Enumerating numerical semigroups using polyhedral geometry

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Joint with Winfried Bruns, Pedro García Sánchez, and Dane Wilburne

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$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{c} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

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Embedding dimension: $\mathsf{e}(S) = \#$ minimal generators
Multiplicity: $\mathsf{m}(S) =$ smallest nonzero element

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Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$.

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- If $S = \langle n_1, n_2, n_3 \rangle$, then there is a fast algorithm for $F(S)$.
- Formulas in a few other special cases.

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For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
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- $|\mathrm{Ap}(S)| = m$

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The Apéry set is a "one stop shop" for computation.

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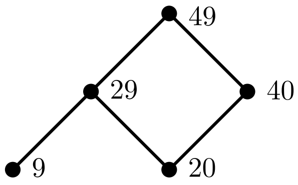

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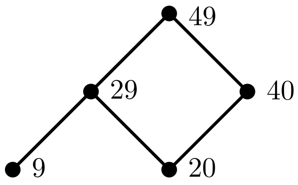

$$\mathrm{e}(S) = \#\text{ min elements} + 1$$

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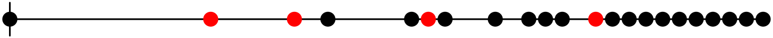

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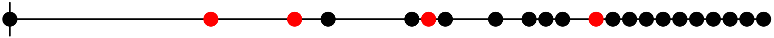

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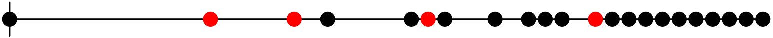

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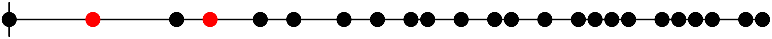

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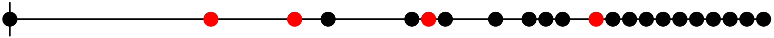

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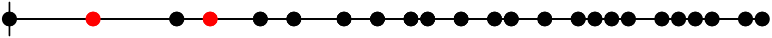

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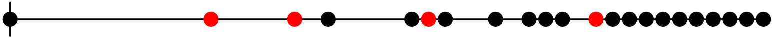

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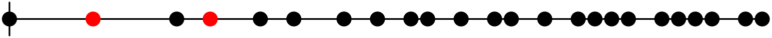

- $S = \langle m, m + 1, \ldots, 2m - 1 \rangle$ (*maximal embedding dimension*)

Proved in many special cases, including $\mathrm{g}(S) \le 60$.

Polyhedral geometry enters the picture

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Theorem (Kunz)

A point $(x_1, \ldots, x_{m-1}) \in \mathbb{Z}^{m-1}$ is the Kunz coordinates of a numerical semigroup if and only if for $1 \leq i, j \leq m-1$,

$$\begin{array}{rcl} x_i \geq 1 & & \\ x_i + x_j \geq x_{i+j} & \textit{for} & i + j < m \\ 1 + x_i + x_j \geq x_{i+j-m} & \textit{for} & i + j > m \end{array}$$

Polyhedral geometry enters the picture

Fix a numerical semigroup S with $\mathrm{m}(S) = m$, and write

$$\mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} \quad \text{with} \quad a_i = mx_i + i.$$

The *Kunz coordinates* of S: $(x_1, \ldots, x_{m-1}) \in \mathbb{Z}_{\geq 1}$.

Key observation: $a_i + a_j \geq a_{i+j}$

Theorem (Kunz)

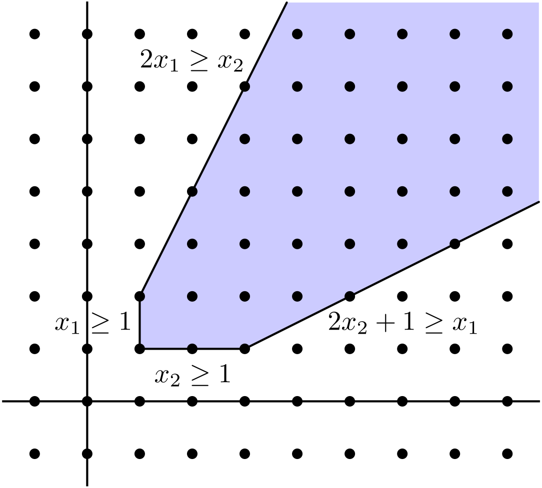
A point $(x_1, \ldots, x_{m-1}) \in \mathbb{Z}^{m-1}$ is the Kunz coordinates of a numerical semigroup if and only if for $1 \leq i, j \leq m-1$,

$$\begin{aligned} x_i &\geq 1 & & \\ x_i + x_j &\geq x_{i+j} & \textit{for} \quad & i + j < m \\ 1 + x_i + x_j &\geq x_{i+j-m} & \textit{for} \quad & i + j > m \end{aligned}$$

Numerical semigroups $\longleftrightarrow$ integer points in rational polyhedra!

Kunz polyhedra

Kunz polyhedron $P_3 \subset \mathbb{R}^2$



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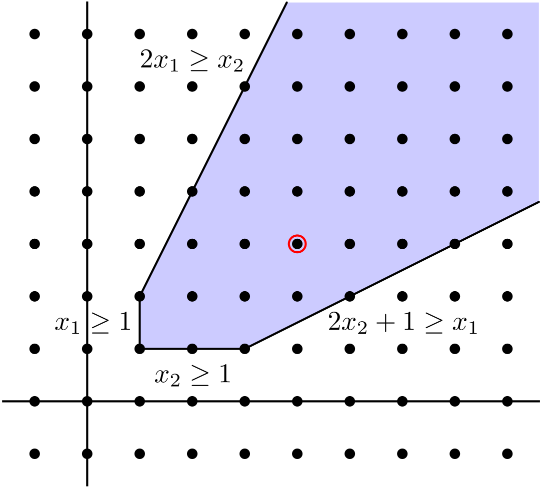

Example: $(4,3) \in P_3$

$\mathrm{Ap}(S) = \{0, 3 \cdot 4 + 1, 3 \cdot 3 + 2\}$
$\Rightarrow S = \langle 3, 11, 13 \rangle$

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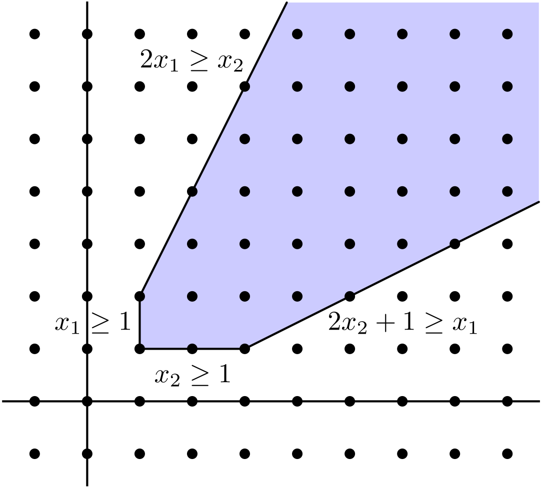

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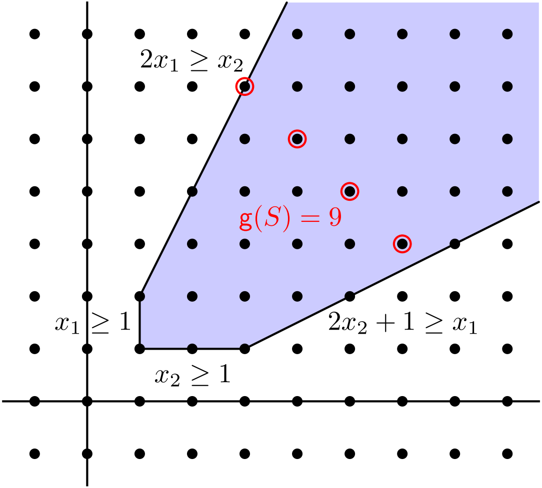

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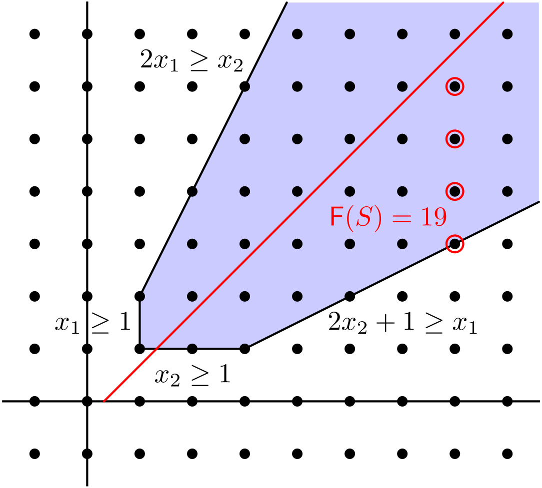

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Observations:

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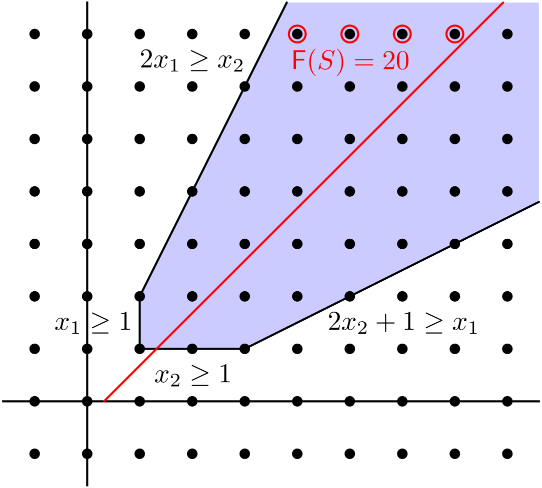
- $\mathsf{F}(S) = \max\{m x_i + i\} - m$

Kunz polyhedra

Kunz polyhedron $P_3 \subset \mathbb{R}^2$



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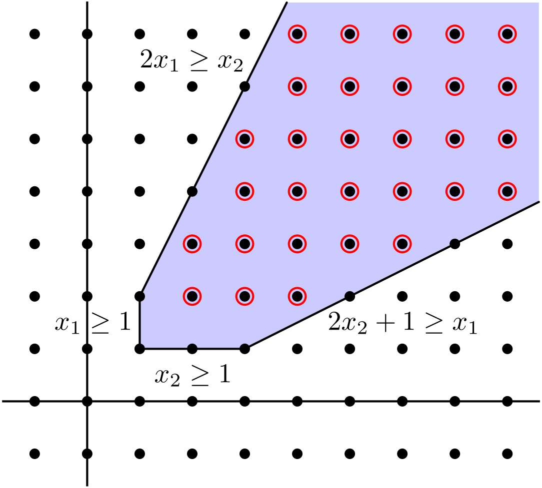

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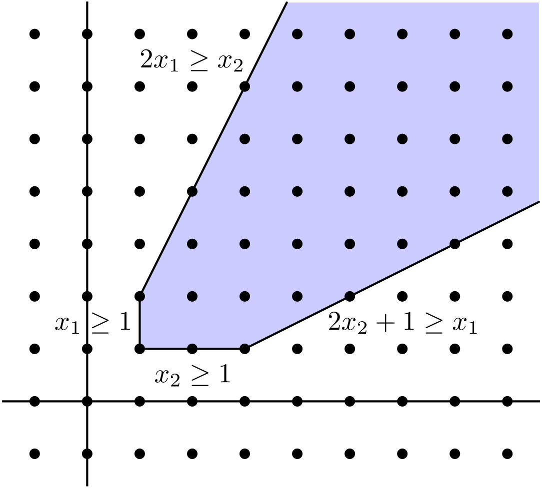

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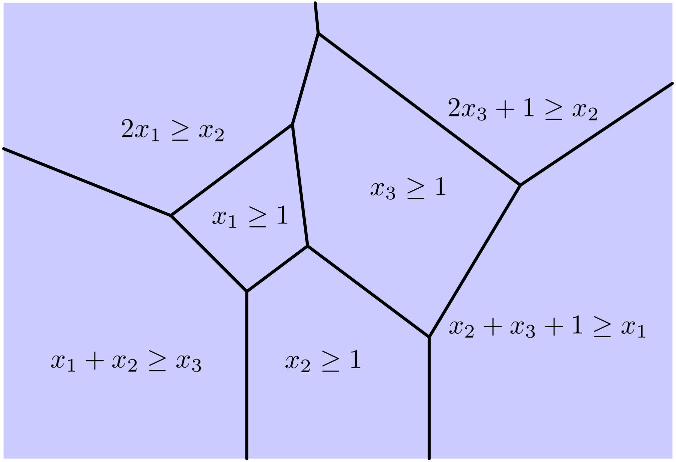
Kunz polyhedra

Kunz Polyhedron $P_4 \subset \mathbb{R}^3$ (boundary only)



Faces of the Kunz polyhedron

Question

When are 2 numerical semigroups in the relative interior of the same face?

Faces of the Kunz polyhedron

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$$S = \langle 6, 9, 20 \rangle$$
$$\mathrm{Ap}(S) = \{0, 49, 20, \;\; 9, 40, 29\}$$

$$S = \langle 6, 26, 27 \rangle$$
$$\mathrm{Ap}(S) = \{0, 79, 26, 27, 52, 53\}$$

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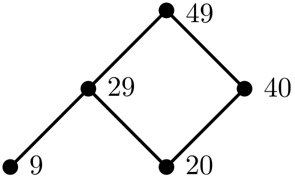

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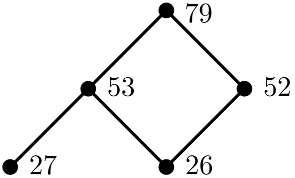

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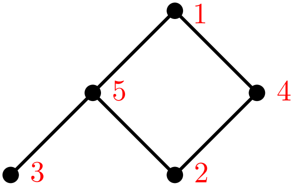

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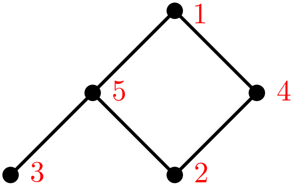

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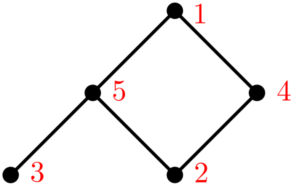

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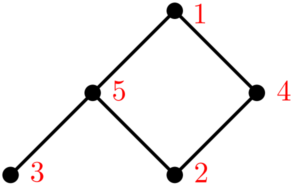

The *Kunz poset* of S: use ground set $\mathbb{Z}_m \setminus \{0\}$ instead of $\mathrm{Ap}(S) \setminus \{0\}$.

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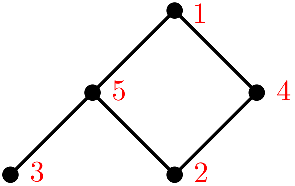

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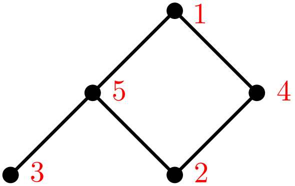

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Theorem (Bruns, García-Sánchez, O., Wilburne)

Two numerical semigroups lie in the relative interior of the same face of P_m if and only if their Kunz posets are identical.

Faces of the Kunz polyhedron

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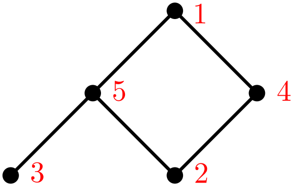

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Theorem (Bruns, García-Sánchez, O., Wilburne)

If two numerical semigroups lie in the relative interior of the same face of P_m, then their Kunz posets are identical.

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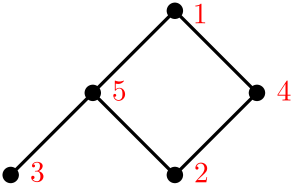

Defining facet equations:

$$2x_2 = x_4$$
$$x_2 + x_3 = x_5$$
$$x_2 + x_5 = x_1 - 1$$
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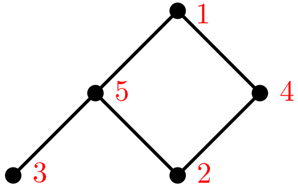

Defining facet equations:

$$\begin{aligned}
2x_2 &= x_4 && 2 \preceq 4 \\
x_2 + x_3 &= x_5 && 2 \preceq 5 \\
& && 3 \preceq 5 \\
x_2 + x_5 &= x_1 - 1 && 2 \preceq 1 \\
& && 5 \preceq 1 \\
x_3 + x_4 &= x_1 - 1 && 3 \preceq 1 \\
& && 4 \preceq 1
\end{aligned}$$

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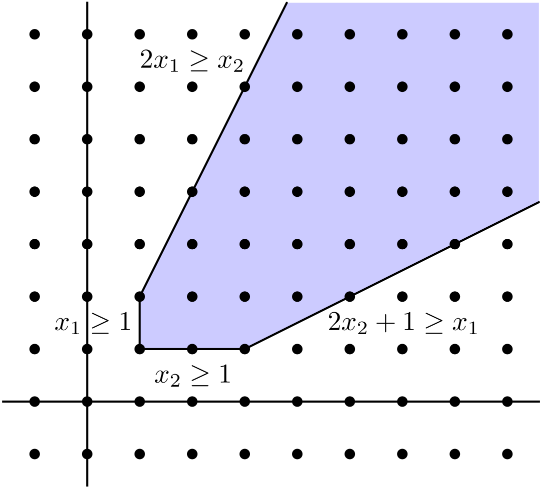

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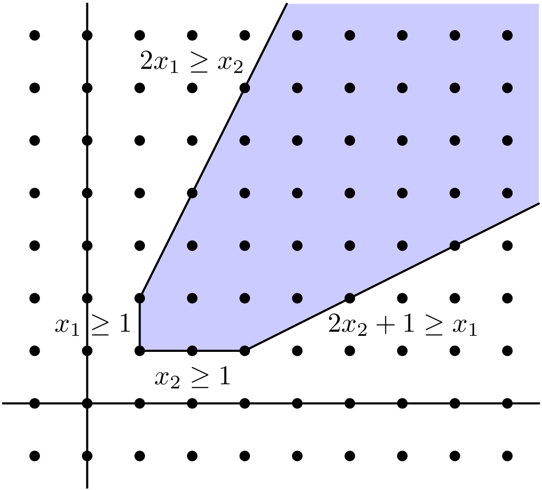
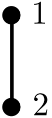


$S = \langle 3, 7 \rangle$



$S = \langle 3, 8 \rangle$

$S = \langle 3, 5, 7 \rangle$

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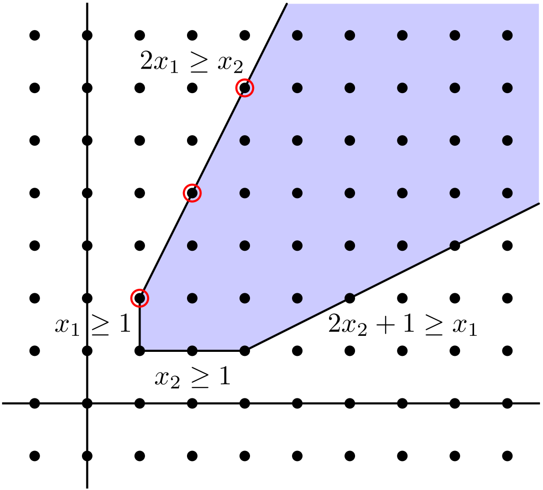

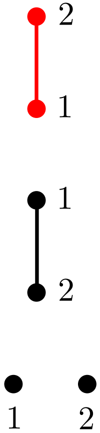

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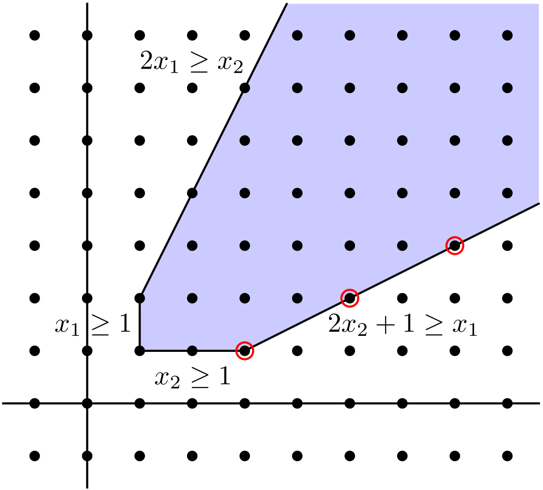

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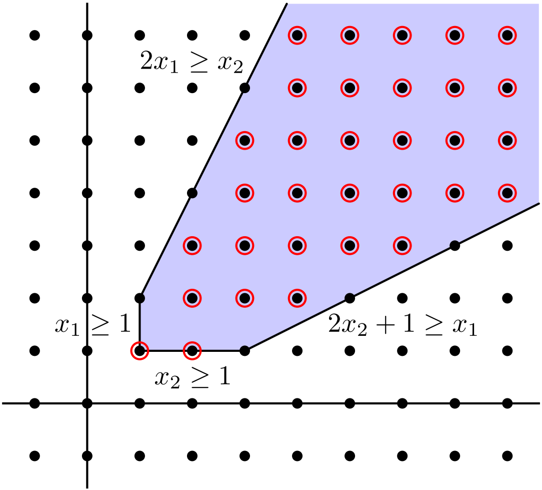

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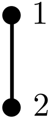

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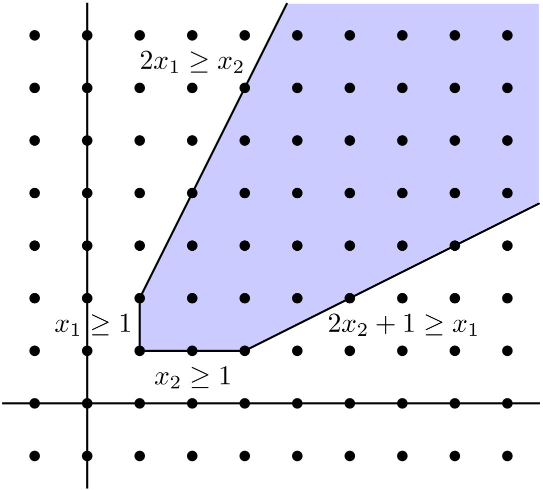

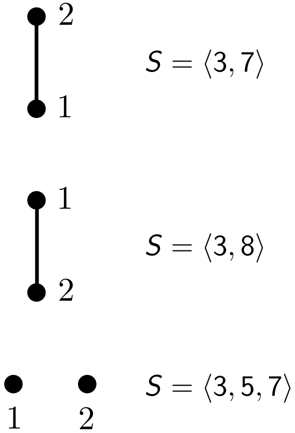

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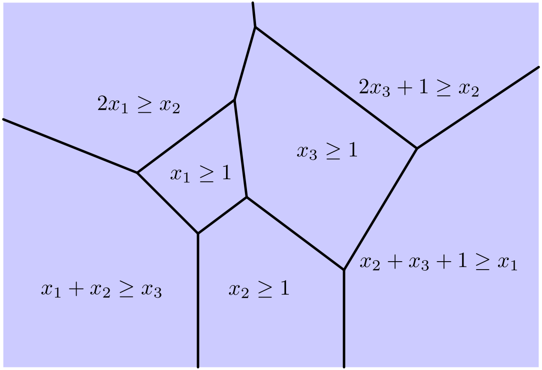

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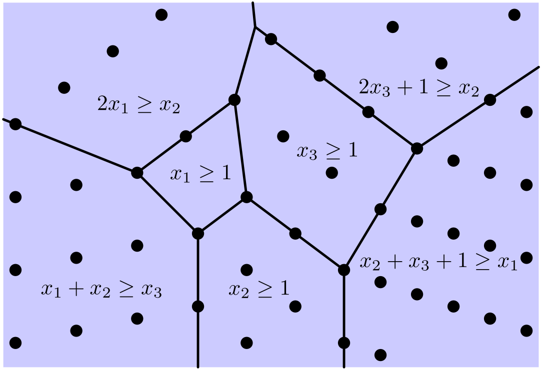

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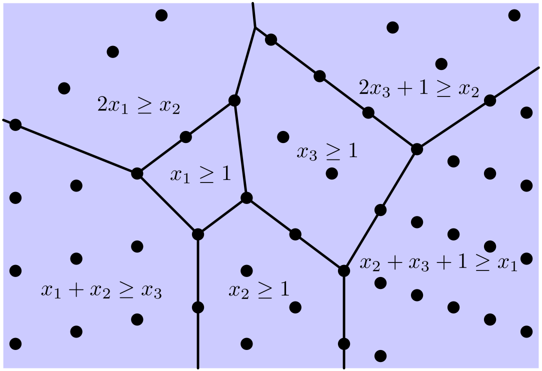

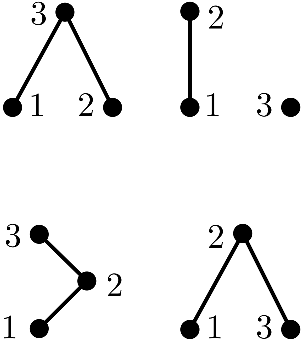

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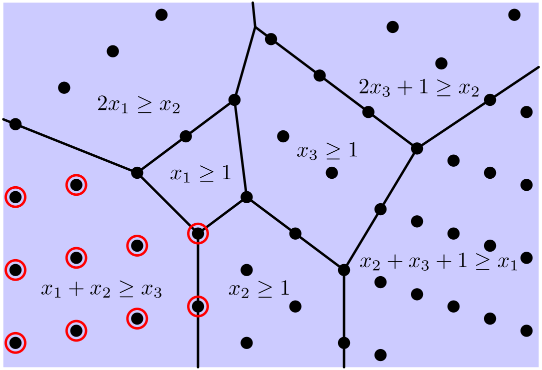

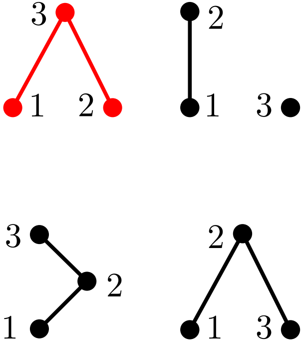

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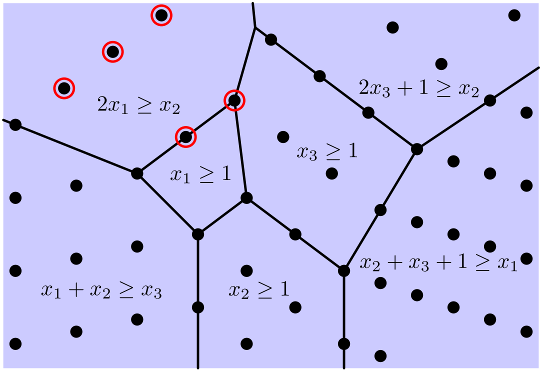

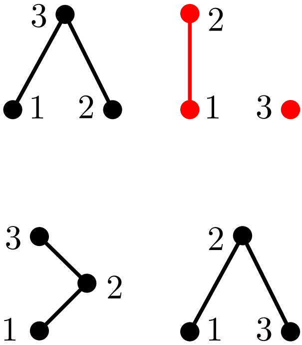

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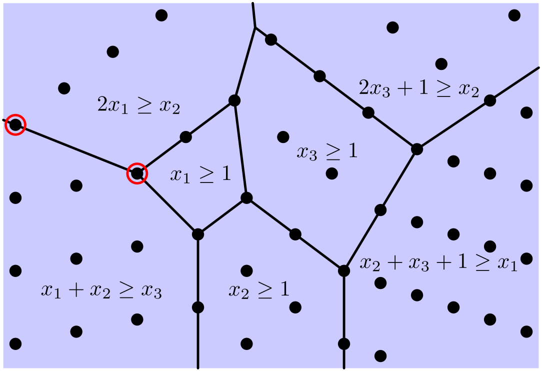

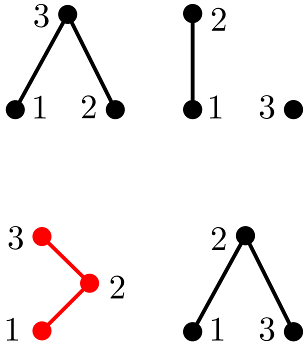

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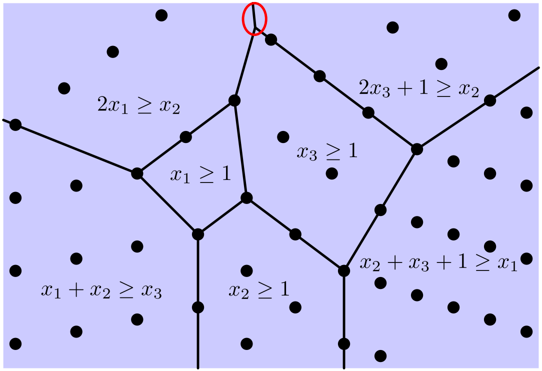

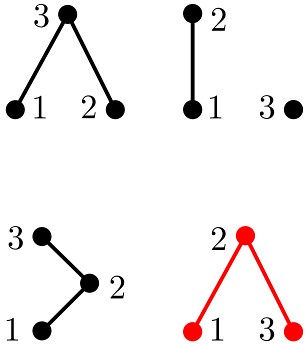

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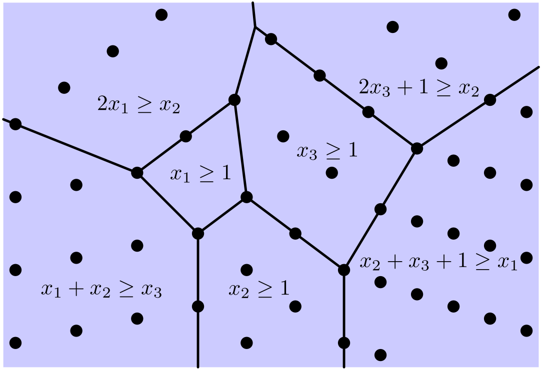

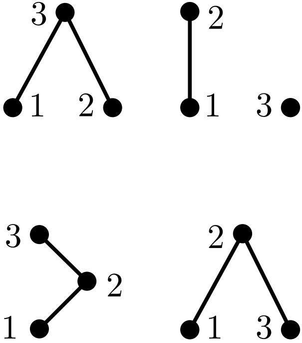

Verifying Wilf's conjecture in fixed multiplicity

Wilf's Conjecture

For any numerical semigroup S, $\mathsf{F}(S) + 1 \le \mathsf{e}(S)(\mathsf{F}(S) + 1 - \mathsf{g}(S))$.

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Example: $S = \langle 6, 9, 20 \rangle$ is a counterexample to Wilf's conjecture iff

$$\begin{array}{rrrr}
2x_2 = x_4 & 2x_1 > x_2 & 2x_4 + 1 > x_2 & x_1 - x_2 \ge 1 \\
x_2 + x_3 = x_5 & x_1 + x_2 > x_3 & 2x_5 + 1 > x_4 & x_1 - x_3 \ge 1 \\
x_2 + x_5 = x_1 - 1 & x_1 + x_3 > x_4 & x_3 + x_5 + 1 > x_2 & x_1 - x_4 \ge 1 \\
x_3 + x_4 = x_1 - 1 & x_1 + x_4 > x_5 & x_4 + x_5 + 1 > x_3 & x_1 - x_5 \ge 1
\end{array}$$

$$-11x_1 + 3x_2 + 3x_3 + 3x_4 + 3x_5 > -7$$

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Algorithm for checking Wilf's conjecture in multiplicity m:

- For each face $F \subset P_m$ and each $f \in [1, m-1]$, search region
  - defining equalities for F,
  - remaining inequalities for P_m (strict),
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  - negation of Wilf's inequality

  for positive integer points.
- Any integer points found are counterexamples to Wilf's conjecture.

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Theorem

Wilf's conjecture holds for all numerical semigroups S with $\mathrm{m}(S) \leq 18$.

Runtimes

m	# ineqs	# extremal rays	faces	total time	≈ RAM
11	50	812	155,944	0.7 s	6 MB
12	60	1,864	669,794	2.5 s	35 MB
13	72	7,005	4,389,234	23 s	80 MB
14	84	15,585	21,038,016	1:19 m	603 MB
15	98	67,262	137,672,474	19:43 m	2.6 GB
16	112	184,025	751,497,188	1:35 h	12 GB
17	128	851,890	5,342,388,604	38:46 h	48 GB
18	144	2,158,379	28,275,375,292	29:05 d	720 GB
19	162	11,665,781	??	??	??

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