Computing the delta set of an affine semigroup: a status report

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* = undergraduate student

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Affine semigroups

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A *numerical semigroup* $S \subset \mathbb{Z}_{\geq 0}$: closed under **addition**.

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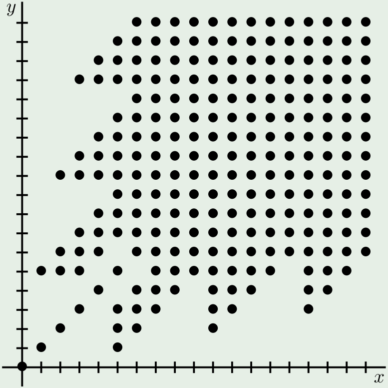

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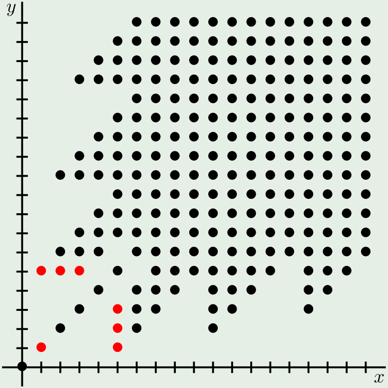

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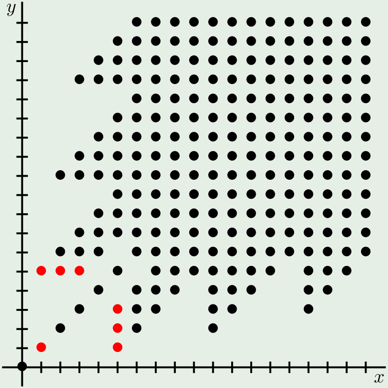

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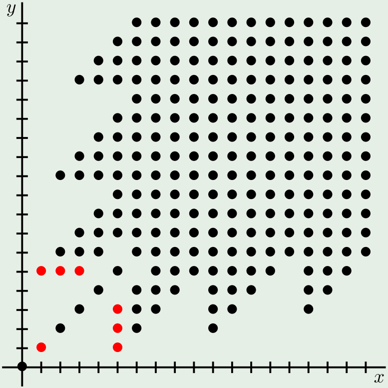

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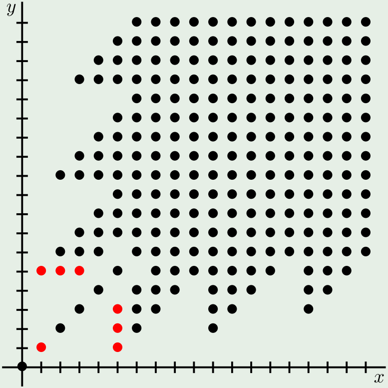

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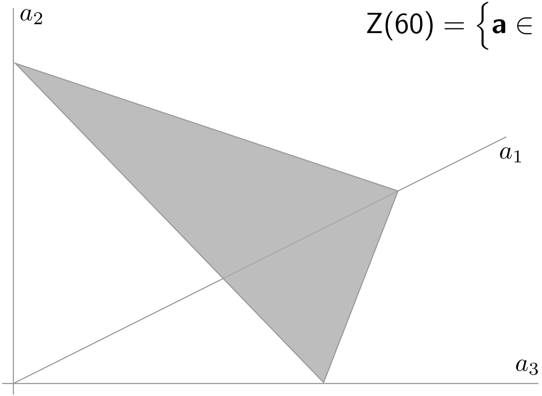

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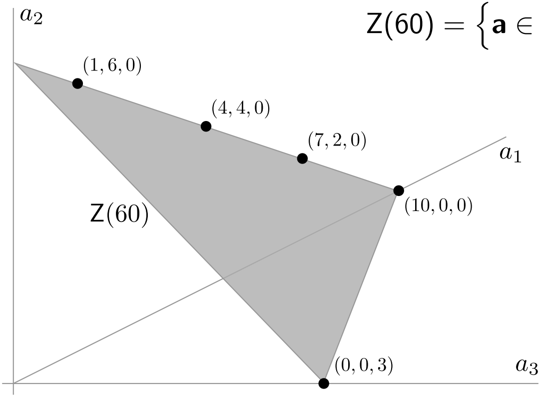

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$$\mathsf{L}(110) = \{16, 18, \ldots, 34, 36\}, \quad \Delta(110) = \{2\}$$

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$S = \langle 42, 86, 245, 285, 365, 463 \rangle$:

$$\mathsf{L}(3023) = \{7, 9, 11, 12, \ldots, 46, 47, 58, 62, 64\}, \qquad \Delta(3023) = \{1, 2, 4, 9\}$$

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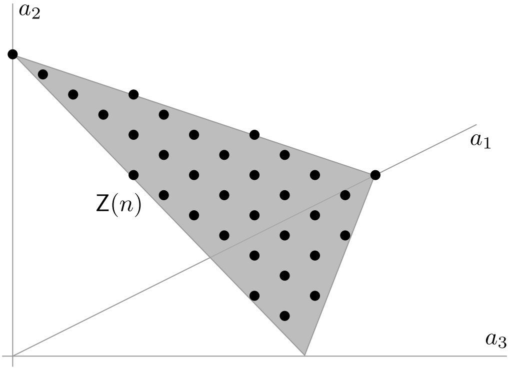

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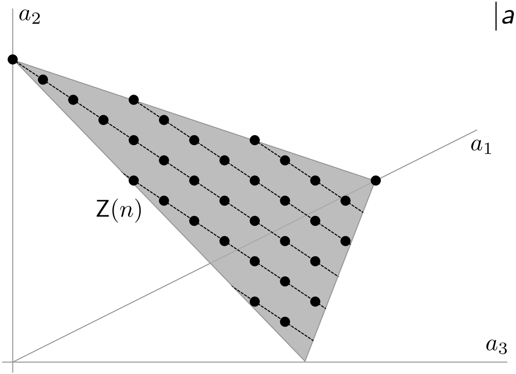

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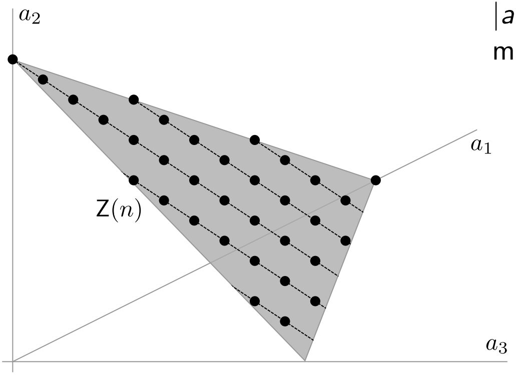

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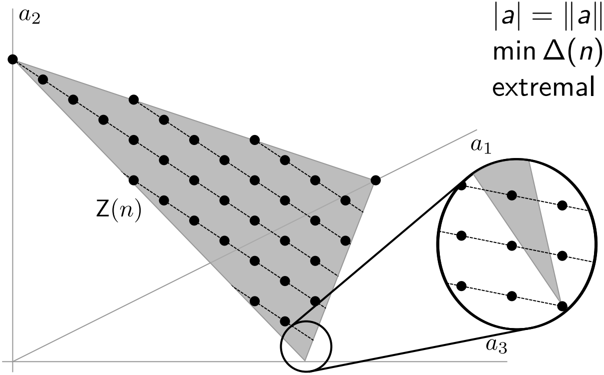

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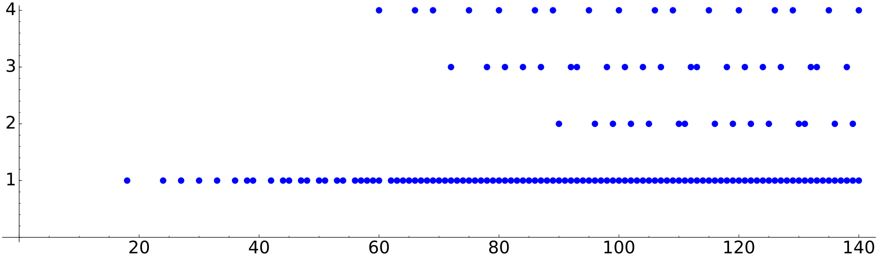

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GAP Numerical Semigroups Package, available at

`http://www.gap-system.org/Packages/numericalsgps.html`.

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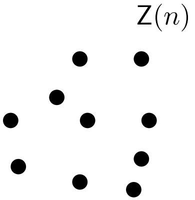

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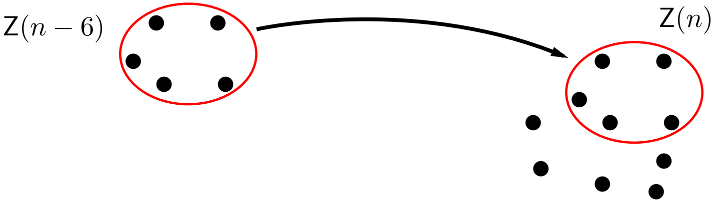

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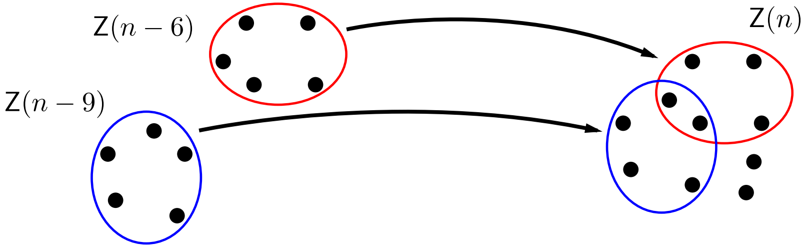

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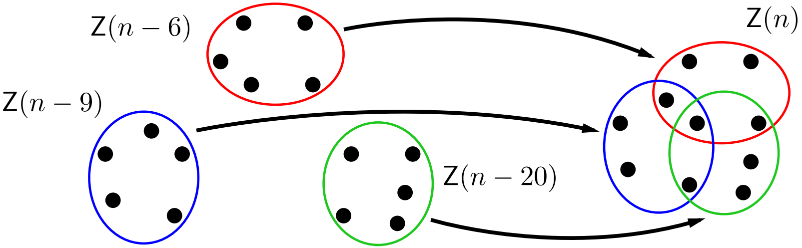

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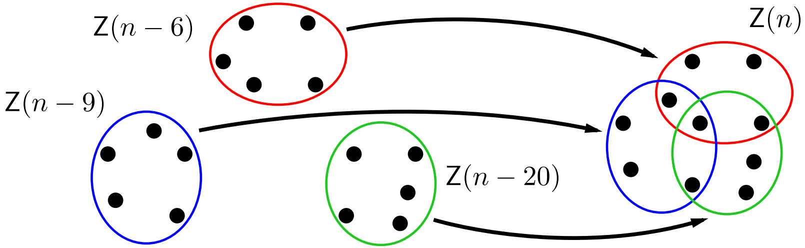

A faster solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \le k$,

$$\begin{aligned} \phi_i : \mathsf{Z}(n - n_i) &\longrightarrow \mathsf{Z}(n) \\ \mathbf{a} &\longmapsto \mathbf{a} + \mathbf{e}_i \end{aligned}$$

$$\mathsf{Z}(n) = \bigcup_{i \le k} \phi_i(\mathsf{Z}(n - n_i))$$

A faster solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \le k$,

$$\begin{array}{rcl} \phi_i : \mathsf{Z}(n - n_i) & \longrightarrow & \mathsf{Z}(n) \\ \mathbf{a} & \longmapsto & \mathbf{a} + \mathbf{e}_i \end{array}$$

$$\mathsf{Z}(n) = \bigcup_{i \le k} \phi_i(\mathsf{Z}(n - n_i))$$

$n \in S = \langle 6, 9, 20 \rangle$	$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0	$\{\mathbf{0}\}$	$\{0\}$

A faster solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \le k$,

$$\begin{array}{rcl} \phi_i : \mathsf{Z}(n - n_i) & \longrightarrow & \mathsf{Z}(n) \\ \mathbf{a} & \longmapsto & \mathbf{a} + \mathbf{e}_i \end{array}$$

$$\mathsf{Z}(n) = \bigcup_{i \le k} \phi_i(\mathsf{Z}(n - n_i))$$

$n \in S = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$

A faster solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \le k$,

$$\begin{array}{rcl} \phi_i : \mathsf{Z}(n - n_i) & \longrightarrow & \mathsf{Z}(n) \\ \mathbf{a} & \longmapsto & \mathbf{a} + \mathbf{e}_i \end{array}$$

$$\mathsf{Z}(n) = \bigcup_{i \le k} \phi_i(\mathsf{Z}(n - n_i))$$

$n \in S = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$
9	$\mathbf{0} \overset{9}{\rightsquigarrow} \mathbf{e}_2$	$\{\mathbf{e}_2\}$	$\{1\}$

A faster solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \leq k$,

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$n \in S = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$
9	$\mathbf{0} \overset{9}{\rightsquigarrow} \mathbf{e}_2$	$\{\mathbf{e}_2\}$	$\{1\}$
12	$\mathbf{e}_1 \overset{6}{\rightsquigarrow} 2\mathbf{e}_1$	$\{2\mathbf{e}_1\}$	$\{2\}$

A faster solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \leq k$,

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$n \in S = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$
9	$\mathbf{0} \overset{9}{\rightsquigarrow} \mathbf{e}_2$	$\{\mathbf{e}_2\}$	$\{1\}$
12	$\mathbf{e}_1 \overset{6}{\rightsquigarrow} 2\mathbf{e}_1$	$\{2\mathbf{e}_1\}$	$\{2\}$
15	$\mathbf{e}_2 \overset{6}{\rightsquigarrow} (1,1,0)$	$\{(1,1,0)\}$	$\{2\}$

A faster solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \leq k$,

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$n \in S = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$
9	$\mathbf{0} \overset{9}{\rightsquigarrow} \mathbf{e}_2$	$\{\mathbf{e}_2\}$	$\{1\}$
12	$\mathbf{e}_1 \overset{6}{\rightsquigarrow} 2\mathbf{e}_1$	$\{2\mathbf{e}_1\}$	$\{2\}$
15	$\mathbf{e}_2 \overset{6}{\rightsquigarrow} (1,1,0)$	$\{(1,1,0)\}$	$\{2\}$
	$\mathbf{e}_1 \overset{9}{\rightsquigarrow} (1,1,0)$		

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Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \le k$,

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$n \in S = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$
9	$\mathbf{0} \overset{9}{\rightsquigarrow} \mathbf{e}_2$	$\{\mathbf{e}_2\}$	$\{1\}$
12	$\mathbf{e}_1 \overset{6}{\rightsquigarrow} 2\mathbf{e}_1$	$\{2\mathbf{e}_1\}$	$\{2\}$
15	$\mathbf{e}_2 \overset{6}{\rightsquigarrow} (1,1,0)$	$\{(1,1,0)\}$	$\{2\}$
	$\mathbf{e}_1 \overset{9}{\rightsquigarrow} (1,1,0)$		
18	$2\mathbf{e}_1 \overset{6}{\rightsquigarrow} 3\mathbf{e}_1$	$\{3\mathbf{e}_1, 2\mathbf{e}_2\}$	$\{2, 3\}$

A faster solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \le k$,

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$n \in S = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$
9	$\mathbf{0} \overset{9}{\rightsquigarrow} \mathbf{e}_2$	$\{\mathbf{e}_2\}$	$\{1\}$
12	$\mathbf{e}_1 \overset{6}{\rightsquigarrow} 2\mathbf{e}_1$	$\{2\mathbf{e}_1\}$	$\{2\}$
15	$\mathbf{e}_2 \overset{6}{\rightsquigarrow} (1, 1, 0)$	$\{(1, 1, 0)\}$	$\{2\}$
	$\mathbf{e}_1 \overset{9}{\rightsquigarrow} (1, 1, 0)$		
18	$2\mathbf{e}_1 \overset{6}{\rightsquigarrow} 3\mathbf{e}_1$	$\{3\mathbf{e}_1, 2\mathbf{e}_2\}$	$\{2, 3\}$
	$\mathbf{e}_2 \overset{9}{\rightsquigarrow} 2\mathbf{e}_2$		

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$$\mathsf{Z}(n) = \textstyle\bigcup_{i \le k} \phi_i(\mathsf{Z}(n - n_i))$$

$n \in S = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$
9	$\mathbf{0} \overset{9}{\rightsquigarrow} \mathbf{e}_2$	$\{\mathbf{e}_2\}$	$\{1\}$
12	$\mathbf{e}_1 \overset{6}{\rightsquigarrow} 2\mathbf{e}_1$	$\{2\mathbf{e}_1\}$	$\{2\}$
15	$\mathbf{e}_2 \overset{6}{\rightsquigarrow} (1,1,0)$	$\{(1,1,0)\}$	$\{2\}$
	$\mathbf{e}_1 \overset{9}{\rightsquigarrow} (1,1,0)$		
18	$2\mathbf{e}_1 \overset{6}{\rightsquigarrow} 3\mathbf{e}_1$	$\{3\mathbf{e}_1, 2\mathbf{e}_2\}$	$\{2, 3\}$
	$\mathbf{e}_2 \overset{9}{\rightsquigarrow} 2\mathbf{e}_2$		
20	$\mathbf{0} \overset{20}{\rightsquigarrow} \mathbf{e}_3$	$\{\mathbf{e}_3\}$	$\{1\}$

A faster solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \le k$,

$$\begin{array}{rcl} \phi_i : \mathsf{Z}(n - n_i) & \longrightarrow & \mathsf{Z}(n) \\ \mathbf{a} & \longmapsto & \mathbf{a} + \mathbf{e}_i \end{array}$$

$$\mathsf{Z}(n) = \textstyle\bigcup_{i \le k} \phi_i(\mathsf{Z}(n - n_i))$$

$n \in S = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$
9	$\mathbf{0} \overset{9}{\rightsquigarrow} \mathbf{e}_2$	$\{\mathbf{e}_2\}$	$\{1\}$
12	$\mathbf{e}_1 \overset{6}{\rightsquigarrow} 2\mathbf{e}_1$	$\{2\mathbf{e}_1\}$	$\{2\}$
15	$\mathbf{e}_2 \overset{6}{\rightsquigarrow} (1,1,0)$	$\{(1,1,0)\}$	$\{2\}$
	$\mathbf{e}_1 \overset{9}{\rightsquigarrow} (1,1,0)$		
18	$2\mathbf{e}_1 \overset{6}{\rightsquigarrow} 3\mathbf{e}_1$	$\{3\mathbf{e}_1, 2\mathbf{e}_2\}$	$\{2, 3\}$
	$\mathbf{e}_2 \overset{9}{\rightsquigarrow} 2\mathbf{e}_2$		
20	$\mathbf{0} \overset{20}{\rightsquigarrow} \mathbf{e}_3$	$\{\mathbf{e}_3\}$	$\{1\}$
$\vdots$		$\vdots$	$\vdots$

A faster solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \leq k$,

$$\begin{array}{rcl} \phi_i : \mathsf{Z}(n - n_i) & \longrightarrow & \mathsf{Z}(n) \\ \mathbf{a} & \longmapsto & \mathbf{a} + \mathbf{e}_i \end{array} \qquad \psi_i : \mathsf{L}(n - n_i) \longrightarrow \mathsf{L}(n)$$

$$\mathsf{Z}(n) = \textstyle\bigcup_{i \leq k} \phi_i(\mathsf{Z}(n - n_i))$$

$n \in S = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$
9	$\mathbf{0} \overset{9}{\rightsquigarrow} \mathbf{e}_2$	$\{\mathbf{e}_2\}$	$\{1\}$
12	$\mathbf{e}_1 \overset{6}{\rightsquigarrow} 2\mathbf{e}_1$	$\{2\mathbf{e}_1\}$	$\{2\}$
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	$\mathbf{e}_1 \overset{9}{\rightsquigarrow} (1,1,0)$		
18	$2\mathbf{e}_1 \overset{6}{\rightsquigarrow} 3\mathbf{e}_1$	$\{3\mathbf{e}_1, 2\mathbf{e}_2\}$	$\{2, 3\}$
	$\mathbf{e}_2 \overset{9}{\rightsquigarrow} 2\mathbf{e}_2$		
20	$\mathbf{0} \overset{20}{\rightsquigarrow} \mathbf{e}_3$	$\{\mathbf{e}_3\}$	$\{1\}$
$\vdots$		$\vdots$	$\vdots$

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$$\mathsf{Z}(n) = \textstyle\bigcup_{i \le k} \phi_i(\mathsf{Z}(n - n_i))$$

$n \in S = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$
9	$\mathbf{0} \overset{9}{\rightsquigarrow} \mathbf{e}_2$	$\{\mathbf{e}_2\}$	$\{1\}$
12	$\mathbf{e}_1 \overset{6}{\rightsquigarrow} 2\mathbf{e}_1$	$\{2\mathbf{e}_1\}$	$\{2\}$
15	$\mathbf{e}_2 \overset{6}{\rightsquigarrow} (1,1,0)$	$\{(1,1,0)\}$	$\{2\}$
	$\mathbf{e}_1 \overset{9}{\rightsquigarrow} (1,1,0)$		
18	$2\mathbf{e}_1 \overset{6}{\rightsquigarrow} 3\mathbf{e}_1$	$\{3\mathbf{e}_1, 2\mathbf{e}_2\}$	$\{2, 3\}$
	$\mathbf{e}_2 \overset{9}{\rightsquigarrow} 2\mathbf{e}_2$		
20	$\mathbf{0} \overset{20}{\rightsquigarrow} \mathbf{e}_3$	$\{\mathbf{e}_3\}$	$\{1\}$
$\vdots$		$\vdots$	$\vdots$

A faster solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \leq k$,

$$\begin{array}{rccc} \phi_i : \mathsf{Z}(n - n_i) & \longrightarrow & \mathsf{Z}(n) \\ \mathbf{a} & \longmapsto & \mathbf{a} + \mathbf{e}_i \end{array} \qquad \begin{array}{rccc} \psi_i : \mathsf{L}(n - n_i) & \longrightarrow & \mathsf{L}(n) \\ \ell & \longmapsto & \ell + 1 \end{array}$$

$$\mathsf{Z}(n) = \textstyle\bigcup_{i \leq k} \phi_i(\mathsf{Z}(n - n_i)) \qquad \mathsf{L}(n) = \textstyle\bigcup_{i \leq k} \psi_i(\mathsf{L}(n - n_i))$$

$n \in S = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$
9	$\mathbf{0} \overset{9}{\rightsquigarrow} \mathbf{e}_2$	$\{\mathbf{e}_2\}$	$\{1\}$
12	$\mathbf{e}_1 \overset{6}{\rightsquigarrow} 2\mathbf{e}_1$	$\{2\mathbf{e}_1\}$	$\{2\}$
15	$\mathbf{e}_2 \overset{6}{\rightsquigarrow} (1,1,0)$	$\{(1,1,0)\}$	$\{2\}$
	$\mathbf{e}_1 \overset{9}{\rightsquigarrow} (1,1,0)$		
18	$2\mathbf{e}_1 \overset{6}{\rightsquigarrow} 3\mathbf{e}_1$	$\{3\mathbf{e}_1, 2\mathbf{e}_2\}$	$\{2, 3\}$
	$\mathbf{e}_2 \overset{9}{\rightsquigarrow} 2\mathbf{e}_2$		
20	$\mathbf{0} \overset{20}{\rightsquigarrow} \mathbf{e}_3$	$\{\mathbf{e}_3\}$	$\{1\}$
$\vdots$		$\vdots$	$\vdots$

A faster solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \le k$,

$$\begin{array}{rcl} \phi_i : \mathsf{Z}(n - n_i) & \longrightarrow & \mathsf{Z}(n) \\ \mathbf{a} & \longmapsto & \mathbf{a} + \mathbf{e}_i \end{array} \qquad \begin{array}{rcl} \psi_i : \mathsf{L}(n - n_i) & \longrightarrow & \mathsf{L}(n) \\ \ell & \longmapsto & \ell + 1 \end{array}$$

$$\mathsf{Z}(n) = \textstyle\bigcup_{i \le k} \phi_i(\mathsf{Z}(n - n_i)) \qquad \mathsf{L}(n) = \textstyle\bigcup_{i \le k} \psi_i(\mathsf{L}(n - n_i))$$

$n \in S = \langle 6, 9, 20 \rangle$	$\mathsf{L}(n)$
0	$\{0\}$
6	
9	
12	
15	
18	
20	
$\vdots$	

A faster solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \le k$,

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$$\mathsf{Z}(n) = \textstyle\bigcup_{i \le k} \phi_i(\mathsf{Z}(n - n_i)) \qquad \mathsf{L}(n) = \textstyle\bigcup_{i \le k} \psi_i(\mathsf{L}(n - n_i))$$

$n \in S = \langle 6, 9, 20 \rangle$	$\mathsf{L}(n)$	
0	$\{0\}$	
6	$\{1\}$	$0 \overset{6}{\rightsquigarrow} 1$
9		
12		
15		
18		
20		
$\vdots$		

A faster solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \leq k$,

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$n \in S = \langle 6, 9, 20 \rangle$	$\mathsf{L}(n)$	
0	$\{0\}$	
6	$\{1\}$	$0 \overset{6}{\rightsquigarrow} 1$
9	$\{1\}$	$0 \overset{9}{\rightsquigarrow} 1$
12		
15		
18		
20		
$\vdots$		

A faster solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \le k$,

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$n \in S = \langle 6, 9, 20 \rangle$	$\mathsf{L}(n)$	
0	$\{0\}$	
6	$\{1\}$	$0 \overset{6}{\rightsquigarrow} 1$
9	$\{1\}$	$0 \overset{9}{\rightsquigarrow} 1$
12	$\{2\}$	$1 \overset{6}{\rightsquigarrow} 2$
15		
18		
20		
$\vdots$		

A faster solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \le k$,

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$$\mathsf{Z}(n) = \textstyle\bigcup_{i \le k} \phi_i(\mathsf{Z}(n - n_i)) \qquad \mathsf{L}(n) = \textstyle\bigcup_{i \le k} \psi_i(\mathsf{L}(n - n_i))$$

$n \in S = \langle 6, 9, 20 \rangle$	$\mathsf{L}(n)$	
0	$\{0\}$	
6	$\{1\}$	$0 \overset{6}{\rightsquigarrow} 1$
9	$\{1\}$	$0 \overset{9}{\rightsquigarrow} 1$
12	$\{2\}$	$1 \overset{6}{\rightsquigarrow} 2$
15	$\{2\}$	$1 \overset{6}{\rightsquigarrow} 2$
18		
20		
$\vdots$		

A faster solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \le k$,

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$n \in S = \langle 6, 9, 20 \rangle$	$\mathsf{L}(n)$	
0	$\{0\}$	
6	$\{1\}$	$0 \overset{6}{\rightsquigarrow} 1$
9	$\{1\}$	$0 \overset{9}{\rightsquigarrow} 1$
12	$\{2\}$	$1 \overset{6}{\rightsquigarrow} 2$
15	$\{2\}$	$1 \overset{6}{\rightsquigarrow} 2$
		$1 \overset{9}{\rightsquigarrow} 2$
18		
20		
$\vdots$		

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Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \le k$,

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$n \in S = \langle 6, 9, 20 \rangle$	$\mathsf{L}(n)$	
0	$\{0\}$	
6	$\{1\}$	$0 \overset{6}{\rightsquigarrow} 1$
9	$\{1\}$	$0 \overset{9}{\rightsquigarrow} 1$
12	$\{2\}$	$1 \overset{6}{\rightsquigarrow} 2$
15	$\{2\}$	$1 \overset{6}{\rightsquigarrow} 2$
		$1 \overset{9}{\rightsquigarrow} 2$
18	$\{2, 3\}$	$2 \overset{6}{\rightsquigarrow} 3$
20		
$\vdots$		

A faster solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \le k$,

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Theorem (García-García–Moreno-Frías–Vigneron-Tenorio, 2014)

$S = \langle n_1, \ldots, n_k \rangle$. *For* $n \geq N_S$, $\Delta(n) = \Delta(n + \operatorname{lcm}(n_1, n_k))$.

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$\langle 11, 53, 73, 87 \rangle$	14381	$\{2, 4, 6, 8, 10, 22\}$	0m 49s	2.5s
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GAP Numerical Semigroups Package, available at

`http://www.gap-system.org/Packages/numericalsgps.html`.

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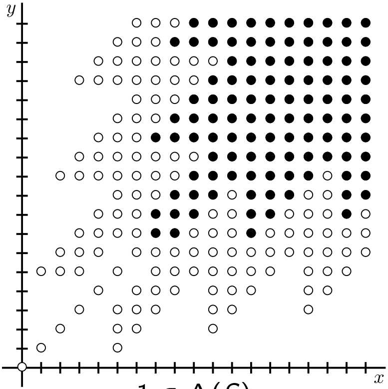

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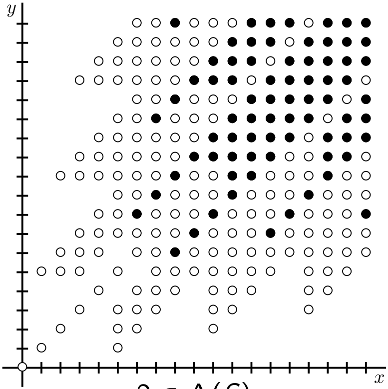

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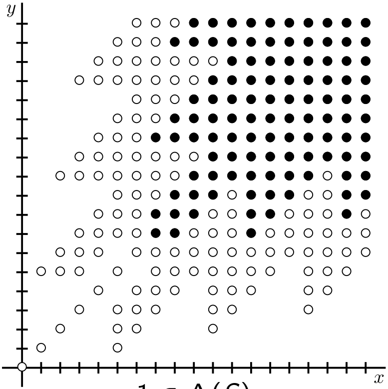

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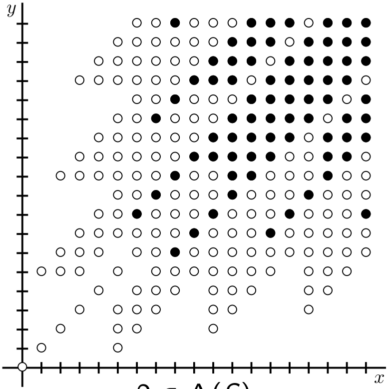

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Need a new approach!

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Fix an **affine** semigroup $S = \langle n_1, \ldots, n_k \rangle \subset \mathbb{Z}_{\geq 0}^d$.

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$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{Z}_{\geq 0}^k & \longrightarrow & \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array}$$

Monomial map:

$$\begin{array}{rcl} \varphi : \Bbbk[x_1, \ldots, x_k] & \longrightarrow & \Bbbk[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

Definition

The *kernel* $\ker \pi$ is the relation $\sim$ on $\mathbb{Z}_{\geq 0}^k$ with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b}) \qquad\qquad x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S = \ker \varphi$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a} \qquad\qquad x^{\mathbf{a}} - x^{\mathbf{a}} = 0 \in I_S$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a} \qquad\qquad x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S \Rightarrow x^{\mathbf{b}} - x^{\mathbf{a}} \in I_S$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c} \qquad\qquad (x^{\mathbf{a}} - x^{\mathbf{b}}) + (x^{\mathbf{b}} - x^{\mathbf{c}}) = x^{\mathbf{a}} - x^{\mathbf{c}}$$

that is closed under *translation*.

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

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$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c} \qquad\qquad x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S \Rightarrow x^{\mathbf{c}}(x^{\mathbf{a}} - x^{\mathbf{b}}) \in I_S$$

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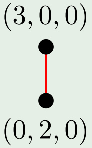

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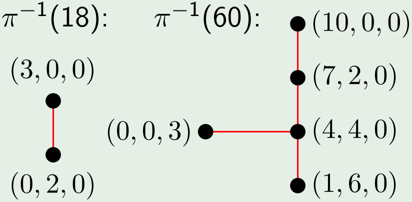

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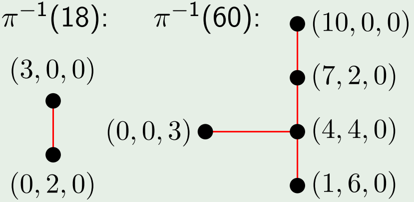

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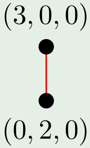

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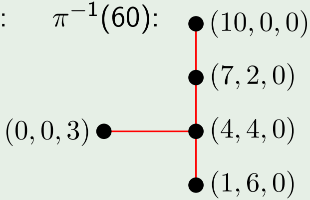

$$x^7y^2 - x^4y^4 = x^4y^2(x^3 - y^2)$$

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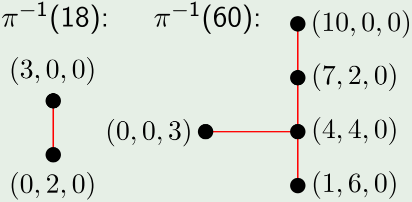

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Generating set for I_S $\Leftrightarrow$ $\pi^{-1}(n)$ connected for all $n \in S$

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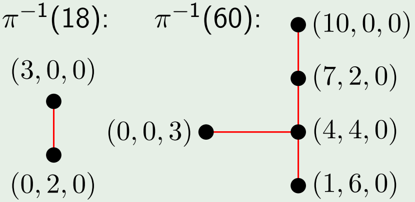

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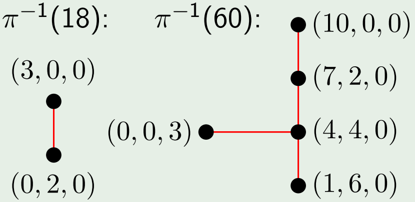

All minimal generating sets:

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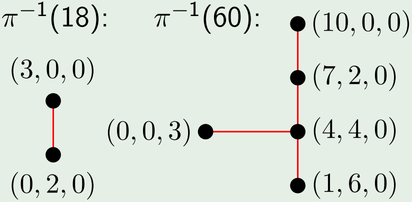

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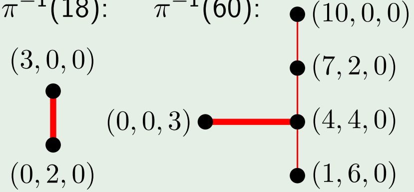

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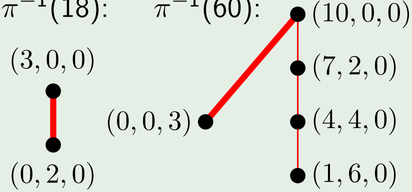

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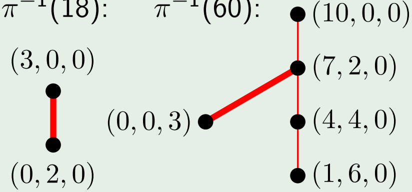

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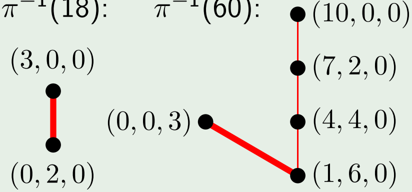

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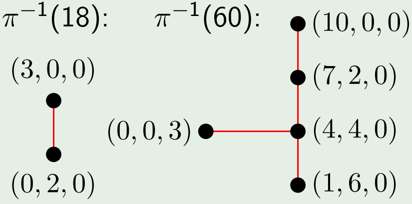

All minimal generating sets:

$$\begin{aligned} I_S &= \langle x^3 - y^2, \, x^{10} - z^3 \rangle \\ &= \langle x^3 - y^2, \, x^7 y^2 - z^3 \rangle \\ &= \langle x^3 - y^2, \, x^4 y^4 - z^3 \rangle \\ &= \langle x^3 - y^2, \, x^6 y - z^3 \rangle \end{aligned}$$

Commutative algebra hiding in the background

$$S = \langle n_1, \ldots, n_k \rangle \subset \mathbb{Z}^d_{\geq 0} \;\; \text{(affine)} \qquad \pi : \mathbb{Z}^k_{\geq 0} \;\longrightarrow\; S$$

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A larger example: $S = \langle 13, 44, 106, 120 \rangle$.

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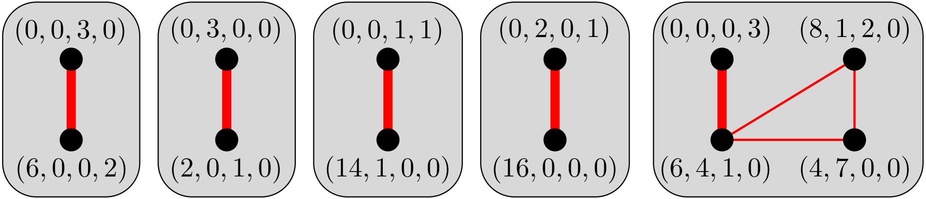

$\pi^{-1}(132)$ $\pi^{-1}(318)$ $\pi^{-1}(226)$ $\pi^{-1}(208)$ $\pi^{-1}(360)$

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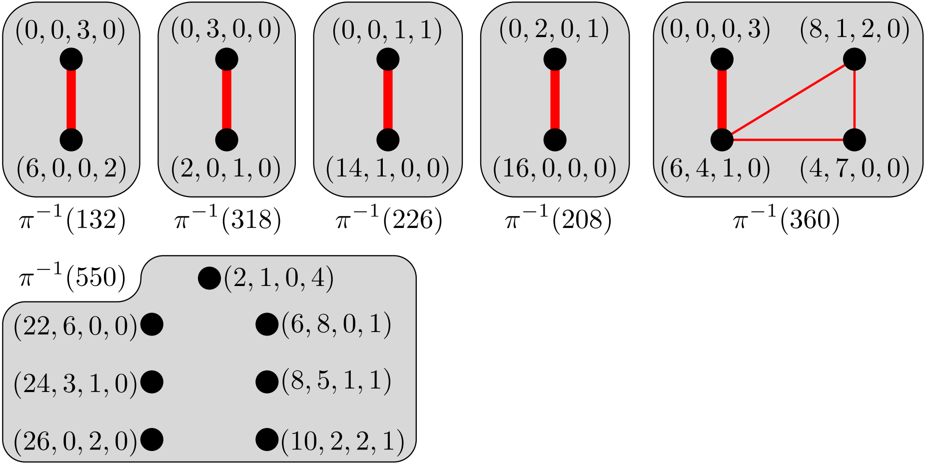

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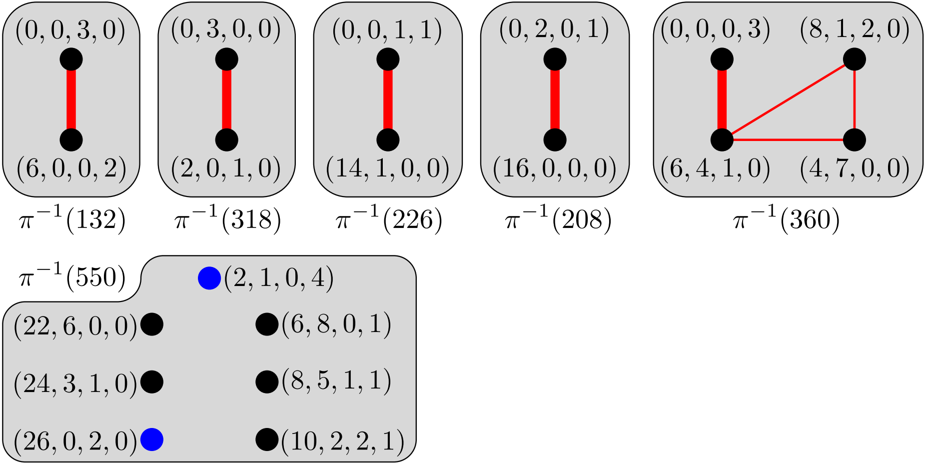

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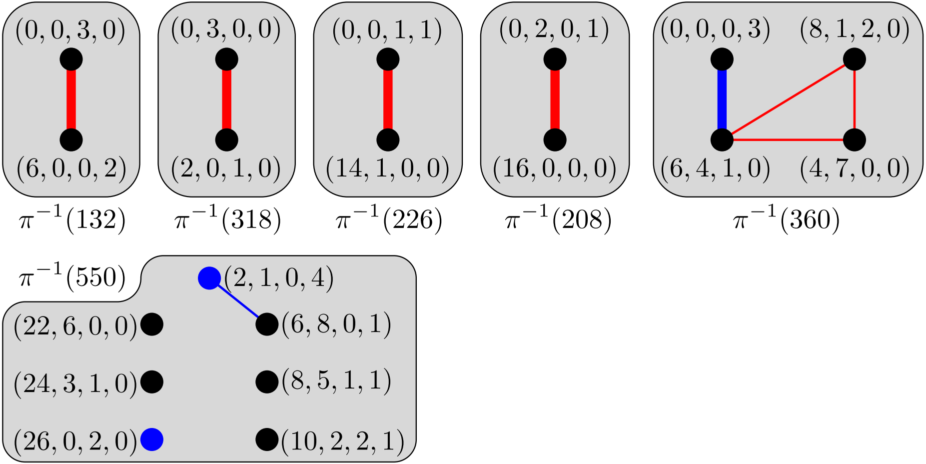

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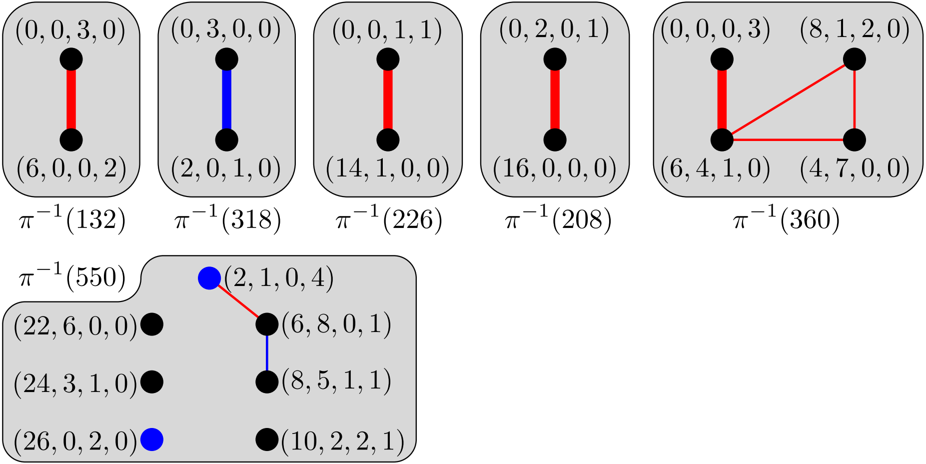

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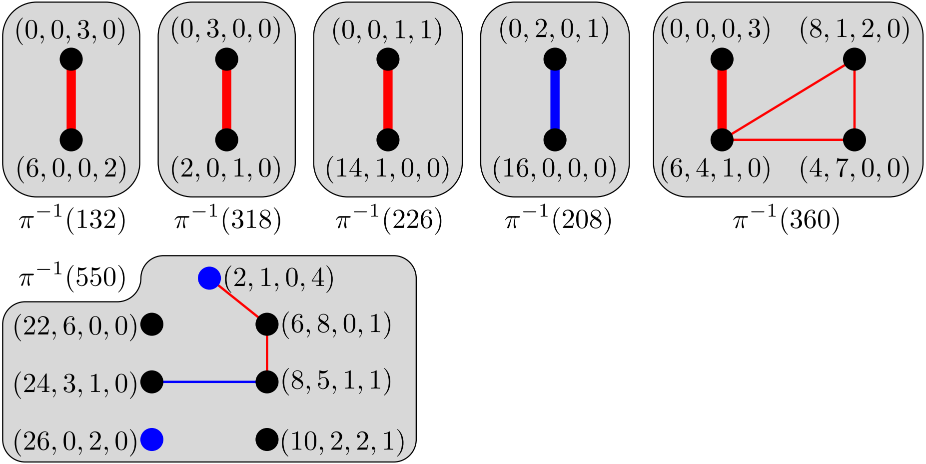

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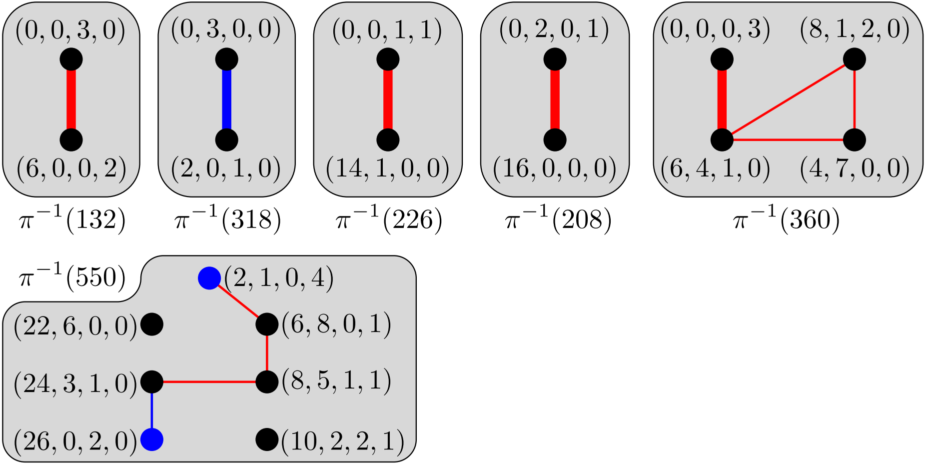

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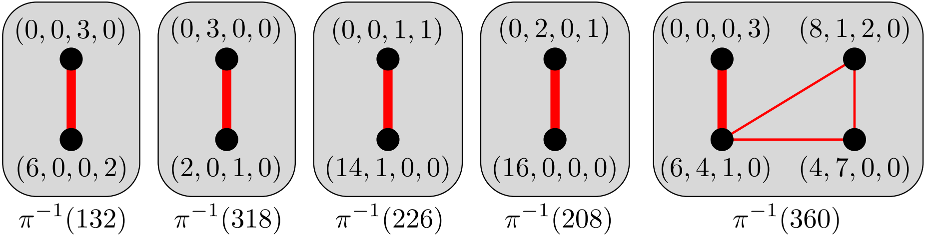

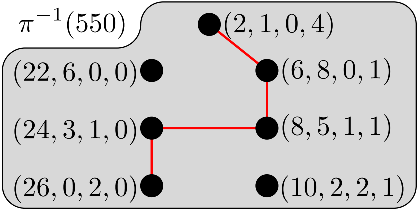

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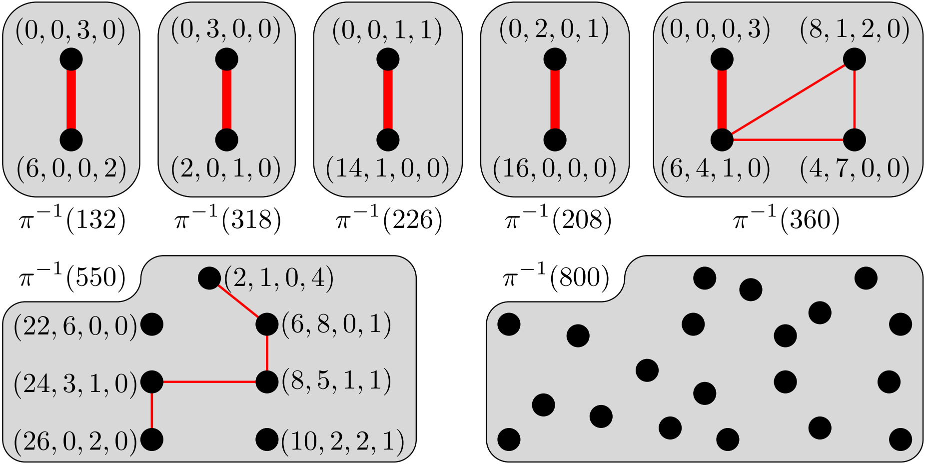

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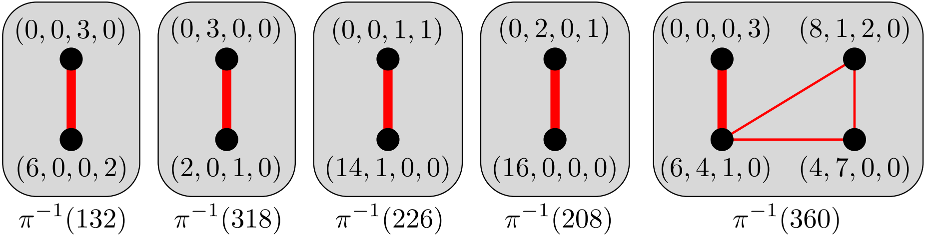

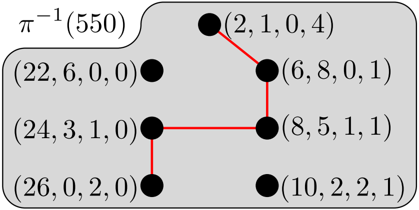

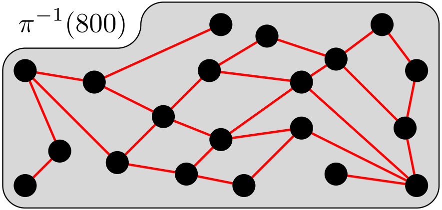

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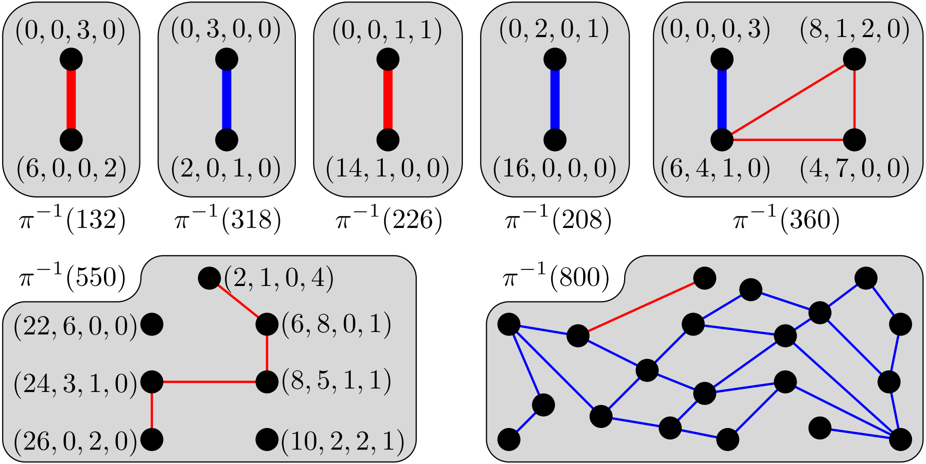

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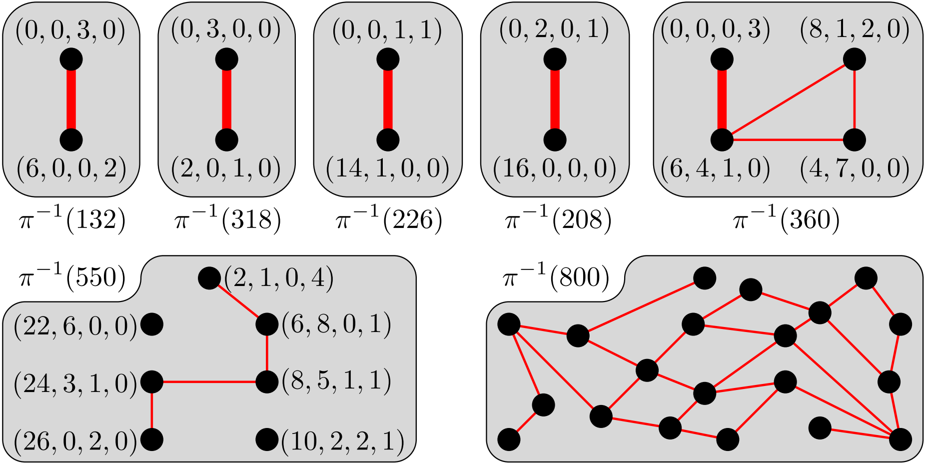

The delta set via commutative algebra

$$\begin{array}{rcl} \pi : \mathbb{Z}_{\geq 0}^k & \longrightarrow & S = \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array} \qquad \begin{array}{rcl} \varphi : \Bbbk[x_1, \ldots, x_k] & \longrightarrow & \Bbbk[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

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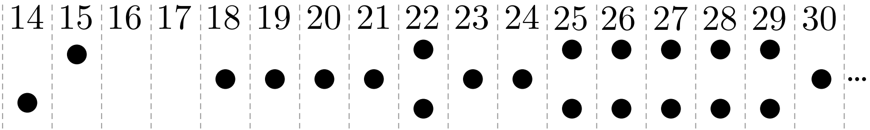
Z(244): connected components: 28



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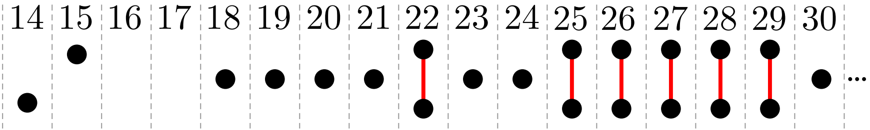
Z(244): connected components: 21



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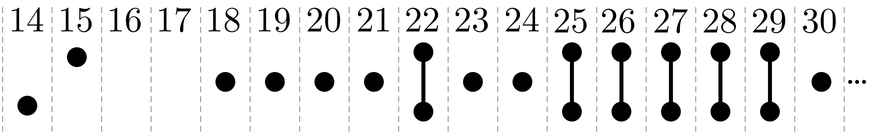
Z(244): connected components: 21



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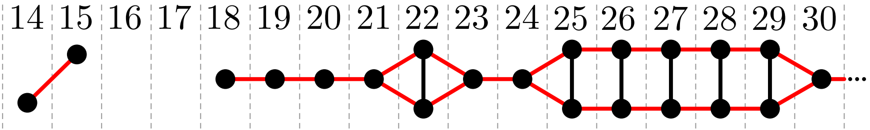
Example: $S\langle 6, 9, 20 \rangle$ $\quad I_S = \langle x^3 - y^2, x^4 y^4 - z^3 \rangle \subset \Bbbk[x, y, z]$

Z(244): connected components: 2



The delta set via commutative algebra

$$\begin{array}{rcl} \pi : \mathbb{Z}_{\geq 0}^k & \longrightarrow & S = \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array} \qquad \begin{array}{rcl} \varphi : \Bbbk[x_1, \ldots, x_k] & \longrightarrow & \Bbbk[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

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Idea: only connect *some* of the factorizations

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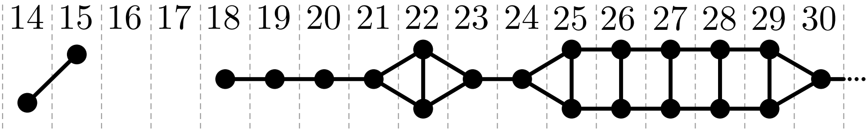

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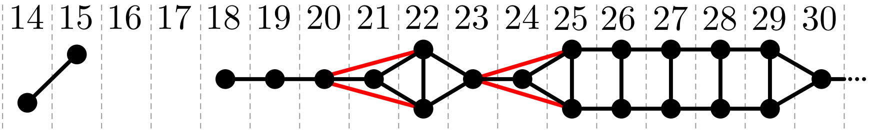

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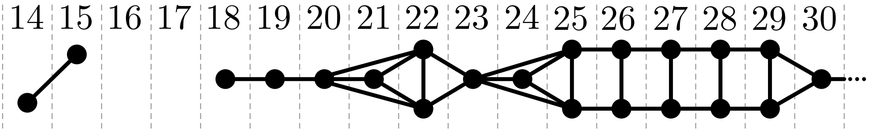

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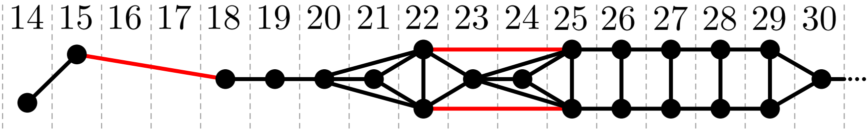

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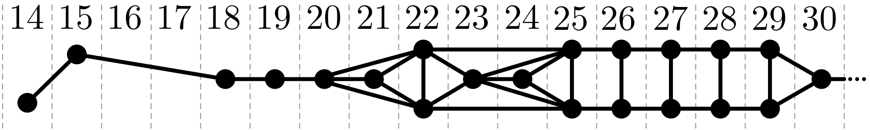

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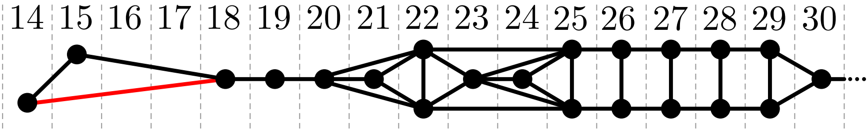

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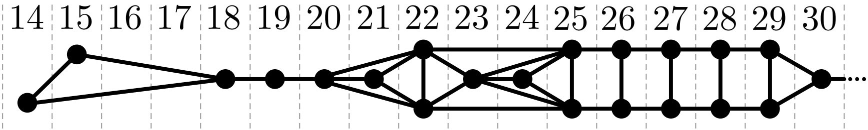

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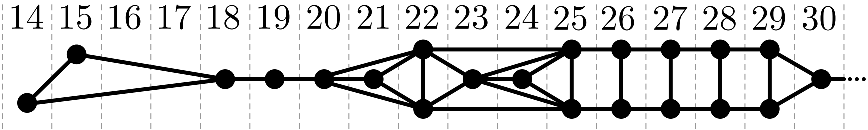

$I_0 = \langle x^{11} z^3 - y^{14} \rangle$

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$I_3 = I_2 + \langle x^2 z^3 - y^8 \rangle$

$I_4 = I_3 + \langle x^4 y^4 - z^3 \rangle$

$\quad = I_5 = I_6 = \cdots = I_S$

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Theorem (O, 2016)

In the ascending chain $I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_S$,

$$j \in \Delta(S) \quad \textit{if and only if} \quad I_{j-1} \subsetneq I_j$$

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Algorithm for computing $\Delta(S)$:

- Compute generators for $I_0, I_1, \ldots$
- At each step, check if $I_{j-1} \neq I_j$
- Stop when I_S reached

The delta set via commutative algebra

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I_{hom}: homogenization of I_S

$$x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S \qquad \longrightarrow \qquad x^{\mathbf{a}} - t^{|\mathbf{a}| - |\mathbf{b}|} x^{\mathbf{b}} \in I_{\text{hom}}$$

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Example: $S\langle 6, 9, 20 \rangle$ $\quad I_S = \langle x^3 - y^2, xy^6 - z^3 \rangle \subset \Bbbk[x, y, z]$

Lex Gröbner basis for I_{hom}:

$$\begin{aligned} I_{\text{hom}} = \langle & x^{11} z^3 - y^{14}, \\ & x^3 - ty^2, x^8 z^3 - ty^{12}, \\ & t^2 x^5 z^3 - y^{10}, \\ & t^3 x^2 z^3 - y^8, \\ & xy^6 - t^4 z^3 \rangle \end{aligned}$$

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Algorithm to compute $\Delta(S)$ (García-Sánchez–O–Webb, 2018)

- Homogenize the ideal I_S with a new variable t
- Compute a reduced lex Gröbner basis G with $t < x_i$
- $\Delta(S) = \{d : t^d x^{\mathbf{a}} - x^{\mathbf{b}} \in G\}$

The delta set via commutative algebra

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- $\Delta(S) = \{d : t^d x^{\mathbf{a}} - x^{\mathbf{b}} \in G\}$

S	$\Delta(S)$	Manual	Dynamic	Algebraic
$\langle 100, 121, 142, 163, 284 \rangle$	$\{21\}$	Days	0m 3.6s	< 10 ms
$\langle 1001, 1211, 1421, 1631, 2841 \rangle$	$\{10, 20, 30\}$	Days	1m 56s	< 10 ms

The delta set via commutative algebra

$$\begin{array}{rcl} \pi : \mathbb{Z}^k_{\geq 0} & \longrightarrow & S = \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array} \qquad \begin{array}{rcl} \varphi : \Bbbk[x_1, \ldots, x_k] & \longrightarrow & \Bbbk[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

$$I_S = \ker(\varphi) = \langle x^{\mathbf{a}} - x^{\mathbf{b}} : \pi(\mathbf{a}) = \pi(\mathbf{b}) \rangle$$

$$I_j = \langle x^{\mathbf{a}} - x^{\mathbf{b}} : \pi(\mathbf{a}) = \pi(\mathbf{b}) \text{ and } \big||\mathbf{a}| - |\mathbf{b}|\big| \leq j \rangle \subset I_S$$

Algorithm to compute $\Delta(S)$ (García-Sánchez–O–Webb, 2018)

- Homogenize the ideal I_S with a new variable t
- Compute a reduced lex Gröbner basis G with $t < x_i$
- $\Delta(S) = \{d : t^d x^{\mathbf{a}} - x^{\mathbf{b}} \in G\}$

S	$\Delta(S)$	Manual	Dynamic	Algebraic
$\langle 100, 121, 142, 163, 284 \rangle$	$\{21\}$	Days	0m 3.6s	< 10 ms
$\langle 1001, 1211, 1421, 1631, 2841 \rangle$	$\{10, 20, 30\}$	Days	1m 56s	< 10 ms
$\langle 550, 1060, 1600, 1781, 4126, 4139, 4407, 5167, 6073, 6079, 6169, 7097, 7602, 8782, 8872 \rangle$	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19\}$	Years	Days	< 1 min

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M. Delgado, P. García-Sánchez, J. Morais
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http://www.gap-system.org/Packages/numericalsgps.html.

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Thanks!