Numerical semigroup invariants and eventually quasipolynomial behavior

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Joint with Thomas Barron* and Roberto Pelayo

Joint with Rebecca Conaway*, Felix Gotti, Jesse Horton*, Roberto Pelayo, Mesa Williams*, and Brian Wissman

Joint with Stephan Ramon Garcia and Samuel Yih*

. . . plus several more REU groups

* = undergraduate student

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Numerical semigroups

Definition

A *numerical semigroup* $S \subset \mathbb{Z}_{\geq 0}$: closed under **addition**.

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Factorizations:

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$$\begin{array}{rclcc} 60 & = & 7(6) + 2(9) & \rightsquigarrow & (7, 2, 0) \\ & = & 3(20) & \rightsquigarrow & (0, 0, 3) \end{array}$$

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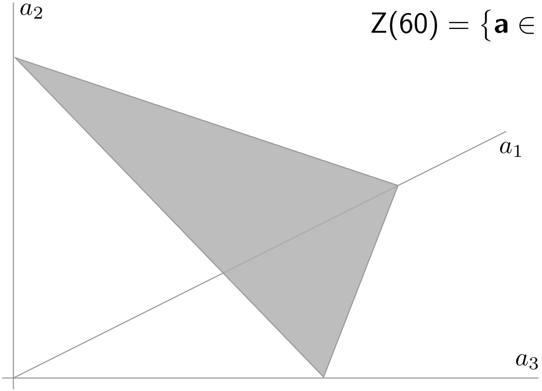

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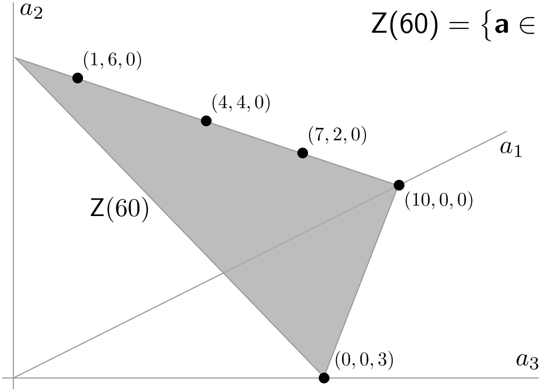

Maximum and minimum factorization length

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$,

$$\mathsf{L}(n) = \{a_1 + \cdots + a_k : (a_1, \ldots, a_k) \in \mathsf{Z}(n)\}$$

denotes the *length set* of n, and

$$\mathsf{M}(n) = \max \mathsf{L}(n) \qquad \text{and}$$
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Theorem (Barron–O–Pelayo, 2014)

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n > n_k(n_{k-1} - 1)$,

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for periodic functions $a_0(n)$, $b_0(n)$.

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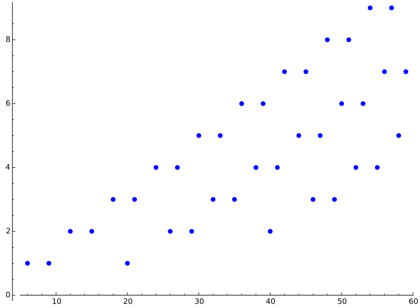
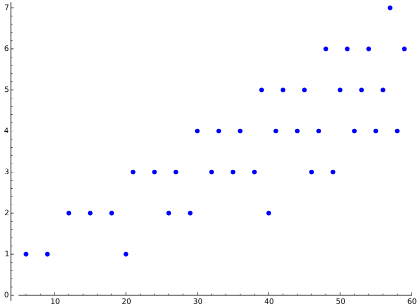


$\mathsf{M}(n) : S \to \mathbb{N}$



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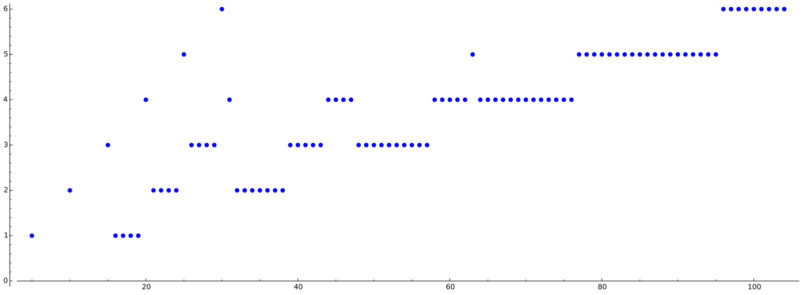

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The delta set of $n \in S$ is $\Delta(n) = \{\ell_2 - \ell_1, \ \ldots, \ \ell_r - \ell_{r-1}\}$

Example

$S = \langle 6, 9, 20 \rangle$: $\mathsf{Z}(60) = \{(10,0,0), (7,2,0), (4,4,0), (1,6,0), (0,0,3)\}$

$\mathsf{L}(60) = \{3, 7, 8, 9, 10\}$ $\Delta(60) = \{1, 4\}$

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Delta set (a geometric viewpoint)

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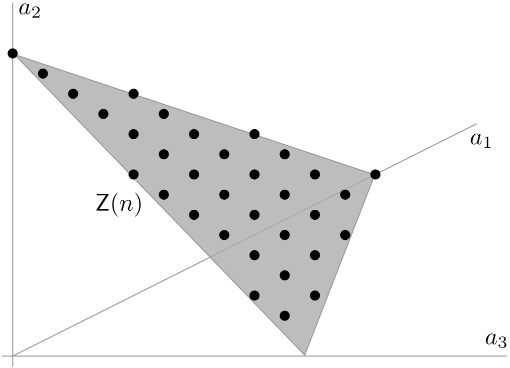

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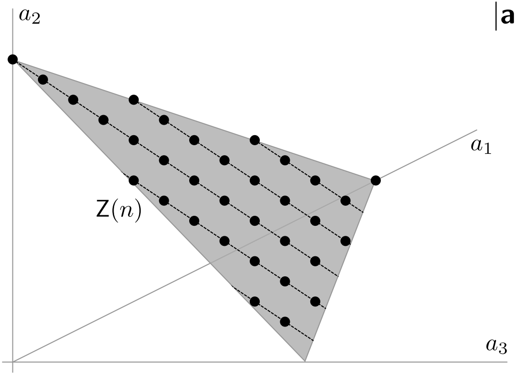

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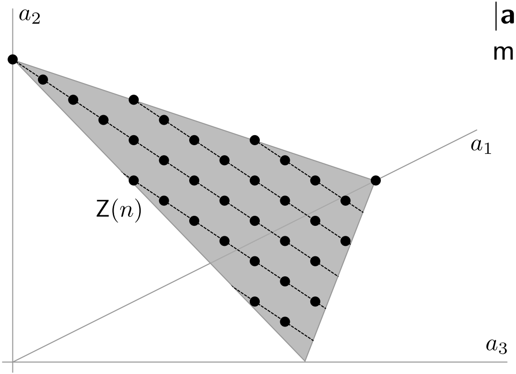

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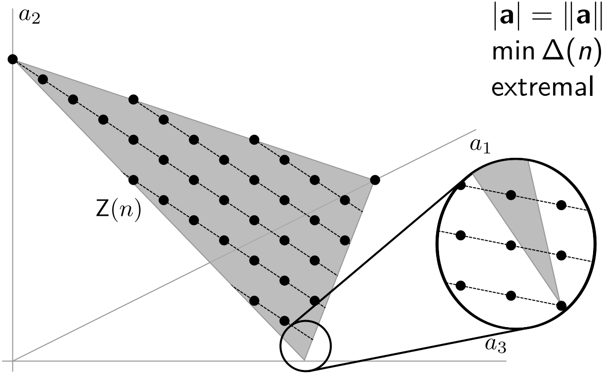

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extremal lengths near vertices

Eventual behavior?

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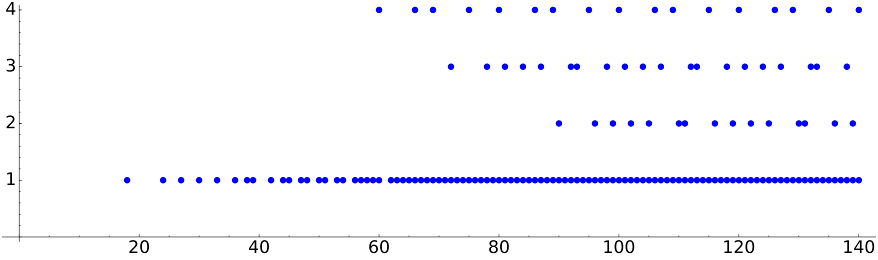
Example: $S = \langle 6, 9, 20 \rangle$: $2kn_2n_k^2 = 21600$



$(n, d) \quad \leftrightsquigarrow \quad d \in \Delta(n)$

Plenty more where that came from!

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Interpretation:

Factorizations are chaotic for small semigroup elements,
but stabalize for large semigroup elements

To shift a numerical semigroup...

Fix $r_1 < \cdots < r_k \in \mathbb{Z}_{\geq 1}$, and consider the *parametrized family*

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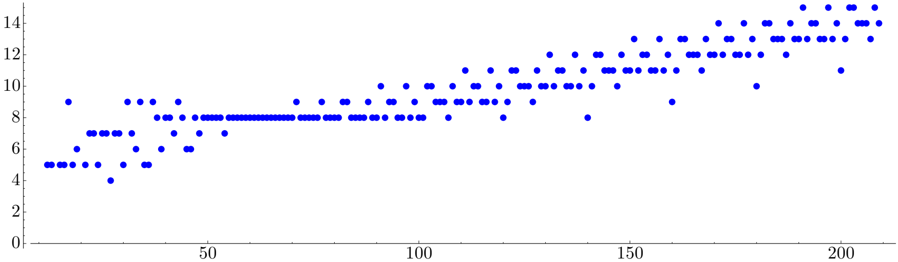

To shift a numerical semigroup. . .

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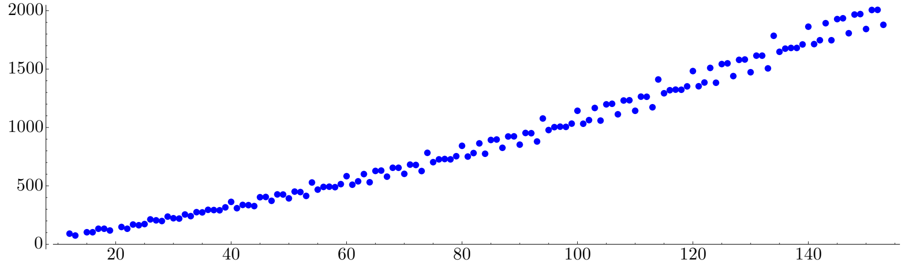

So. . . what *isn't* eventually quasipolynomial?

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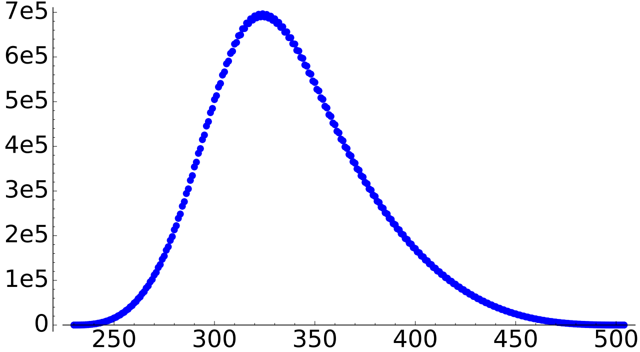
Example: $S = \langle 4, 5, 6, 7 \rangle$, $n = 2520$.



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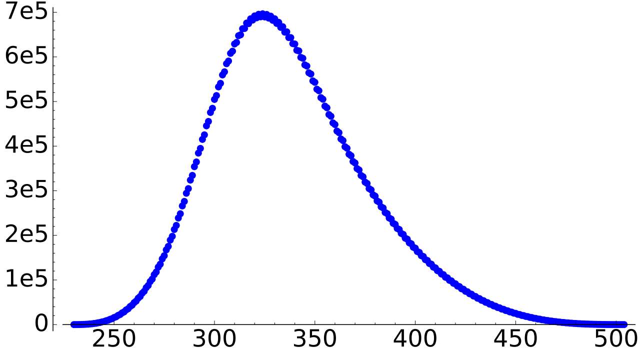

Mean length? Median length? Mode length?

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Theorem (García–O–Yih)

For $k = 3$, the mean length is eventually quasirational for $n \gg 0$:

$$\mu(n) = \frac{\sum_{\mathbf{a} \in \mathsf{Z}(n)} |\mathbf{a}|}{|\mathsf{Z}(n)|},$$

and

$$\lim_{n \to \infty} \frac{\mu(n)}{n} = \frac{1}{3}\left(\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}\right).$$

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Theorem (García–O–Yih)

For $k = 3$, the median length $\eta(n)$ is asymptotically linear:

$$\lim_{n \to \infty} \frac{\eta(n)}{n} = \text{a very complicated formula}$$

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$$\mathsf{Z}(n) = \{\mathbf{a} \in \mathbb{Z}^k_{\geq 0} : n = a_1 n_1 + \cdots + a_k n_k\}$$
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Define the *length multiset* $\mathsf{L}[\![n]\!] = \{\!\{a_1 + \cdots + a_k : \mathbf{a} \in \mathsf{Z}(n)\}\!\}$.

Theorem (García–O–Yih)

For $k = 3$, the median length $\eta(n)$ is asymptotically linear:

$$\lim_{n \to \infty} \frac{\eta(n)}{n} = \text{a very complicated formula}$$

The point: the very complicated formula is *sometimes* irrational.

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For $S = \langle 12, 15, 20 \rangle$, $\lim_{n \to \infty} \eta(n)/n = \frac{1}{15}$.

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