Augmented Hilbert series of numerical semigroups

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A *numerical semigroup* $S \subset \mathbb{N}$: **additive** submsemigroup, $|\mathbb{N} \setminus S| < \infty$.

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Maximum and minimum factorization length

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$$\begin{aligned} \mathsf{m}(82) &= 5 \quad \text{and} \quad \mathsf{Z}(82) = \{(0,3,2,0,0),(5,0,0,0,3),\ldots\} \\ \mathsf{m}(462) &= 25 \quad \text{and} \quad \mathsf{Z}(462) = \{(0,3,2,0,20),\ldots\} \end{aligned}$$

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Theorem (Barron–O.–Pelayo, 2014)

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n > n_k(n_{k-1} - 1)$,

$$\begin{aligned} \mathsf{M}(n + n_1) &= 1 + \mathsf{M}(n) \\ \mathsf{m}(n + n_k) &= 1 + \mathsf{m}(n) \end{aligned}$$

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Equivalently: $\mathsf{M}(n)$, $\mathsf{m}(n)$ eventually *quasilinear*

$$\begin{aligned} \mathsf{M}(n) &= \tfrac{1}{n_1} n + a_0(n) \\ \mathsf{m}(n) &= \tfrac{1}{n_k} n + b_0(n) \end{aligned}$$

for periodic functions $a_0(n)$, $b_0(n)$.

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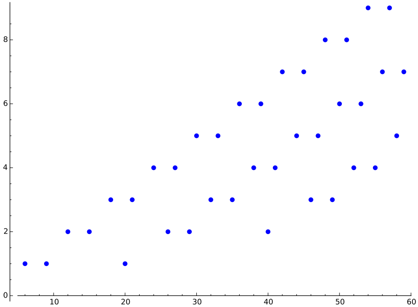

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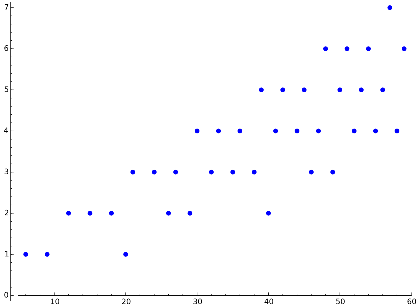

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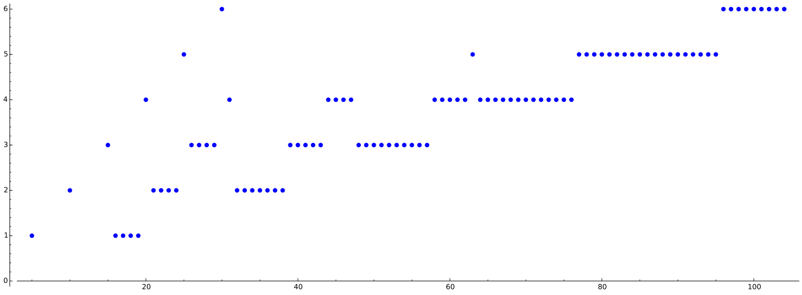

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Hilbert series

Let $S = \langle n_1, \ldots, n_k \rangle$. The *Hilbert series* of S is the formal power series

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Example: $S = \langle 6, 9, 20 \rangle$

$$\mathcal{H}(S;t) = 1 + t^6 + t^9 + t^{12} + t^{15} + t^{18} + t^{20} + \cdots = \frac{f(t)}{1-t}$$

$$\begin{aligned} f(t) = 1 - t + t^6 - t^7 + t^9 - t^{10} + t^{12} - t^{13} + t^{15} - t^{16} + t^{18} - t^{19} \\ + t^{20} - t^{22} + t^{24} - t^{25} + t^{26} - t^{28} + t^{29} - t^{31} + t^{32} - t^{34} \\ + t^{35} - t^{37} + t^{38} - t^{43} + t^{44} \end{aligned}$$

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A more concise expression:

$$\mathcal{H}(S;t) = \frac{\sum_{n \in \mathrm{Ap}(S;n_1)} t^n}{1 - t^{n_1}} = \frac{1 + t^9 + t^{20} + t^{29} + t^{40} + t^{49}}{1 - t^6}$$

where $\mathrm{Ap}(S; n_1) =$ the *Apéry set* of S.

The "Big Theorem"

The Big Theorem (Bruns, Herzog)

For any numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$,

$$\mathcal{H}(S; t) = \frac{\sum_{n \in S} \chi(\Delta_n) t^n}{(1 - t^{n_1}) \cdots (1 - t^{n_k})}$$

Δ_n is the *squarefree divisor complex*: simplicial complex on $\{n_1, \ldots, n_k\}$,

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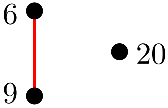
Δ_{60}: 

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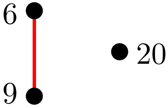
Δ_{60}: 

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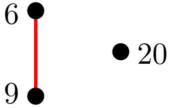

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Disconnected complexes $\longleftrightarrow$ minimal relations between generators

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with $a_0(n)$ n_1-periodic ($\mathsf{M}(n)$ is eventually quasilinear).

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$$\mathcal{H}_{\mathsf{M}}(S; t) = \frac{f(t)}{(1 - t^{n_1})^2}, \qquad f(t) = \sum_{n \in S} \big(\mathsf{M}(n) - 2\mathsf{M}(n - n_1) + \mathsf{M}(n - 2n_1)\big) t^n$$

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$$\mathcal{H}_{\mathsf{M}}(S; t) = \frac{f(t)}{(1 - t^{n_1})^2}, \qquad f(t) = \sum_{n \in S} \big(\mathsf{M}(n) - 2\mathsf{M}(n - n_1) + \mathsf{M}(n - 2n_1)\big) t^n$$

Example: $S = \langle 6, 9, 20 \rangle$

$$\mathcal{H}_{\mathsf{M}}(S; t) = 1 + t^6 + t^9 + 2t^{12} + 2t^{15} + 3t^{18} + t^{20} + \cdots = \frac{f(t)}{(1 - t^6)^2}$$

where $f(t) = t^6 + t^9 + t^{20} + 2t^{29} - t^{35} + 2t^{40} - t^{46} + 3t^{49} - 2t^{55}$

Augmented Hilbert series

Proposition (Glenn–O.–Ponomarenko–Sepanski)

For any numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$,

$$\mathcal{H}_{\mathrm{M}}(S; t) = \frac{\sum_{n \in S} \widehat{\chi}_{\mathrm{M}}(\Delta_n) t^n}{(1 - t^{n_1}) \cdots (1 - t^{n_k})}$$

$\widehat{\chi}_{\mathrm{M}}(\Delta_n)$ is the *weighted Euler characteristic*:

$$F \in \Delta_n \text{ has weight } \mathrm{M}(n - \textstyle\sum F)$$

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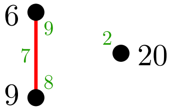
Δ_{18}: 

Δ_{60}: 

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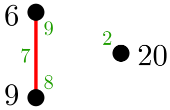
Δ_{60}: 

$\widehat{\chi}_{\mathsf{M}}(\Delta_{18}) = 3 - (2 + 1) = 0$

$\widehat{\chi}_{\mathsf{M}}(\Delta_{60}) = 10 - 19 + 7 = -2$

Augmented Hilbert series

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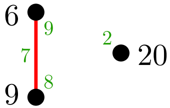
Δ_{60}: 

$\widehat{\chi}_{\mathrm{M}}(\Delta_{18}) = 3 - (2 + 1) = 0$

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$$\mathrm{H}_{\mathrm{M}}(S;t) = \frac{t^6 + t^9 + t^{12} + t^{20} - t^{38} + t^{40} - t^{58} - 2t^{60} - t^{66} - t^{69} - t^{72} + 2t^{78}}{(1 - t^6)(1 - t^9)(1 - t^{20})}$$

Augmented Hilbert series

Theorem (Glenn–O.–Ponomarenko–Sepanski)

For any numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$,

$$\mathcal{H}_{\mathrm{M}}(S; t) = \frac{\sum_{n \in S} \chi_{\mathrm{M}}(\Delta_n) t^n}{(1 - t^{n_1}) \cdots (1 - t^{n_k})} + \mathcal{H}(S; t) \sum_{i=1}^{k} \frac{t^{n_i}}{1 - t^{n_i}}$$

This time, $\chi_{\mathrm{M}}(\Delta_n)$ is computed as follows:

$$F \in \Delta_n \text{ has weight } |F| + \mathrm{M}(n - \textstyle\sum F)$$

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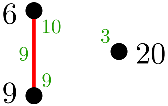
Δ_{60}: 

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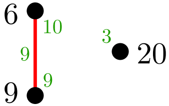
Example: $S = \langle 6, 9, 20 \rangle$

Δ_{18}: 

$\chi_{\mathsf{M}}(\Delta_{18}) = 3 - (3 + 2) = -2$

Δ_{60}: 

$\chi_{\mathsf{M}}(\Delta_{60}) = 10 - 22 + 9 = -3$

Augmented Hilbert series

Theorem (Glenn–O.–Ponomarenko–Sepanski)

For any numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$,

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This time, $\chi_{\mathsf{M}}(\Delta_n)$ is computed as follows:

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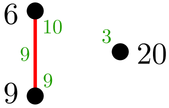
Example: $S = \langle 6, 9, 20 \rangle$

$$\sum_{n \in S} \chi_{\mathsf{M}}(\Delta_n) t^n = -2t^{18} - 3t^{60} + 5t^{78}$$

Δ_{18}:

Δ_{60}:



$\chi_{\mathsf{M}}(\Delta_{18}) = 3 - (3 + 2) = -2$

$\chi_{\mathsf{M}}(\Delta_{60}) = 10 - 22 + 9 = -3$

Augmented Hilbert series

Theorem (Glenn–O.–Ponomarenko–Sepanski)

For any numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$,

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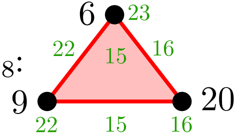
Example: $S = \langle 6, 9, 20 \rangle$ $\qquad \displaystyle\sum_{n \in S} \chi_{\mathsf{M}}(\Delta_n) t^n = -2t^{18} - 3t^{60} + 5t^{78}$

Δ_{138}:



$\chi_{\mathsf{M}}(\Delta_{138}) = 0$

Augmented Hilbert series

Let $S = \langle n_1, \ldots, n_k \rangle$.

$$\begin{aligned}\mathcal{H}_{\mathsf{M}}(S;t) &= \frac{\sum_{n \in S} \chi_{\mathsf{M}}(\Delta_n) t^n}{(1 - t^{n_1}) \cdots (1 - t^{n_k})} + \mathcal{H}(S;t) \sum_{i=1}^{k} \frac{t^{n_i}}{1 - t^{n_i}} \\ &= \frac{\sum_{n \in S} \widehat{\chi}_{\mathsf{M}}(\Delta_n) t^n}{(1 - t^{n_1}) \cdots (1 - t^{n_k})}\end{aligned}$$

Augmented Hilbert series

Let $S = \langle n_1, \ldots, n_k \rangle$.

$$\mathcal{H}_{\mathrm{M}}(S; t) = \frac{\sum_{n \in S} \chi_{\mathrm{M}}(\Delta_n) t^n}{(1 - t^{n_1}) \cdots (1 - t^{n_k})} + \mathcal{H}(S; t) \sum_{i=1}^{k} \frac{t^{n_i}}{1 - t^{n_i}}$$
$$= \frac{\sum_{n \in S} \widehat{\chi}_{\mathrm{M}}(\Delta_n) t^n}{(1 - t^{n_1}) \cdots (1 - t^{n_k})}$$

$S = \langle 9, 10, 23 \rangle$:

$$\sum_{n \in S} \chi(\Delta_n) t^n = 1 - t^{46} - t^{50} - t^{63} + t^{73} + t^{86}$$
$$\sum_{n \in S} \chi_{\mathrm{M}}(\Delta_n) t^n = -2t^{46} - 4t^{50} - 5t^{63} + 5t^{73} + 6t^{86} - t^{90} + t^{113}$$
$$\sum_{n \in S} \widehat{\chi}_{\mathrm{M}}(\Delta_n) t^n = t^9 + t^{10} + t^{18} + t^{20} + t^{23} + t^{27} + t^{30} + t^{36} + t^{40} + t^{45} - t^{46} - 3t^{50} + t^{54} - t^{55} - t^{56} - t^{59} - 4t^{63} - t^{64} - t^{66} - t^{68} + 2t^{73} - t^{76} - t^{77} + 3t^{86} - t^{90} + t^{113}$$

Augmented Hilbert series

Let $S = \langle n_1, \ldots, n_k \rangle$.

$$\begin{aligned}\mathcal{H}_{\mathsf{M}}(S;t) &= \frac{\sum_{n \in S} \chi_{\mathsf{M}}(\Delta_n) t^n}{(1 - t^{n_1}) \cdots (1 - t^{n_k})} + \mathcal{H}(S;t) \sum_{i=1}^{k} \frac{t^{n_i}}{1 - t^{n_i}} \\ &= \frac{\sum_{n \in S} \widehat{\chi}_{\mathsf{M}}(\Delta_n) t^n}{(1 - t^{n_1}) \cdots (1 - t^{n_k})}\end{aligned}$$

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$S = \langle 11, 18, 24 \rangle$:

$$\begin{aligned}\sum_{n \in S} \chi(\Delta_n) t^n &= 1 - t^{66} - t^{72} + t^{138} \\ \sum_{n \in S} \chi_{\mathrm{M}}(\Delta_n) t^n &= -3t^{66} - 3t^{72} - t^{90} + 7t^{138} \\ \sum_{n \in S} \widehat{\chi}_{\mathrm{M}}(\Delta_n) t^n &= t^{11} + t^{18} + t^{22} + t^{24} + t^{33} + t^{36} + t^{44} + t^{48} + \\ &\quad t^{54} + t^{55} - 2t^{66} - t^{72} - t^{83} - t^{84} - 2t^{90} - t^{94} - \\ &\quad t^{102} - t^{105} - t^{114} - t^{116} - t^{120} - t^{127} + 4t^{138}\end{aligned}$$

Augmented Hilbert series

Let $S = \langle n_1, \ldots, n_k \rangle$.

$$\mathcal{H}_{\mathrm{M}}(S;t) = \frac{\sum_{n \in S} \chi_{\mathrm{M}}(\Delta_n) t^n}{(1-t^{n_1})\cdots(1-t^{n_k})} + \mathcal{H}(S;t) \sum_{i=1}^{k} \frac{t^{n_i}}{1-t^{n_i}}$$
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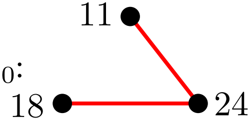
$S = \langle 11, 18, 24 \rangle$:

Δ_{90}:



$$\sum_{n \in S} \chi(\Delta_n) t^n = 1 - t^{66} - t^{72} + t^{138}$$

$$\sum_{n \in S} \chi_{\mathrm{M}}(\Delta_n) t^n = -3t^{66} - 3t^{72} - t^{90} + 7t^{138}$$

$$\sum_{n \in S} \widehat{\chi}_{\mathrm{M}}(\Delta_n) t^n = t^{11} + t^{18} + t^{22} + t^{24} + t^{33} + t^{36} + t^{44} + t^{48} + t^{54} + t^{55} - 2t^{66} - t^{72} - t^{83} - t^{84} - 2t^{90} - t^{94} - t^{102} - t^{105} - t^{114} - t^{116} - t^{120} - t^{127} + 4t^{138}$$

References

M. Delgado, P. García-Sánchez, and J. Morais,
GAP numerical semigroups package
http://www.gap-system.org/Packages/numericalsgps.html.

J. Glenn, C. O'Neill, V. Ponomarenko, and B. Sepanski (2018)
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Sage (www.sagemath.org)
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References

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