Augmented Hilbert series of numerical semigroups

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Definition

A *numerical semigroup* $S \subset \mathbb{N}$: **additive** submsemigroup, $|\mathbb{N} \setminus S| < \infty$.

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Maximum and minimum factorization length

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$,

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$$\begin{aligned} \mathsf{m}(82) &= 5 \quad \text{and} \quad \mathsf{Z}(82) = \{(0,3,2,0,0),(5,0,0,0,3),\ldots\} \\ \mathsf{m}(462) &= 25 \quad \text{and} \quad \mathsf{Z}(462) = \{(0,3,2,0,20),\ldots\} \end{aligned}$$

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Theorem (Barron–O.–Pelayo, 2014)

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n > n_k(n_{k-1} - 1)$,

$$\begin{aligned} \mathsf{M}(n + n_1) &= 1 + \mathsf{M}(n) \\ \mathsf{m}(n + n_k) &= 1 + \mathsf{m}(n) \end{aligned}$$

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Equivalently: $\mathsf{M}(n)$, $\mathsf{m}(n)$ eventually *quasilinear*

$$\begin{aligned} \mathsf{M}(n) &= \tfrac{1}{n_1} n + a_0(n) \\ \mathsf{m}(n) &= \tfrac{1}{n_k} n + b_0(n) \end{aligned}$$

for periodic functions $a_0(n)$, $b_0(n)$.

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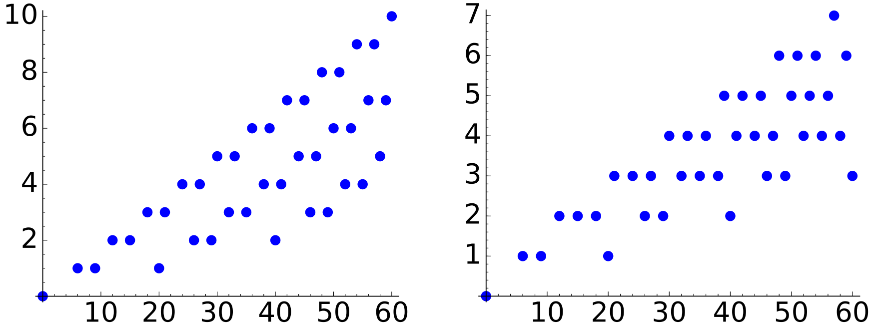

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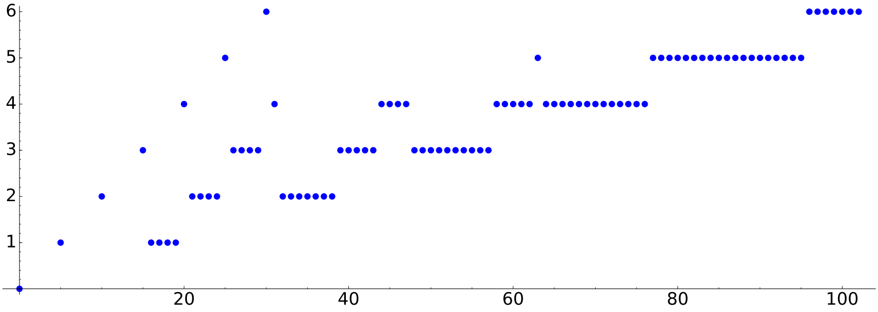

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$$\frac{1}{1-t} = \sum_{n \geq 0} t^n = 1 + t + t^2 + t^3 + \cdots$$

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- Take an integer sequence $a_0, a_1, a_2, \ldots$ you want to study
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$$F(t) = \sum_{n \ge 0} f_n t^n = 0 + t + \sum_{n \ge 2} (f_{n-1} + f_{n-2}) t^n$$

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Classic Example

Fibonacci numbers: $0, 1, 1, 2, 3, 5, 8, 13, \ldots$

$f_0 = 0, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 2$

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Key: $\dfrac{t^9}{1-t^6} = t^9(1 + t^6 + t^{12} + t^{18} + \cdots)$

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Let $S = \langle n_1, \ldots, n_k \rangle$.

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If $\mathrm{Ap}(S) = \{a_0, a_1, \ldots\}$, *then the Hilbert series of* S *can be written as*

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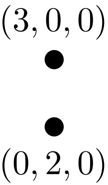

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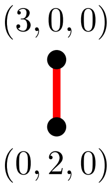

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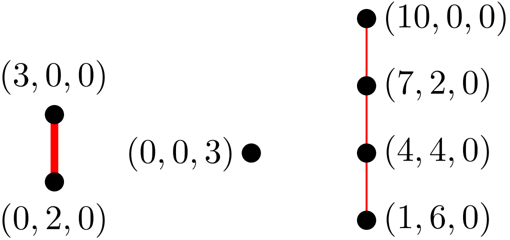

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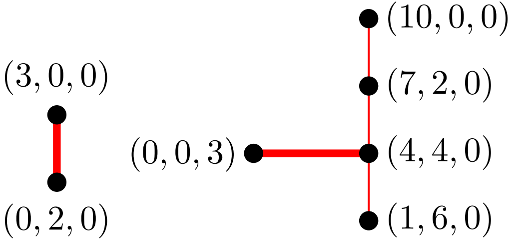

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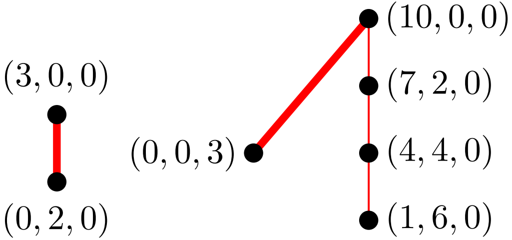

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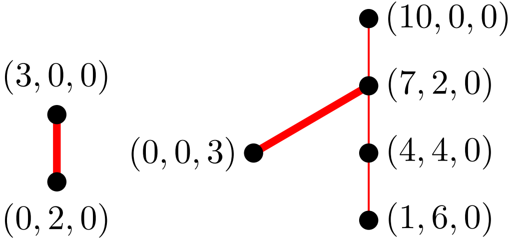

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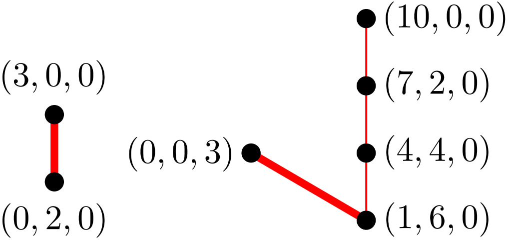

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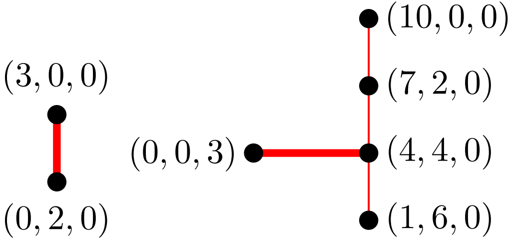

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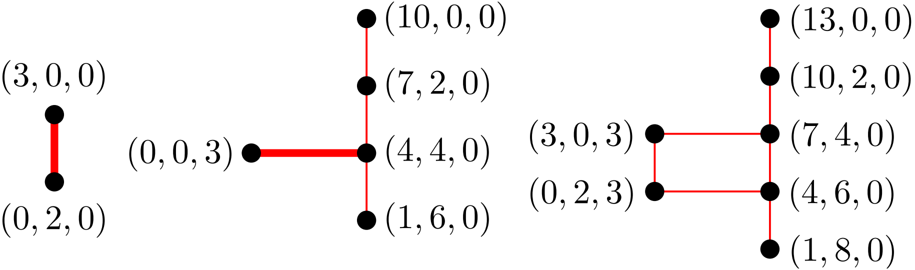

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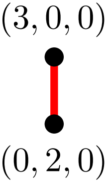

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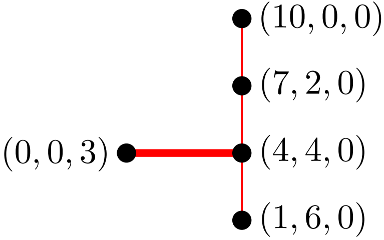

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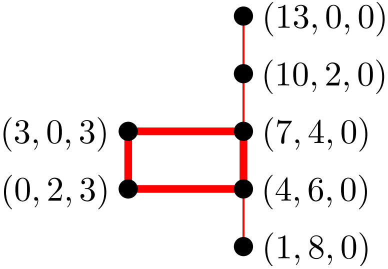

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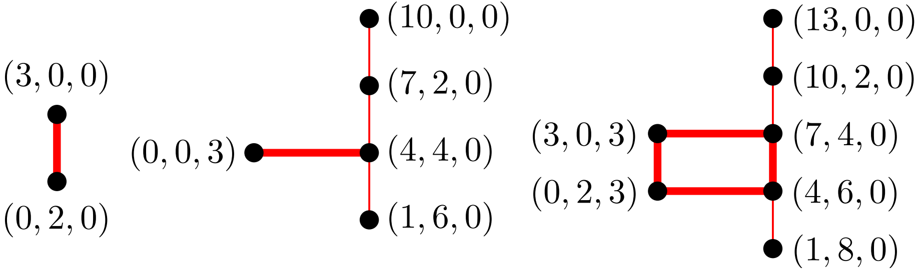

Disconnected $\longleftrightarrow$ minimal relation between generators

Hilbert series

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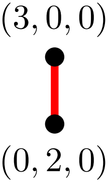

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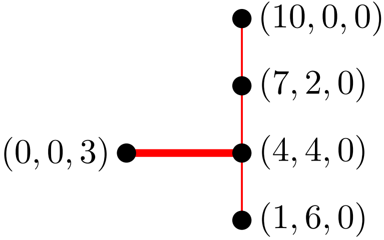

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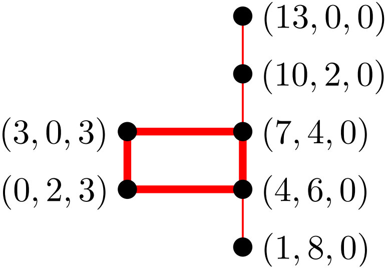

Disconnected $\longleftrightarrow$ minimal relation between generators

Cycles $\longleftrightarrow$ relations between minimal relations

The “Big Theorem”

The Big Theorem (Bruns, Herzog)

For any numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$,

$$\mathcal{H}(S; t) = \frac{\sum_{n \in S} \chi(\Delta_n) t^n}{(1 - t^{n_1}) \cdots (1 - t^{n_k})}$$

Δ_n is the *squarefree divisor complex*: simplicial complex on $\{n_1, \ldots, n_k\}$,

$$F \in \Delta_n \quad \text{if} \quad n - \sum F \in S$$

Euler characteristic: $\chi(\Delta_n) = 1 - \#\text{vertices} + \#\text{edges} - \cdots$

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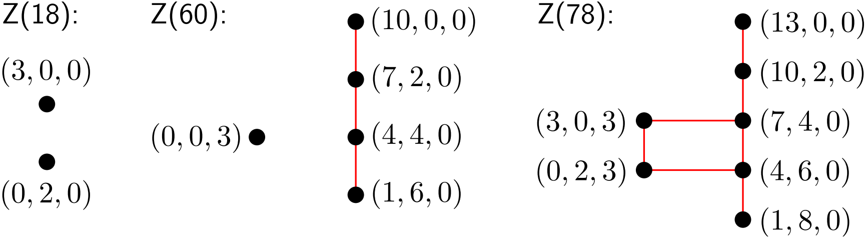
Δ_n is a **contraction** of complex from previous slide:



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For any numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$,

$$\mathcal{H}(S; t) = \frac{f(t)}{1 - t}$$

for some polynomial $f(t)$,

$$\mathcal{H}(S; t) = \frac{t^{a_0} + t^{a_1} + \cdots}{1 - t^{n_1}}$$

where $\mathrm{Ap}(S) = \{a_0, a_1, \ldots\}$ is the *Apéry set* of S, and

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$\longrightarrow$ Algebraic manipulation reveals deep structures! $\longleftarrow$

Augmented Hilbert series

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n \gg 0$, max factorization length $\mathsf{M}(n)$ satisfies

$$\mathsf{M}(n) = \tfrac{1}{n_1} n + a_0(n)$$

with $a_0(n)$ n_1-periodic ($\mathsf{M}(n)$ is eventually quasilinear).

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The *augmented Hilbert series* of S is

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Example: $S = \langle 6, 9, 20 \rangle$

$$\begin{aligned}\mathcal{H}_{\mathsf{M}}(S;t) &= 1 + t^6 + t^9 + 2t^{12} + 2t^{15} + 3t^{18} + t^{20} + \cdots \\ &= \frac{t^6 + t^9 + t^{20} + 2t^{29} - t^{35} + 2t^{40} - t^{46} + 3t^{49} - 2t^{55}}{(1 - t^6)^2}\end{aligned}$$

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Theorem (Glenn–O.–Ponomarenko–Sepanski)

For any numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$,

$$\begin{aligned} \mathcal{H}_{\mathrm{M}}(S;t) &= \frac{g(t)}{(1 - t^{n_1}) \cdots (1 - t^{n_k})} \\ &= \frac{h(t)}{(1 - t^{n_1}) \cdots (1 - t^{n_k})} + \mathcal{H}(S;t) \sum_{i=1}^{k} \frac{t^{n_i}}{1 - t^{n_i}} \end{aligned}$$

where the coefficients of $g(t)$ and $h(t)$ are *weighted* Euler characteristics.

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$S = \langle 9, 10, 23 \rangle$:

$$\mathcal{H}(S;t) = \frac{1 - t^{46} - t^{50} - t^{63} + t^{73} + t^{86}}{(1-t^9)(1-t^{10})(1-t^{23})}$$

$$g(t) = t^9 + t^{10} + t^{18} + t^{20} + t^{23} + t^{27} + t^{30} + t^{36} + t^{40} + t^{45} - t^{46} - 3t^{50} + t^{54} - t^{55} - t^{56} - t^{59} - 4t^{63} - t^{64} - t^{66} - t^{68} + 2t^{73} - t^{76} - t^{77} + 3t^{86} - t^{90} + t^{113}$$

$$h(t) = -2t^{46} - 4t^{50} - 5t^{63} + 5t^{73} + 6t^{86} - t^{90} + t^{113}$$

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$S = \langle 11, 18, 24 \rangle$:

$$\mathcal{H}(S;t) = \frac{1 - t^{66} - t^{72} + t^{138}}{(1-t^{11})(1-t^{18})(1-t^{24})}$$

$$\begin{aligned} g(t) = {} & t^{11} + t^{18} + t^{22} + t^{24} + t^{33} + t^{36} + t^{44} + t^{48} + \\ & t^{54} + t^{55} - 2t^{66} - t^{72} - t^{83} - t^{84} - 2t^{90} - t^{94} - \\ & t^{102} - t^{105} - t^{114} - t^{116} - t^{120} - t^{127} + 4t^{138} \end{aligned}$$

$$h(t) = -3t^{66} - 3t^{72} - t^{90} + 7t^{138}$$

References

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