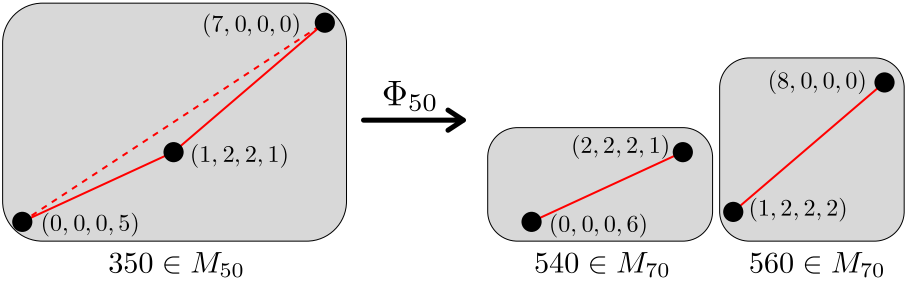

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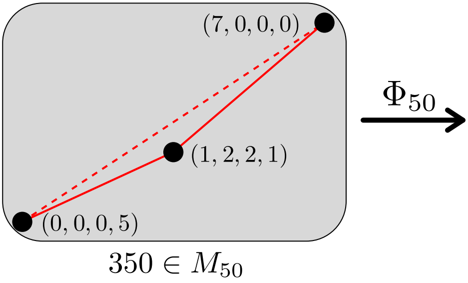

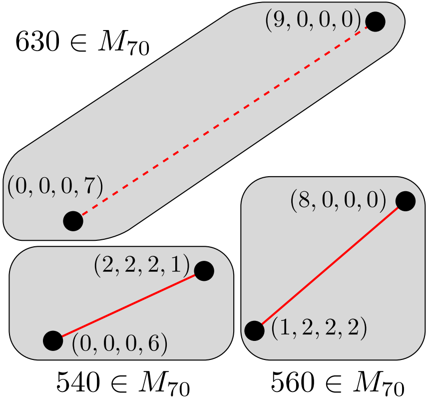

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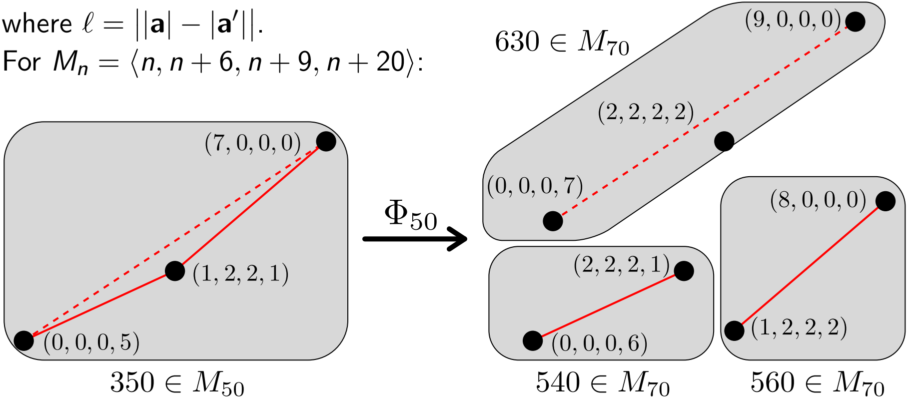

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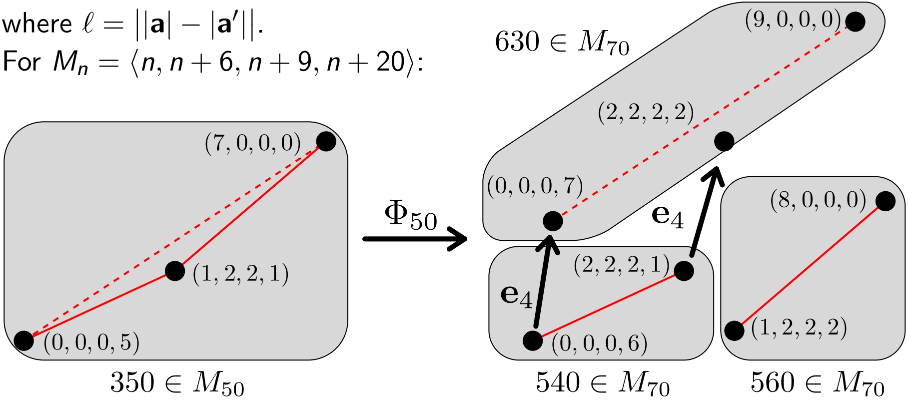

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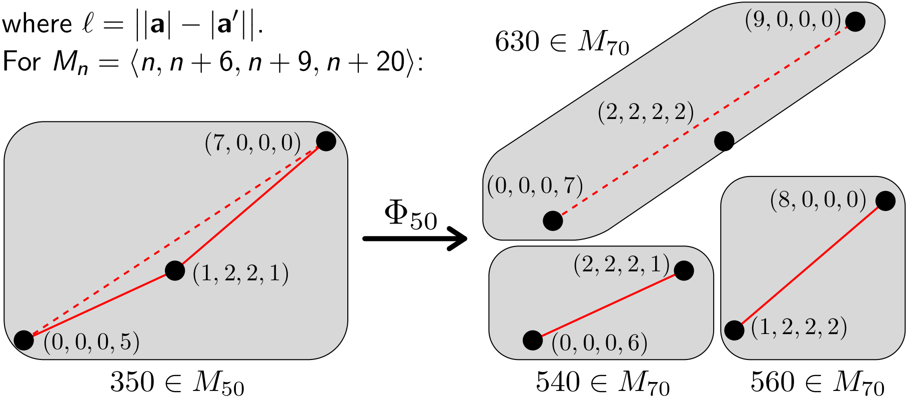

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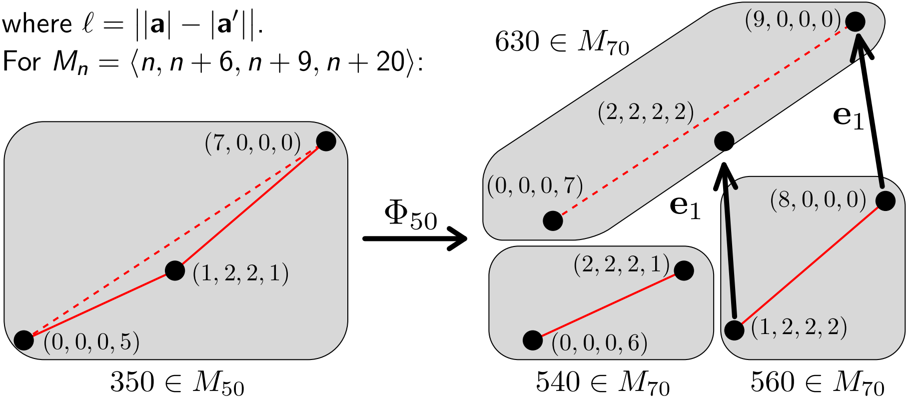

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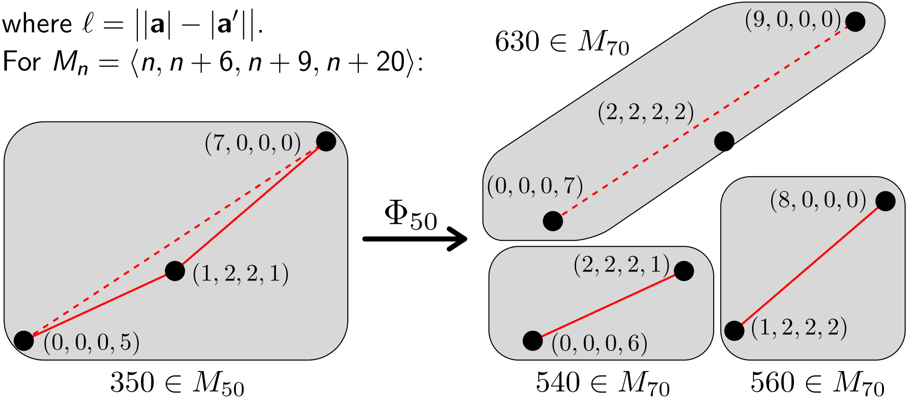
where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

For $M_n = \langle n, n + 6, n + 9, n + 20 \rangle$:



Transitivity before/after shifting

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

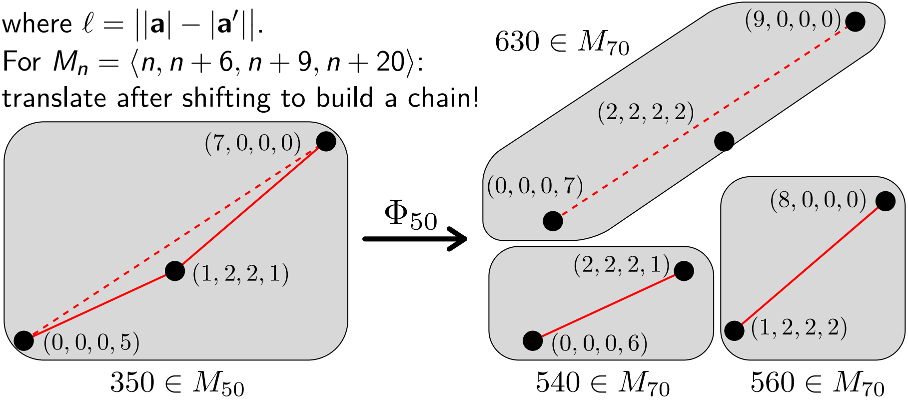
Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ is given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = ||\mathbf{a}| - |\mathbf{a}'||$.

For $M_n = \langle n, n + 6, n + 9, n + 20 \rangle$:

translate after shifting to build a chain!



Monotone chains

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Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}), (\mathbf{a}, \mathbf{c}) \in \ker \pi_n$.

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Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}), (\mathbf{a}, \mathbf{c}) \in \ker \pi_n$.

If $|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|$:

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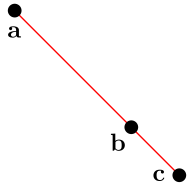

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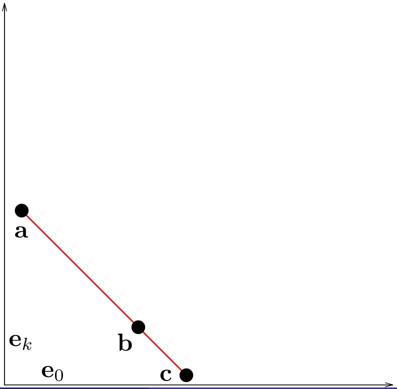

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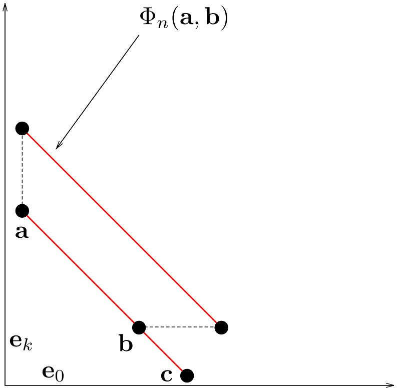

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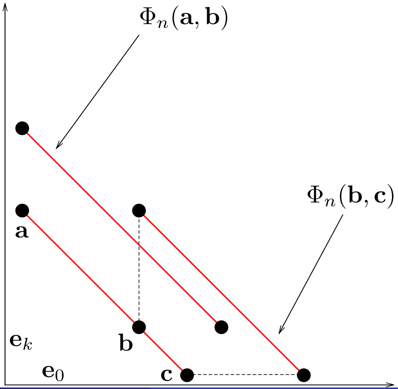

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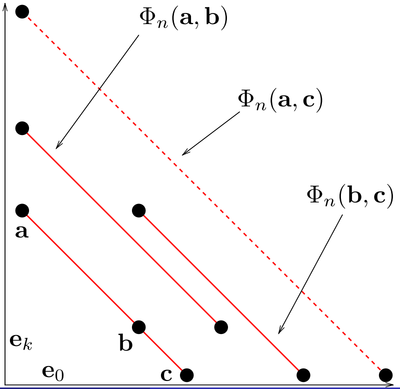

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If $|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|$:

translate $\Phi_n(\mathbf{a}, \mathbf{b})$ and $\Phi_n(\mathbf{b}, \mathbf{c})$
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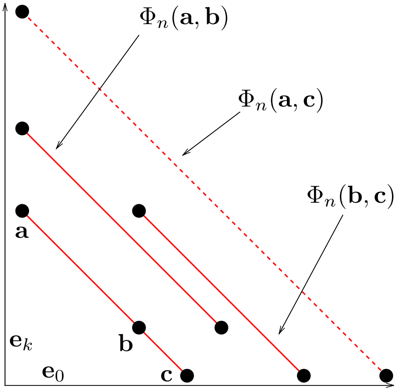

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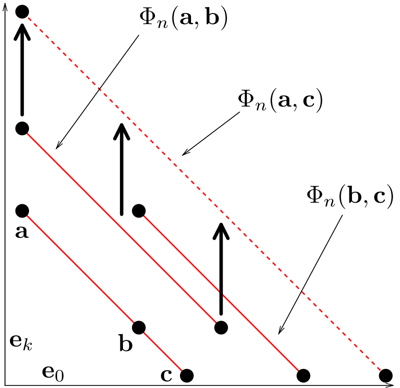

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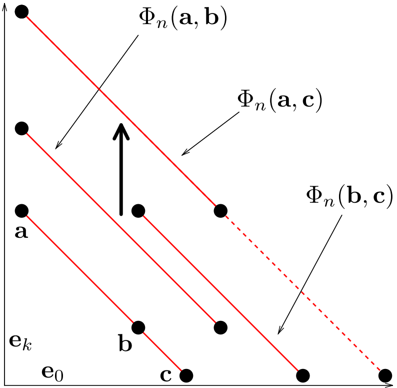

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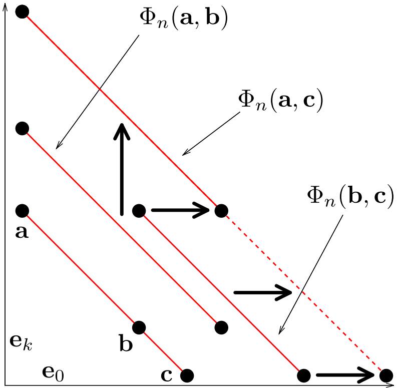

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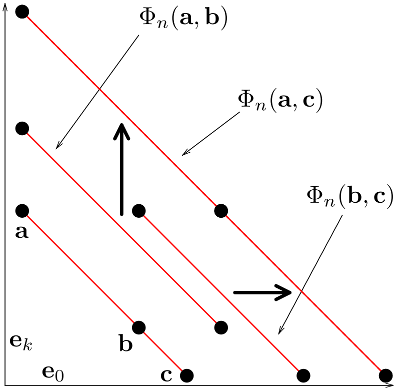

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Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}), (\mathbf{a}, \mathbf{c}) \in \ker \pi_n$.

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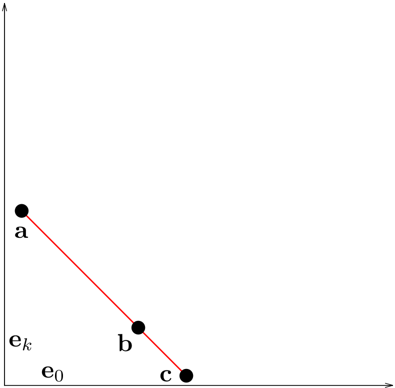

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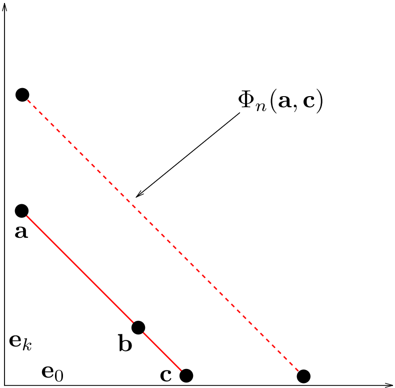

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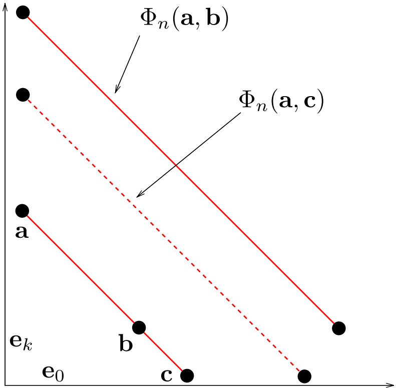

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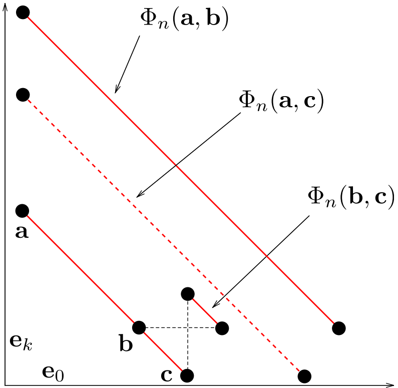

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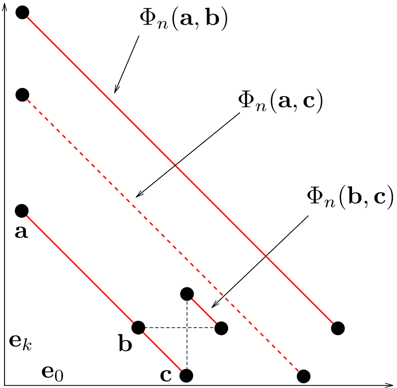
If $|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|$:

translate $\Phi_n(\mathbf{a}, \mathbf{b})$ and $\Phi_n(\mathbf{b}, \mathbf{c})$
$\Rightarrow$ obtain $\Phi_n(\mathbf{a}, \mathbf{c})$.

If $|\mathbf{a}| < |\mathbf{c}| < |\mathbf{b}|$: chaos!



Monotone chains

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ is given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell\mathbf{e}_k, \mathbf{a}' + \ell\mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell\mathbf{e}_0, \mathbf{a}' + \ell\mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}), (\mathbf{a}, \mathbf{c}) \in \ker \pi_n$.

If $|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|$:

translate $\Phi_n(\mathbf{a}, \mathbf{b})$ and $\Phi_n(\mathbf{b}, \mathbf{c})$
$\Rightarrow$ obtain $\Phi_n(\mathbf{a}, \mathbf{c})$.

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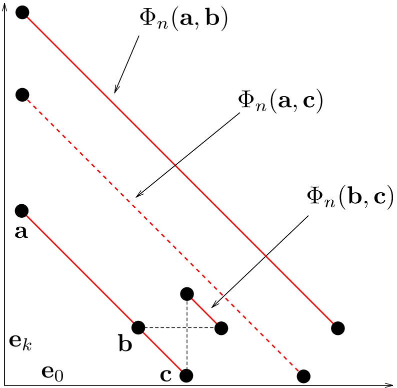
Need: *monotone* chains for Φ_n to preserve transitive closure.



The main result

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If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$ (costly), then $a_0 > 0$ and $a_k' > 0$.

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Consequences:

- The Betti numbers $n \mapsto \beta_j(M_n)$ are eventually r_k-periodic:

  Graded degrees for $\beta_0(M_n)$ are $\pi_n(\mathbf{a})$ for each $(\mathbf{a}, \mathbf{a}') \in \rho$

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  $\Delta(M_n) = \{d\}$ when $||\mathbf{a}| - |\mathbf{a}'|| \in \{0, d\}$ for all $(\mathbf{a}, \mathbf{a}') \in \rho$

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Application: computing minimal presentations

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$\langle 414, 420, 423, 434 \rangle$:

$((0, 0, 8, 0), (3, 2, 0, 3))$,
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$((0,3,0,0),(1,0,2,0))$,		$((0,3,0,0),(1,0,2,0))$,
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n	M_n	Min. Pres. Runtime
50	$\langle 50, 56, 59, 70 \rangle$	1 ms
200	$\langle 200, 206, 209, 220 \rangle$	40 ms
400	$\langle 400, 406, 409, 420 \rangle$	210 ms
1000	$\langle 1000, 1006, 1009, 1020 \rangle$	3 sec
3000	$\langle 3000, 3006, 3009, 3020 \rangle$	2 min
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GAP Numerical Semigroups Package, available at

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Future shifty work

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Frobenius number: $F(S) = \max(\mathbb{N} \setminus S)$.

Example

If $S = \langle 6, 9, 20 \rangle$, then $F(S) = 43$ since

$$\mathbb{N} \setminus S = \{1, 2, 3, 4, 5, 7, 8, 10, 11, 13, \ldots, 31, 34, 37, 43\}.$$

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Sneak peek for $F(\langle n, n+6, n+9, n+20 \rangle)$:

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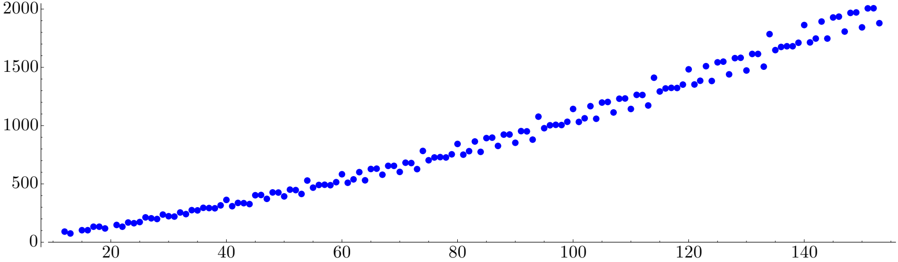

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