Frobenius numbers of shifted numerical monoids

Christopher O'Neill

University of California Davis

coneill@math.ucdavis.edu

Joint with Roberto Pelayo

April 29, 2017

Numerical monoids

Definition

A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

Numerical monoids

Definition

A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$.

Numerical monoids

Definition

A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$. "McNugget Monoid"

Numerical monoids

Definition

A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$. "McNugget Monoid"

Factorizations:

$$60 \quad =$$

Numerical monoids

Definition

A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$. "McNugget Monoid"
Factorizations:

$$60 = 7(6) + 2(9)$$

Numerical monoids

Definition

A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$. "McNugget Monoid"
Factorizations:

$$\begin{aligned} 60 &= 7(6) + 2(9) \\ &= \qquad\qquad\qquad 3(20) \end{aligned}$$

Numerical monoids

Definition

A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$. "McNugget Monoid"

Factorizations:

$$\begin{array}{rclcc} 60 & = & 7(6) + 2(9) & \rightsquigarrow & (7,2,0) \\ & = & 3(20) & \rightsquigarrow & (0,0,3) \end{array}$$

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \cdots + a_k r_k \end{aligned}$$

denotes the *factorization homomorphism* of S.

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \cdots + a_k r_k \end{aligned}$$

denotes the *factorization homomorphism* of S. In particular,

$$\pi^{-1}(n) = \left\{ \mathbf{a} \in \mathbb{N}^k : n = a_1 r_1 + \cdots + a_k r_k \right\}$$

is the *set of factorizations* of $n \in S$.

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

$$\begin{array}{rcl} \pi : \mathbb{N}^k & \longrightarrow & \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} & \longmapsto & a_1 r_1 + \cdots + a_k r_k \end{array}$$

denotes the *factorization homomorphism* of S. In particular,

$$\pi^{-1}(n) = \left\{ \mathbf{a} \in \mathbb{N}^k : n = a_1 r_1 + \cdots + a_k r_k \right\}$$

is the *set of factorizations* of $n \in S$.

$$|\mathbf{a}| = a_1 + \cdots + a_k \text{ (\textit{length} of } \mathbf{a})$$

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

$$\begin{array}{rcl} \pi : \mathbb{N}^k & \longrightarrow & \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} & \longmapsto & a_1 r_1 + \cdots + a_k r_k \end{array}$$

denotes the *factorization homomorphism* of S. In particular,

$$\pi^{-1}(n) = \left\{ \mathbf{a} \in \mathbb{N}^k : n = a_1 r_1 + \cdots + a_k r_k \right\}$$

is the *set of factorizations* of $n \in S$.

$$|\mathbf{a}| = a_1 + \cdots + a_k \text{ (\textit{length} of } \mathbf{a})$$

Example

$S = \langle 6, 9, 20 \rangle$:

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \cdots + a_k r_k \end{aligned}$$

denotes the *factorization homomorphism* of S. In particular,

$$\pi^{-1}(n) = \left\{ \mathbf{a} \in \mathbb{N}^k : n = a_1 r_1 + \cdots + a_k r_k \right\}$$

is the *set of factorizations* of $n \in S$.

$$|\mathbf{a}| = a_1 + \cdots + a_k \ (\textit{length} \text{ of } \mathbf{a})$$

Example

$S = \langle 6, 9, 20 \rangle$:

$$\pi^{-1}(60) = \{(10,0,0), (7,2,0), (4,4,0), (1,6,0), (0,0,3)\}$$

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

$$\begin{array}{rcl} \pi : \mathbb{N}^k & \longrightarrow & \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} & \longmapsto & a_1 r_1 + \cdots + a_k r_k \end{array}$$

denotes the *factorization homomorphism* of S. In particular,

$$\pi^{-1}(n) = \left\{ \mathbf{a} \in \mathbb{N}^k : n = a_1 r_1 + \cdots + a_k r_k \right\}$$

is the *set of factorizations* of $n \in S$.

$$|\mathbf{a}| = a_1 + \cdots + a_k \text{ (\textit{length} of } \mathbf{a})$$

Example

$S = \langle 6, 9, 20 \rangle$:

$$\pi^{-1}(60) = \{(10,0,0), (7,2,0), (4,4,0), (1,6,0), (0,0,3)\}$$

Possible factorization lengths for $n = 60$: $3, 7, 8, 9, 10$.

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

$$\begin{array}{rcl} \pi : \mathbb{N}^k & \longrightarrow & \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} & \longmapsto & a_1 r_1 + \cdots + a_k r_k \end{array}$$

denotes the *factorization homomorphism* of S. In particular,

$$\pi^{-1}(n) = \left\{ \mathbf{a} \in \mathbb{N}^k : n = a_1 r_1 + \cdots + a_k r_k \right\}$$

is the *set of factorizations* of $n \in S$.

$$|\mathbf{a}| = a_1 + \cdots + a_k \text{ (\textit{length} of } \mathbf{a})$$

Example

$S = \langle 6, 9, 20 \rangle$:

$$\pi^{-1}(60) = \{(10,0,0), (7,2,0), (4,4,0), (1,6,0), (0,0,3)\}$$

Possible factorization lengths for $n = 60$: $3, 7, 8, 9, 10$.

$$\pi^{-1}(1000001) =$$

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

$$\begin{aligned} \pi : \mathbb{N}^k & \longrightarrow \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} & \longmapsto a_1 r_1 + \cdots + a_k r_k \end{aligned}$$

denotes the *factorization homomorphism* of S. In particular,

$$\pi^{-1}(n) = \left\{ \mathbf{a} \in \mathbb{N}^k : n = a_1 r_1 + \cdots + a_k r_k \right\}$$

is the *set of factorizations* of $n \in S$.

$$|\mathbf{a}| = a_1 + \cdots + a_k \text{ (\textit{length} of } \mathbf{a})$$

Example

$S = \langle 6, 9, 20 \rangle$:

$$\pi^{-1}(60) = \{(10,0,0), (7,2,0), (4,4,0), (1,6,0), (0,0,3)\}$$

Possible factorization lengths for $n = 60$: $3, 7, 8, 9, 10$.

$$\pi^{-1}(1000001) = \{\underbrace{\qquad\qquad}_{\text{shortest}}, \quad \ldots, \quad \underbrace{\qquad\qquad}_{\text{longest}}\}$$

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

$$\begin{aligned} \pi : \mathbb{N}^k & \longrightarrow \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} & \longmapsto a_1 r_1 + \cdots + a_k r_k \end{aligned}$$

denotes the *factorization homomorphism* of S. In particular,

$$\pi^{-1}(n) = \left\{ \mathbf{a} \in \mathbb{N}^k : n = a_1 r_1 + \cdots + a_k r_k \right\}$$

is the *set of factorizations* of $n \in S$.

$$|\mathbf{a}| = a_1 + \cdots + a_k \text{ (\textit{length} of } \mathbf{a})$$

Example

$S = \langle 6, 9, 20 \rangle$:

$$\pi^{-1}(60) = \{(10,0,0), (7,2,0), (4,4,0), (1,6,0), (0,0,3)\}$$

Possible factorization lengths for $n = 60$: $3, 7, 8, 9, 10$.

$$\pi^{-1}(1000001) = \{ \underbrace{(2,1,49999)}_{\text{shortest}}, \quad \ldots, \quad \underbrace{\qquad\qquad\qquad}_{\text{longest}} \}$$

Numerical monoids

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

$$\begin{array}{rcl} \pi : \mathbb{N}^k & \longrightarrow & \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} & \longmapsto & a_1 r_1 + \cdots + a_k r_k \end{array}$$

denotes the *factorization homomorphism* of S. In particular,

$$\pi^{-1}(n) = \left\{ \mathbf{a} \in \mathbb{N}^k : n = a_1 r_1 + \cdots + a_k r_k \right\}$$

is the *set of factorizations* of $n \in S$.

$$|\mathbf{a}| = a_1 + \cdots + a_k \text{ (\textit{length} of } \mathbf{a})$$

Example

$S = \langle 6, 9, 20 \rangle$:

$$\pi^{-1}(60) = \{(10, 0, 0), (7, 2, 0), (4, 4, 0), (1, 6, 0), (0, 0, 3)\}$$

Possible factorization lengths for $n = 60$: $3, 7, 8, 9, 10$.

$$\pi^{-1}(1000001) = \{ \underbrace{(2, 1, 49999)}_{\text{shortest}}, \quad \ldots, \quad \underbrace{(166662, 1, 1)}_{\text{longest}} \}$$

Frobenius number

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Frobenius number

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

$F(S) = \max(\mathbb{N} \setminus S)$ is the *Frobenius number* of S.

Frobenius number

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

$F(S) = \max(\mathbb{N} \setminus S)$ is the *Frobenius number* of S.

Example

If $S = \langle 6, 9, 20 \rangle$, then $F(S) = 43$ since

$$\mathbb{N} \setminus S = \{1, 2, 3, 4, 5, 7, 8, 10, 11, 13, \ldots, 31, 34, 37, 43\}.$$

Frobenius number

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

$F(S) = \max(\mathbb{N} \setminus S)$ is the *Frobenius number* of S.

Example

If $S = \langle 6, 9, 20 \rangle$, then $F(S) = 43$ since

$$\mathbb{N} \setminus S = \{1, 2, 3, 4, 5, 7, 8, 10, 11, 13, \ldots, 31, 34, 37, 43\}.$$

Computing the Frobenius number for general S is **hard**.

Frobenius number

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

$F(S) = \max(\mathbb{N} \setminus S)$ is the *Frobenius number* of S.

Example

If $S = \langle 6, 9, 20 \rangle$, then $F(S) = 43$ since

$$\mathbb{N} \setminus S = \{1, 2, 3, 4, 5, 7, 8, 10, 11, 13, \ldots, 31, 34, 37, 43\}.$$

Computing the Frobenius number for general S is **hard**.

- If $S = \langle r_1, r_2 \rangle$, then $F(S) = r_1 r_2 - (r_1 + r_2)$.

Frobenius number

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

$F(S) = \max(\mathbb{N} \setminus S)$ is the *Frobenius number* of S.

Example

If $S = \langle 6, 9, 20 \rangle$, then $F(S) = 43$ since

$$\mathbb{N} \setminus S = \{1, 2, 3, 4, 5, 7, 8, 10, 11, 13, \ldots, 31, 34, 37, 43\}.$$

Computing the Frobenius number for general S is **hard**.

- If $S = \langle r_1, r_2 \rangle$, then $F(S) = r_1 r_2 - (r_1 + r_2)$.
- If $S = \langle r_1, r_2, r_3 \rangle$, then there is a fast algorithm for $F(S)$.

Frobenius number

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

$F(S) = \max(\mathbb{N} \setminus S)$ is the *Frobenius number* of S.

Example

If $S = \langle 6, 9, 20 \rangle$, then $F(S) = 43$ since

$$\mathbb{N} \setminus S = \{1, 2, 3, 4, 5, 7, 8, 10, 11, 13, \ldots, 31, 34, 37, 43\}.$$

Computing the Frobenius number for general S is **hard**.

- If $S = \langle r_1, r_2 \rangle$, then $F(S) = r_1 r_2 - (r_1 + r_2)$.
- If $S = \langle r_1, r_2, r_3 \rangle$, then there is a fast algorithm for $F(S)$.
- Formulas in a few other special cases.

The Apéry set

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

The Apéry set

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

For $n \in S$, the *Apéry set* of n is

$$\mathrm{Ap}(S; n) = \{a \in S : a - n \notin S\}$$

The Apéry set

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

For $n \in S$, the *Apéry set* of n is

$$\mathrm{Ap}(S; n) = \{a \in S : a - n \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$ and $n = 6$:

$$\mathrm{Ap}(S; 6) = \{0, 49, 20, 9, 40, 29\}$$

The Apéry set

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

For $n \in S$, the *Apéry set* of n is

$$\mathrm{Ap}(S; n) = \{a \in S : a - n \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$ and $n = 6$:

$$\mathrm{Ap}(S; 6) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

The Apéry set

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

For $n \in S$, the *Apéry set* of n is

$$\mathrm{Ap}(S; n) = \{a \in S : a - n \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$ and $n = 6$:

$$\mathrm{Ap}(S; 6) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

The Apéry set

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

For $n \in S$, the *Apéry set* of n is

$$\mathrm{Ap}(S; n) = \{a \in S : a - n \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$ and $n = 6$:

$$\mathrm{Ap}(S; 6) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

The Apéry set

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

For $n \in S$, the *Apéry set* of n is

$$\mathrm{Ap}(S; n) = \{a \in S : a - n \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$ and $n = 6$:

$$\mathrm{Ap}(S; 6) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

Observations:

The Apéry set

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

For $n \in S$, the *Apéry set* of n is

$$\mathrm{Ap}(S; n) = \{a \in S : a - n \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$ and $n = 6$:

$$\mathrm{Ap}(S; 6) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

Observations:

- The elements of $\mathrm{Ap}(S; n)$ are distinct modulo n

The Apéry set

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

For $n \in S$, the *Apéry set* of n is

$$\mathrm{Ap}(S; n) = \{a \in S : a - n \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$ and $n = 6$:

$$\mathrm{Ap}(S; 6) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

Observations:

- The elements of $\mathrm{Ap}(S; n)$ are distinct modulo n
- $|\mathrm{Ap}(S; n)| = n$

The Apéry set

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

For $n \in S$, the *Apéry set* of n is

$$\mathrm{Ap}(S; n) = \{a \in S : a - n \notin S\}$$

The Apéry set

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

For $n \in S$, the *Apéry set* of n is

$$\mathrm{Ap}(S; n) = \{a \in S : a - n \notin S\}$$

Many things can be easily recovered from the Apéry set.

The Apéry set

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

For $n \in S$, the *Apéry set* of n is

$$\mathrm{Ap}(S; n) = \{a \in S : a - n \notin S\}$$

Many things can be easily recovered from the Apéry set.

- Fast membership test:

$$m \in S \text{ if } m \geq a \text{ for } a \in \mathrm{Ap}(S; n) \text{ and } a \equiv m \bmod n$$

The Apéry set

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

For $n \in S$, the *Apéry set* of n is

$$\mathrm{Ap}(S; n) = \{a \in S : a - n \notin S\}$$

Many things can be easily recovered from the Apéry set.

- Fast membership test:

$$m \in S \text{ if } m \geq a \text{ for } a \in \mathrm{Ap}(S; n) \text{ and } a \equiv m \bmod n$$

- Frobenius number: $F(S) = \max(\mathrm{Ap}(S; n)) - n$

The Apéry set

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

For $n \in S$, the *Apéry set* of n is

$$\mathrm{Ap}(S; n) = \{a \in S : a - n \notin S\}$$

Many things can be easily recovered from the Apéry set.

- Fast membership test:

$$m \in S \text{ if } m \geq a \text{ for } a \in \mathrm{Ap}(S; n) \text{ and } a \equiv m \bmod n$$

- Frobenius number: $F(S) = \max(\mathrm{Ap}(S; n)) - n$
- Number of gaps:

$$|\mathbb{N} \setminus S| = \sum_{a \in \mathrm{Ap}(S;n)} \left\lfloor \frac{a}{n} \right\rfloor$$

The Apéry set

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

For $n \in S$, the *Apéry set* of n is

$$\mathrm{Ap}(S; n) = \{a \in S : a - n \notin S\}$$

Many things can be easily recovered from the Apéry set.

- Fast membership test:

$$m \in S \text{ if } m \geq a \text{ for } a \in \mathrm{Ap}(S; n) \text{ and } a \equiv m \bmod n$$

- Frobenius number: $F(S) = \max(\mathrm{Ap}(S; n)) - n$
- Number of gaps:

$$|\mathbb{N} \setminus S| = \sum_{a \in \mathrm{Ap}(S;n)} \left\lfloor \frac{a}{n} \right\rfloor$$

- Other Apéry sets: recover $\mathrm{Ap}(S; m)$ from $\mathrm{Ap}(S; n)$.

The Apéry set

Fix a numerical monoid $S = \langle r_1, \ldots, r_k \rangle$.

Definition

For $n \in S$, the *Apéry set* of n is

$$\mathrm{Ap}(S; n) = \{a \in S : a - n \notin S\}$$

Many things can be easily recovered from the Apéry set.

- Fast membership test:
$$m \in S \text{ if } m \geq a \text{ for } a \in \mathrm{Ap}(S; n) \text{ and } a \equiv m \bmod n$$
- Frobenius number: $F(S) = \max(\mathrm{Ap}(S; n)) - n$
- Number of gaps:
$$|\mathbb{N} \setminus S| = \sum_{a \in \mathrm{Ap}(S;n)} \left\lfloor \frac{a}{n} \right\rfloor$$
- Other Apéry sets: recover $\mathrm{Ap}(S; m)$ from $\mathrm{Ap}(S; n)$.

The Apéry set is a "one stop shop" for computation.

To shift a numerical monoid…

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Question

How does $n \mapsto F(M_n)$ behave for $n \gg 0$?

To shift a numerical monoid…

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Question

How does $n \mapsto F(M_n)$ behave for $n \gg 0$?

For $M_n = \langle n, n+6, n+9, n+20 \rangle$:

To shift a numerical monoid...

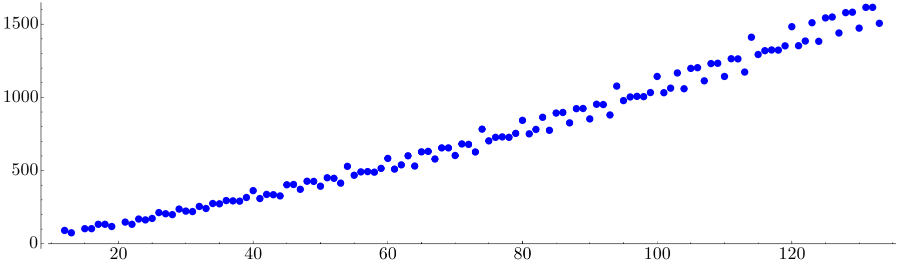
Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Question

How does $n \mapsto F(M_n)$ behave for $n \gg 0$?

For $M_n = \langle n, n+6, n+9, n+20 \rangle$:



To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

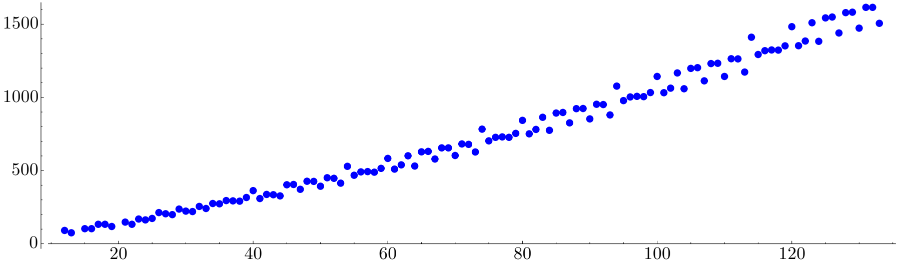
$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Question

How does $n \mapsto F(M_n)$ behave for $n \gg 0$?

For $M_n = \langle n, n+6, n+9, n+20 \rangle$:

$F(M_n)$ is *quasi-quadratic* for $n \geq 44$.



To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

To shift a numerical monoid…

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

The parametric Frobenius problem: study $F(P_n)$, where

$$P_n = \langle q_0(n), q_1(n), \ldots, q_k(n) \rangle$$

for $q_i(n)$ polynomial and eventually positive.

To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

The parametric Frobenius problem: study $F(P_n)$, where

$$P_n = \langle q_0(n), q_1(n), \ldots, q_k(n) \rangle$$

for $q_i(n)$ polynomial and eventually positive.

Conjecture (Roune-Woods)

$F(P_n)$ eventually quasipolynomial.

To shift a numerical monoid. . .

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

The parametric Frobenius problem: study $F(P_n)$, where

$$P_n = \langle q_0(n), q_1(n), \ldots, q_k(n) \rangle$$

for $q_i(n)$ polynomial and eventually positive.

Conjecture (Roune-Woods)

$F(P_n)$ eventually quasipolynomial.

- Proved by Roune-Woods when
  - $P_n = \langle b_0 n + a_0, b_1 n + a_1, \ldots, b_k n + a_k \rangle$
  - $P_n = \langle q_0(n), q_1(n), q_2(n) \rangle$

To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

The parametric Frobenius problem: study $F(P_n)$, where

$$P_n = \langle q_0(n), q_1(n), \ldots, q_k(n) \rangle$$

for $q_i(n)$ polynomial and eventually positive.

Conjecture (Roune-Woods)

$F(P_n)$ eventually quasipolynomial.

- Proved by Roune-Woods when
  - $P_n = \langle b_0 n + a_0, b_1 n + a_1, \ldots, b_k n + a_k \rangle$
  - $P_n = \langle q_0(n), q_1(n), q_2(n) \rangle$
- Proved in general by Shen

To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

The parametric Frobenius problem: study $F(P_n)$, where

$$P_n = \langle q_0(n), q_1(n), \ldots, q_k(n) \rangle$$

for $q_i(n)$ polynomial and eventually positive.

Conjecture (Roune-Woods)

$F(P_n)$ eventually quasipolynomial.

- Proved by Roune-Woods when
  - $P_n = \langle b_0 n + a_0, b_1 n + a_1, \ldots, b_k n + a_k \rangle$
  - $P_n = \langle q_0(n), q_1(n), q_2(n) \rangle$
- Proved in general by Shen

Revised Question

What about Apéry sets?

Apéry sets under shifting

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Apéry sets under shifting

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Example: let $S = \langle 6, 9, 20 \rangle$.

$$\mathrm{Ap}(M_{400}; 400) = \left\{ \begin{array}{l} 0, 406, 409, 420, 812, 815, 818, 826, 829, 840, \\ 1221, 1224, 1227, 1232, 1235, 1238, 1246, 1249, 1260, \\ 1630, 1633, 1636, 1641, 1644, 1647, 1652, 1655, \ldots \end{array} \right\}$$

$$\mathrm{Ap}(M_{420}; 420) = \left\{ \begin{array}{l} 0, 426, 429, 440, 852, 855, 858, 866, 869, 880, \\ 1281, 1284, 1287, 1292, 1295, 1298, 1306, 1309, 1320, \\ 1710, 1713, 1716, 1721, 1724, 1727, 1732, 1735, \ldots \end{array} \right\}$$

Apéry sets under shifting

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Example: let $S = \langle 6, 9, 20 \rangle$.

$$\mathrm{Ap}(M_{400}; 400) = \left\{ \begin{array}{l} 0, 406, 409, 420, 812, 815, 818, 826, 829, 840, \\ 1221, 1224, 1227, 1232, 1235, 1238, 1246, 1249, 1260, \\ 1630, 1633, 1636, 1641, 1644, 1647, 1652, 1655, \ldots \end{array} \right\}$$

$$\mathrm{Ap}(M_{420}; 420) = \left\{ \begin{array}{l} 0, 426, 429, 440, 852, 855, 858, 866, 869, 880, \\ 1281, 1284, 1287, 1292, 1295, 1298, 1306, 1309, 1320, \\ 1710, 1713, 1716, 1721, 1724, 1727, 1732, 1735, \ldots \end{array} \right\}$$

$$\mathrm{Ap}(M_{440}; 440) = \left\{ \begin{array}{l} 0, 446, 449, 460, 892, 895, 898, 906, 909, 920, \\ 1341, 1344, 1347, 1352, 1355, 1358, 1366, 1369, 1380, \\ 1790, 1793, 1796, 1801, 1804, 1807, 1812, 1815, \ldots \end{array} \right\}$$

Apéry sets under shifting

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Example: let $S = \langle 6, 9, 20 \rangle$.

$$\mathrm{Ap}(M_{400}; 400) = \left\{ \begin{array}{l} 0, 406, 409, 420, 812, 815, 818, 826, 829, 840, \\ 1221, 1224, 1227, 1232, 1235, 1238, 1246, 1249, 1260, \\ 1630, 1633, 1636, 1641, 1644, 1647, 1652, 1655, \ldots \end{array} \right\}$$

$$\mathrm{Ap}(M_{420}; 420) = \left\{ \begin{array}{l} 0, 426, 429, 440, 852, 855, 858, 866, 869, 880, \\ 1281, 1284, 1287, 1292, 1295, 1298, 1306, 1309, 1320, \\ 1710, 1713, 1716, 1721, 1724, 1727, 1732, 1735, \ldots \end{array} \right\}$$

$$\mathrm{Ap}(M_{440}; 440) = \left\{ \begin{array}{l} 0, 446, 449, 460, 892, 895, 898, 906, 909, 920, \\ 1341, 1344, 1347, 1352, 1355, 1358, 1366, 1369, 1380, \\ 1790, 1793, 1796, 1801, 1804, 1807, 1812, 1815, \ldots \end{array} \right\}$$

Spoiler: for $n \gg 0$, each $a \in \mathrm{Ap}(M_n; n)$ has unique factorization length.

Apéry sets under shifting

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Example: let $S = \langle 6, 9, 20 \rangle$.

$$\mathrm{Ap}(M_{400}; 400) = \left\{ \begin{array}{l} 0, 406, 409, 420, 812, 815, 818, 826, 829, 840, \\ 1221, 1224, 1227, 1232, 1235, 1238, 1246, 1249, 1260, \\ 1630, 1633, 1636, 1641, 1644, 1647, 1652, 1655, \ldots \end{array} \right\}$$

Apéry sets under shifting

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Example: let $S = \langle 6, 9, 20 \rangle$.

$$\mathrm{Ap}(M_{400}; 400) = \left\{ \begin{array}{l} 0, 406, 409, 420, 812, 815, 818, 826, 829, 840, \\ 1221, 1224, 1227, 1232, 1235, 1238, 1246, 1249, 1260, \\ 1630, 1633, 1636, 1641, 1644, 1647, 1652, 1655, \ldots \end{array} \right\}$$

$$\mathrm{Ap}(S; 400) = \left\{ \begin{array}{l} 0, \quad 6, \quad 9, \quad 20, \quad 12, \quad 15, \quad 18, \quad 26, \quad 29, \quad 40, \\ \quad 21, \quad 24, \quad 27, \quad 32, \quad 35, \quad 38, \quad 46, \quad 49, \quad 60, \\ \quad 30, \quad 33, \quad 36, \quad 41, \quad 44, \quad 47, \quad 52, \quad 55, \ldots \end{array} \right\}$$

Apéry sets under shifting

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Example: let $S = \langle 6, 9, 20 \rangle$.

$$\mathrm{Ap}(M_{400}; 400) = \left\{ \begin{array}{l} 0, 406, 409, 420, 812, 815, 818, 826, 829, 840, \\ 1221, 1224, 1227, 1232, 1235, 1238, 1246, 1249, 1260, \\ 1630, 1633, 1636, 1641, 1644, 1647, 1652, 1655, \ldots \end{array} \right\}$$

$$\mathrm{Ap}(S; 400) = \left\{ \begin{array}{l} 0, \quad 6, \quad 9, \quad 20, \quad 12, \quad 15, \quad 18, \quad 26, \quad 29, \quad 40, \\ \quad 21, \quad 24, \quad 27, \quad 32, \quad 35, \quad 38, \quad 46, \quad 49, \quad 60, \\ \quad 30, \quad 33, \quad 36, \quad 41, \quad 44, \quad 47, \quad 52, \quad 55, \ldots \end{array} \right\}$$

Approach: relate $\mathrm{Ap}(M_n; n)$ to $\mathrm{Ap}(S; n)$ for $n \gg 0$.

Apéry sets under shifting

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Example: let $S = \langle 6, 9, 20 \rangle$.

$$\mathrm{Ap}(M_{400}; 400) = \left\{ \begin{array}{l} 0, 406, 409, 420, 812, 815, 818, 826, 829, 840, \\ 1221, 1224, 1227, 1232, 1235, 1238, 1246, 1249, 1260, \\ 1630, 1633, 1636, 1641, 1644, 1647, 1652, 1655, \ldots \end{array} \right\}$$

$$\mathrm{Ap}(S; 400) = \left\{ \begin{array}{l} 0, \;\; 6, \;\; 9, \;\; 20, \;\; 12, \;\; 15, \;\; 18, \;\; 26, \;\; 29, \;\; 40, \\ \;\; 21, \;\; 24, \;\; 27, \;\; 32, \;\; 35, \;\; 38, \;\; 46, \;\; 49, \;\; 60, \\ \;\; 30, \;\; 33, \;\; 36, \;\; 41, \;\; 44, \;\; 47, \;\; 52, \;\; 55, \ldots \end{array} \right\}$$

Approach: relate $\mathrm{Ap}(M_n; n)$ to $\mathrm{Ap}(S; n)$ for $n \gg 0$.

$$\begin{array}{rcl} \mathrm{Ap}(S; n) & \longrightarrow & \mathrm{Ap}(M_n; n) \\ i & \longmapsto & i + \underline{\qquad} n \end{array}$$

Relating Apéry sets of S and M_n

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Relating Apéry sets of S and M_n

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

$$m = b_0 n \;+\; b_1(n{+}r_1) \;+\; \cdots \;+\; b_k(n{+}r_k)$$

Relating Apéry sets of S and M_n

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

$$m = b_0 n \; + \; b_1(n+r_1) \; + \; \cdots \; + \; b_k(n+r_k)$$

Relating Apéry sets of S and M_n

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

$$m = b_0 n \; + \; b_1(n+r_1) \; + \; \cdots \; + \; b_k(n+r_k) \;\; = \;\; |\mathbf{b}| n \; + \; b_1 r_1 \; + \; \cdots \; + \; b_k r_k$$

Relating Apéry sets of S and M_n

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

$$m = b_0 n \; + \; b_1(n{+}r_1) \; + \; \cdots \; + \; b_k(n{+}r_k) \;\; = \;\; |\mathbf{b}| n \; + \; b_1 r_1 \; + \; \cdots \; + \; b_k r_k$$

Relating Apéry sets of S and M_n

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

$$m = b_0 n + b_1(n+r_1) + \cdots + b_k(n+r_k) = |\mathbf{b}| n + b_1 r_1 + \cdots + b_k r_k$$

$$\left\{ \begin{array}{c} \text{Factorizations of } m \in M_n \\ \text{with length } = \ell \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Factorizations of } m - \ell n \in S \\ \text{with length } \leq \ell \end{array} \right\}$$

Relating Apéry sets of S and M_n

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

$$m = b_0 n \,+\, b_1(n{+}r_1) \,+\, \cdots \,+\, b_k(n{+}r_k) \;=\; |\mathbf{b}|n \,+\, b_1 r_1 \,+\, \cdots \,+\, b_k r_k$$

$$\left\{ \begin{array}{c} \text{Factorizations of } m \in M_n \\ \text{with length } = \ell \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Factorizations of } m - \ell n \in S \\ \text{with length } \le \ell \end{array} \right\}$$

$$(b_0, b_1, \ldots, b_k) \qquad \longmapsto \qquad (b_1, \ldots, b_k)$$

Relating Apéry sets of S and M_n

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

$$m = b_0 n \, + \, b_1(n{+}r_1) \, + \, \cdots \, + \, b_k(n{+}r_k) \; = \; |\mathbf{b}| n \, + \, b_1 r_1 \, + \, \cdots \, + \, b_k r_k$$

$$\left\{ \begin{array}{c} \text{Factorizations of } m \in M_n \\ \text{with length } = \ell \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{Factorizations of } m - \ell n \in S \\ \text{with length } \leq \ell \end{array} \right\}$$

$$(b_0, b_1, \ldots, b_k) \qquad \longmapsto \qquad (b_1, \ldots, b_k)$$

$\mathrm{Ap}(M_n; n) = \{m \in M_n : a_0 = 0 \text{ for all factorizations } \mathbf{a} \text{ of } m\}$

Relating Apéry sets of S and M_n

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

$$m = b_0 n \; + \; b_1(n{+}r_1) \; + \; \cdots \; + \; b_k(n{+}r_k) \; = \; |\mathbf{b}|n \; + \; b_1 r_1 \; + \; \cdots \; + \; b_k r_k$$

$$\left\{ \begin{array}{c} \text{Factorizations of } m \in M_n \\ \text{with length } = \ell \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Factorizations of } m - \ell n \in S \\ \text{with length } \le \ell \end{array} \right\}$$

$$(b_0, b_1, \ldots, b_k) \longmapsto (b_1, \ldots, b_k)$$

$\mathrm{Ap}(M_n; n) = \{m \in M_n : a_0 = 0 \text{ for all factorizations } \mathbf{a} \text{ of } m\}$

Lemma (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For $n > r_k^2$, if $\mathbf{a}, \mathbf{b}$ are factorizations of m with $|\mathbf{a}| < |\mathbf{b}|$, then $b_0 > 0$.

Relating Apéry sets of S and M_n

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

$$m = b_0 n \,+\, b_1(n+r_1) \,+\, \cdots \,+\, b_k(n+r_k) \;=\; |\mathbf{b}|n \,+\, b_1 r_1 \,+\, \cdots \,+\, b_k r_k$$

$$\left\{ \begin{array}{c} \text{Factorizations of } m \in M_n \\ \text{with length } = \ell \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Factorizations of } m - \ell n \in S \\ \text{with length } \le \ell \end{array} \right\}$$

$$(b_0, b_1, \ldots, b_k) \longmapsto (b_1, \ldots, b_k)$$

$\mathrm{Ap}(M_n; n) = \{m \in M_n : a_0 = 0 \text{ for all factorizations } \mathbf{a} \text{ of } m\}$

Lemma (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For $n > r_k^2$, if $\mathbf{a}, \mathbf{b}$ are factorizations of m with $|\mathbf{a}| < |\mathbf{b}|$, then $b_0 > 0$.

$\Rightarrow$ each $m \in \mathrm{Ap}(M_n; n)$ has unique factorization length ℓ

Relating Apéry sets of S and M_n

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

$$m = b_0 n \,+\, b_1(n{+}r_1) \,+\, \cdots \,+\, b_k(n{+}r_k) \;=\; |\mathbf{b}|n \,+\, b_1 r_1 \,+\, \cdots \,+\, b_k r_k$$

$$\left\{ \begin{array}{c} \text{Factorizations of } m \in M_n \\ \text{with length } = \ell \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Factorizations of } m - \ell n \in S \\ \text{with length } \le \ell \end{array} \right\}$$

$$(b_0, b_1, \ldots, b_k) \longmapsto (b_1, \ldots, b_k)$$

$\mathrm{Ap}(M_n; n) = \{m \in M_n : a_0 = 0 \text{ for all factorizations } \mathbf{a} \text{ of } m\}$

Lemma (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For $n > r_k^2$, if $\mathbf{a}, \mathbf{b}$ are factorizations of m with $|\mathbf{a}| < |\mathbf{b}|$, then $b_0 > 0$.

$\Rightarrow$ each $m \in \mathrm{Ap}(M_n; n)$ has unique factorization length ℓ, so define

$$\begin{array}{rcl} \mathrm{Ap}(S; n) & \longrightarrow & \mathrm{Ap}(M_n; n) \\ i & \longmapsto & i + \underline{\;\ell\;} n \end{array}$$

The main result

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$, where $\mathrm{m}_S(i)$ denotes min factorization length of $i \in S$.

The main result

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$,
where $\mathrm{m}_S(i)$ *denotes min factorization length of* $i \in S$.

$M = \langle 1234, 1240, 1243, 1254 \rangle$

The main result

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$, where $\mathrm{m}_S(i)$ denotes min factorization length of $i \in S$.

$M = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \ldots, n + r_k \rangle$

The main result

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$, where $\mathrm{m}_S(i)$ denotes min factorization length of $i \in S$.

$$M = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \ldots, n + r_k \rangle$$

$$n = 1234$$

The main result

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$, where $\mathrm{m}_S(i)$ denotes min factorization length of $i \in S$.

$$M = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \ldots, n + r_k \rangle$$

$$n = 1234 \qquad r_3 = 1254 - 1234 = 20$$

The main result

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$, *where* $\mathrm{m}_S(i)$ *denotes min factorization length of* $i \in S$.

$M = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \ldots, n + r_k \rangle$

$$n = 1234 \qquad r_3 = 1254 - 1234 = 20$$

Verify $n > r_k^2$:

The main result

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$, where $\mathrm{m}_S(i)$ denotes min factorization length of $i \in S$.

$M = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \ldots, n + r_k \rangle$

$$n = 1234 \qquad r_3 = 1254 - 1234 = 20$$

Verify $n > r_k^2$: $1234 > 400$

The main result

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$, where $\mathrm{m}_S(i)$ denotes min factorization length of $i \in S$.

$$M = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \ldots, n + r_k \rangle$$

$$n = 1234 \qquad r_3 = 1254 - 1234 = 20$$

Verify $n > r_k^2$: $1234 > 400$ ✓

The main result

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$, *where* $\mathrm{m}_S(i)$ *denotes min factorization length of* $i \in S$.

$M = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \ldots, n + r_k \rangle$

$$n = 1234 \qquad r_3 = 1254 - 1234 = 20$$

Verify $n > r_k^2$: $1234 > 400$ ✓

Let $S = \langle r_1, r_2, r_3 \rangle = \langle 6, 9, 20 \rangle$.

The main result

$$M_n = \langle n, n+r_1, \ldots, n+r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$, where $\mathrm{m}_S(i)$ denotes min factorization length of $i \in S$.

$M = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n+r_1, \ldots, n+r_k \rangle$

$$n = 1234 \qquad r_3 = 1254 - 1234 = 20$$

Verify $n > r_k^2$: $1234 > 400$ ✓

Let $S = \langle r_1, r_2, r_3 \rangle = \langle 6, 9, 20 \rangle$. To compute $\mathrm{Ap}(M; 1234)$:

The main result

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$, where $\mathrm{m}_S(i)$ denotes min factorization length of $i \in S$.

$M = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \ldots, n + r_k \rangle$

$$n = 1234 \qquad r_3 = 1254 - 1234 = 20$$

Verify $n > r_k^2$: $1234 > 400$ ✓

Let $S = \langle r_1, r_2, r_3 \rangle = \langle 6, 9, 20 \rangle$. To compute $\mathrm{Ap}(M; 1234)$:

1. Compute $\mathrm{Ap}(S; 6) = \{0, 49, 20, 9, 40, 29\}$

The main result

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$, where $\mathrm{m}_S(i)$ denotes min factorization length of $i \in S$.

$M = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \ldots, n + r_k \rangle$

$$n = 1234 \qquad r_3 = 1254 - 1234 = 20$$

Verify $n > r_k^2$: $1234 > 400$ ✓

Let $S = \langle r_1, r_2, r_3 \rangle = \langle 6, 9, 20 \rangle$. To compute $\mathrm{Ap}(M; 1234)$:

1. Compute $\mathrm{Ap}(S; 6) = \{0, 49, 20, 9, 40, 29\}$
2. Compute $\mathrm{Ap}(S; 1234)$ from $\mathrm{Ap}(S; 6)$ (fast)

The main result

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$, where $\mathrm{m}_S(i)$ denotes min factorization length of $i \in S$.

$$M = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \ldots, n + r_k \rangle$$

$$n = 1234 \qquad r_3 = 1254 - 1234 = 20$$

Verify $n > r_k^2$: $1234 > 400$ ✓

Let $S = \langle r_1, r_2, r_3 \rangle = \langle 6, 9, 20 \rangle$. To compute $\mathrm{Ap}(M; 1234)$:

1. Compute $\mathrm{Ap}(S; 6) = \{0, 49, 20, 9, 40, 29\}$
2. Compute $\mathrm{Ap}(S; 1234)$ from $\mathrm{Ap}(S; 6)$ (fast)
3. Compute $\mathrm{Ap}(M; 1234)$ from $\mathrm{Ap}(S; 1234)$ using Theorem

Application: computing Apéry sets

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$, *where* $\mathrm{m}_S(i)$ *denotes min factorization length of* $i \in S$.

Application: computing Apéry sets

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathsf{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$, where $\mathsf{m}_S(i)$ denotes min factorization length of $i \in S$.

n	M_n	Apéry Set Runtime
50	$\langle 50, 56, 59, 70 \rangle$	1 ms
200	$\langle 200, 206, 209, 220 \rangle$	30 ms
400	$\langle 400, 406, 409, 420 \rangle$	170 ms
1000	$\langle 1000, 1006, 1009, 1020 \rangle$	3 sec
3000	$\langle 3000, 3006, 3009, 3020 \rangle$	2 min
5000	$\langle 5000, 5006, 5009, 5020 \rangle$	17 min
10000	$\langle 10000, 10006, 10009, 10020 \rangle$	3.6 hr

Application: computing Apéry sets

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$,
where $\mathrm{m}_S(i)$ *denotes min factorization length of* $i \in S$.

n	M_n	Apéry Set Runtime
50	$\langle 50, 56, 59, 70 \rangle$	1 ms
200	$\langle 200, 206, 209, 220 \rangle$	30 ms
400	$\langle 400, 406, 409, 420 \rangle$	170 ms
1000	$\langle 1000, 1006, 1009, 1020 \rangle$	~~3 sec~~ 1 ms
3000	$\langle 3000, 3006, 3009, 3020 \rangle$	~~2 min~~ 1 ms
5000	$\langle 5000, 5006, 5009, 5020 \rangle$	~~17 min~~ 1 ms
10000	$\langle 10000, 10006, 10009, 10020 \rangle$	~~3.6 hr~~ 1 ms

Application: computing Apéry sets

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$,
where $\mathrm{m}_S(i)$ *denotes min factorization length of* $i \in S$.

n	M_n	Apéry Set Runtime
50	$\langle 50, 56, 59, 70 \rangle$	1 ms
200	$\langle 200, 206, 209, 220 \rangle$	30 ms
400	$\langle 400, 406, 409, 420 \rangle$	170 ms
1000	$\langle 1000, 1006, 1009, 1020 \rangle$	~~3 sec~~ 1 ms
3000	$\langle 3000, 3006, 3009, 3020 \rangle$	~~2 min~~ 1 ms
5000	$\langle 5000, 5006, 5009, 5020 \rangle$	~~17 min~~ 1 ms
10000	$\langle 10000, 10006, 10009, 10020 \rangle$	~~3.6 hr~~ 1 ms

GAP Numerical Semigroups Package, available at

`http://www.gap-system.org/Packages/numericalsgps.html`.

Frobenius numbers and related quantities

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$,
where $\mathrm{m}_S(i)$ *denotes min factorization length of* $i \in S$.

Frobenius numbers and related quantities

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$, where $\mathrm{m}_S(i)$ denotes min factorization length of $i \in S$.

Let $S = \langle 6, 9, 20 \rangle$. Compute $\mathrm{Ap}(S; 100)$.

Frobenius numbers and related quantities

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$, where $\mathrm{m}_S(i)$ denotes min factorization length of $i \in S$.

Let $S = \langle 6, 9, 20 \rangle$. Compute $\mathrm{Ap}(S; 100)$.

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14,
15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29,
30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44,
45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59,
60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74,
75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89,
90, 91, 92, 93, 94, 95, 96, 97, 98, 99,
100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114,
115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129,
130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144

Frobenius numbers and related quantities

$$M_n = \langle n, n + r_1, \ldots, n + r_k\rangle, \quad S = \langle r_1, \ldots, r_k\rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$, where $\mathrm{m}_S(i)$ denotes min factorization length of $i \in S$.

Let $S = \langle 6, 9, 20\rangle$. Compute $\mathrm{Ap}(S; 100)$.

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14,
15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29,
30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44,
45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59,
60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74,
75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89,
90, 91, 92, 93, 94, 95, 96, 97, 98, 99,
100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114,
115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129,
130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144

Frobenius numbers and related quantities

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$,
where $\mathrm{m}_S(i)$ *denotes min factorization length of* $i \in S$.

Frobenius numbers and related quantities

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$, where $\mathrm{m}_S(i)$ denotes min factorization length of $i \in S$.

Lemma

If $n > F(S)$, then $\mathrm{Ap}(S; n) = \{a_0, \ldots, a_{n-1}\}$, where

$$a_i = \begin{cases} i & i \in S \\ i + n & i \notin S \end{cases}$$

Frobenius numbers and related quantities

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle$$

Theorem (O.–Pelayo)

For $n > r_k^2$, $\mathrm{Ap}(M_n; n) = \{i + \mathsf{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$,
where $\mathsf{m}_S(i)$ *denotes min factorization length of* $i \in S$.

Lemma

If $n > F(S)$, *then* $\mathrm{Ap}(S; n) = \{a_0, \ldots, a_{n-1}\}$, *where*

$$a_i = \begin{cases} i & i \in S \\ i + n & i \notin S \end{cases}$$

Theorem (Barron–O.–Pelayo)

$n \mapsto \mathsf{m}_S(n)$ *is eventually quasilinear with period* r_k.

Frobenius numbers and related quantities

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle, \quad n > r_k^2$$

Facts:

- $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$,
- If $n > F(S)$, then $\mathrm{Ap}(S; n) = \{a_0, \ldots, a_{n-1}\}$, where
$$a_i = \begin{cases} i & i \in S \\ i + n & i \notin S \end{cases}$$
- $n \mapsto \mathrm{m}_S(n)$ is eventually quasilinear with period r_k.

Frobenius numbers and related quantities

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle, \quad n > r_k^2$$

Facts:

- $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$,
- If $n > F(S)$, then $\mathrm{Ap}(S; n) = \{a_0, \ldots, a_{n-1}\}$, where
$$a_i = \begin{cases} i & i \in S \\ i + n & i \notin S \end{cases}$$
- $n \mapsto \mathrm{m}_S(n)$ is eventually quasilinear with period r_k.

Corollary

The Frobenius number $n \mapsto F(M_n)$ is quasiquadratic for $n > r_k^2$.

Frobenius numbers and related quantities

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle, \quad n > r_k^2$$

Facts:

- $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$,
- If $n > F(S)$, then $\mathrm{Ap}(S; n) = \{a_0, \ldots, a_{n-1}\}$, where
$$a_i = \begin{cases} i & i \in S \\ i + n & i \notin S \end{cases}$$
- $n \mapsto \mathrm{m}_S(n)$ is eventually quasilinear with period r_k.

Corollary

The Frobenius number $n \mapsto F(M_n)$ is quasiquadratic for $n > r_k^2$.

$$F(M_n) = \max \mathrm{Ap}(M_n; n) - n$$

Frobenius numbers and related quantities

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle, \quad n > r_k^2$$

Facts:

- $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$,
- If $n > F(S)$, then $\mathrm{Ap}(S; n) = \{a_0, \ldots, a_{n-1}\}$, where
$$a_i = \begin{cases} i & i \in S \\ i + n & i \notin S \end{cases}$$
- $n \mapsto \mathrm{m}_S(n)$ is eventually quasilinear with period r_k.

Corollary

The Frobenius number $n \mapsto F(M_n)$ is quasiquadratic for $n > r_k^2$.

$$\begin{aligned} F(M_n) &= \max \mathrm{Ap}(M_n; n) - n \\ &= F(S) + \mathrm{m}_S(F(S) + n) \cdot n \end{aligned}$$

Frobenius numbers and related quantities

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle, \quad n > r_k^2$$

Facts:

- $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$,
- If $n > F(S)$, then $\mathrm{Ap}(S; n) = \{a_0, \ldots, a_{n-1}\}$, where
$$a_i = \begin{cases} i & i \in S \\ i + n & i \notin S \end{cases}$$
- $n \mapsto \mathrm{m}_S(n)$ is eventually quasilinear with period r_k.

Frobenius numbers and related quantities

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle, \quad n > r_k^2$$

Facts:

- $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$,
- If $n > F(S)$, then $\mathrm{Ap}(S; n) = \{a_0, \ldots, a_{n-1}\}$, where

$$a_i = \begin{cases} i & i \in S \\ i + n & i \notin S \end{cases}$$

- $n \mapsto \mathrm{m}_S(n)$ is eventually quasilinear with period r_k.

Corollary

The number of gaps $n \mapsto G(M_n) = |\mathbb{N} \setminus M_n|$ is quasiquadratic for $n > r_k^2$.

Frobenius numbers and related quantities

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle, \quad n > r_k^2$$

Facts:

- $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$,
- If $n > F(S)$, then $\mathrm{Ap}(S; n) = \{a_0, \ldots, a_{n-1}\}$, where
$$a_i = \begin{cases} i & i \in S \\ i + n & i \notin S \end{cases}$$
- $n \mapsto \mathrm{m}_S(n)$ is eventually quasilinear with period r_k.

Corollary

The number of gaps $n \mapsto G(M_n) = |\mathbb{N} \setminus M_n|$ is quasiquadratic for $n > r_k^2$.

$$G(M_n) = \sum_{a \in \mathrm{Ap}(M_n; n)} \left\lfloor \frac{a}{n} \right\rfloor$$

Frobenius numbers and related quantities

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle, \quad n > r_k^2$$

Facts:

- $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$,
- If $n > F(S)$, then $\mathrm{Ap}(S; n) = \{a_0, \ldots, a_{n-1}\}$, where
$$a_i = \begin{cases} i & i \in S \\ i + n & i \notin S \end{cases}$$
- $n \mapsto \mathrm{m}_S(n)$ is eventually quasilinear with period r_k.

Corollary

The number of gaps $n \mapsto G(M_n) = |\mathbb{N} \setminus M_n|$ is quasiquadratic for $n > r_k^2$.

$$G(M_n) = \sum_{a \in \mathrm{Ap}(M_n;n)} \left\lfloor \frac{a}{n} \right\rfloor = \sum_{i \in \mathrm{Ap}(S;n)} \left\lfloor \frac{i}{n} \right\rfloor + \sum_{i \in \mathrm{Ap}(S;n)} \mathrm{m}_S(i)$$

Frobenius numbers and related quantities

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \quad S = \langle r_1, \ldots, r_k \rangle, \quad n > r_k^2$$

Facts:

- $\mathrm{Ap}(M_n; n) = \{i + \mathrm{m}_S(i) \cdot n \mid i \in \mathrm{Ap}(S; n)\}$,
- If $n > F(S)$, then $\mathrm{Ap}(S; n) = \{a_0, \ldots, a_{n-1}\}$, where
$$a_i = \begin{cases} i & i \in S \\ i + n & i \notin S \end{cases}$$
- $n \mapsto \mathrm{m}_S(n)$ is eventually quasilinear with period r_k.

Corollary

The number of gaps $n \mapsto G(M_n) = |\mathbb{N} \setminus M_n|$ is quasiquadratic for $n > r_k^2$.

$$G(M_n) = \sum_{a \in \mathrm{Ap}(M_n; n)} \left\lfloor \frac{a}{n} \right\rfloor = \sum_{i \in \mathrm{Ap}(S; n)} \left\lfloor \frac{i}{n} \right\rfloor + \sum_{i \in \mathrm{Ap}(S; n)} \mathrm{m}_S(i)$$

$$= G(S) + \sum_{i \in S \cap [1, n-1]} \mathrm{m}_S(i) + \sum_{i \in G(S)} \mathrm{m}_S(i + n)$$

References

R. Conaway, F. Gotti, J. Horton, C. O'Neill, R. Pelayo, M. Williams, and B. Wissman (2016)
Minimal presentations of shifted numerical monoids
submitted. Available at [arXiv:1701.08555].

M. Delgado, P. García-Sánchez, and J. Morais,
GAP numerical semigroups package
http://www.gap-system.org/Packages/numericalsgps.html.

C. O'Neill and R. Pelayo (2017)
Apéry sets of shifted numerical monoids
in preparation.

B. Roune and K. Woods (2015),
The parametric Frobenius problem
Electronic J. Combinatorics 22 (2) (2015), #P2.36.

References

R. Conaway, F. Gotti, J. Horton, C. O'Neill, R. Pelayo, M. Williams, and B. Wissman (2016)
Minimal presentations of shifted numerical monoids
submitted. Available at [arXiv:1701.08555].

M. Delgado, P. García-Sánchez, and J. Morais,
GAP numerical semigroups package
http://www.gap-system.org/Packages/numericalsgps.html.

C. O'Neill and R. Pelayo (2017)
Apéry sets of shifted numerical monoids
in preparation.

B. Roune and K. Woods (2015),
The parametric Frobenius problem
Electronic J. Combinatorics 22 (2) (2015), #P2.36.

Thanks!