Numerical semigroups: a sales pitch

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Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$ and an element $n \in S$.

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$$\begin{aligned} \mathsf{m}(82) = 5 \quad &\text{with} \quad 82 = 3(16) + 2(17) \\ \mathsf{m}(462) = 25 \quad &\text{with} \quad 462 = 3(16) + 2(17) + 20(19) \end{aligned}$$

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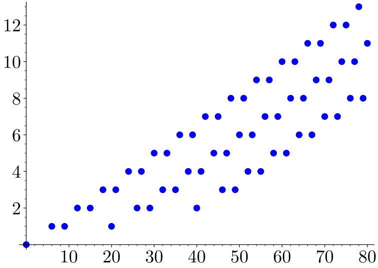

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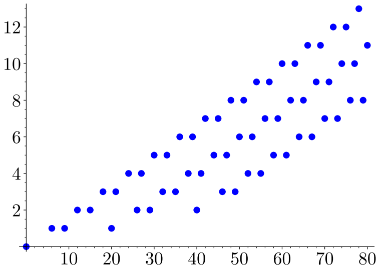

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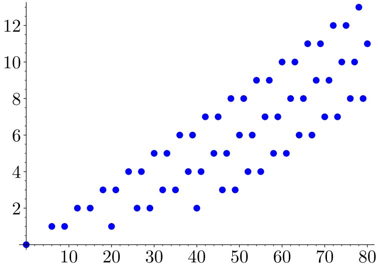

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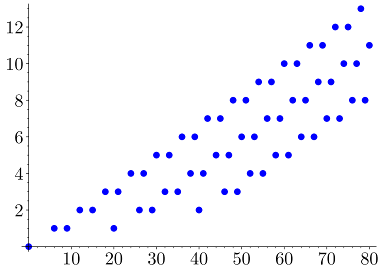

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$$\mathsf{M}(n) = \tfrac{1}{6}n + a_0(n) \qquad \text{for some periodic function } a_0(n)$$

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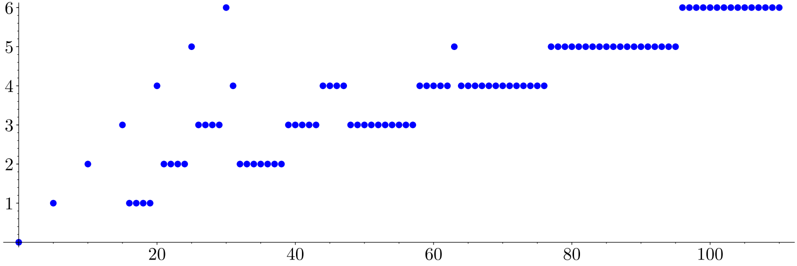

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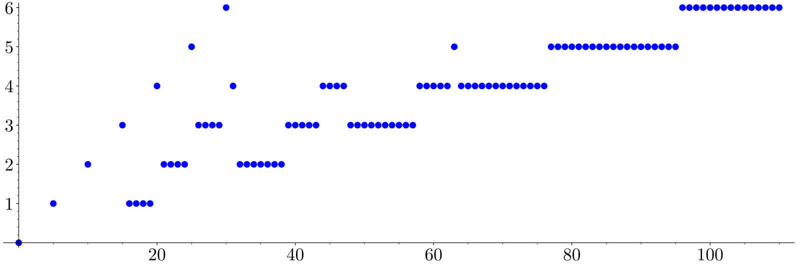

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Conclusion: $\mathsf{m}(n)$ is quasilinear for $n \geq 64$, with period 19.

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Let $S = \langle n_1, \ldots, n_k \rangle$. For $n \gg 0$ (i.e., for n sufficiently large),

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$$\begin{aligned} \mathsf{M}(n) &= \tfrac{1}{n_1} n + a_0(n) \\ \mathsf{m}(n) &= \tfrac{1}{n_k} n + b_0(n) \end{aligned}$$

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Parametrized families of numerical semigroups

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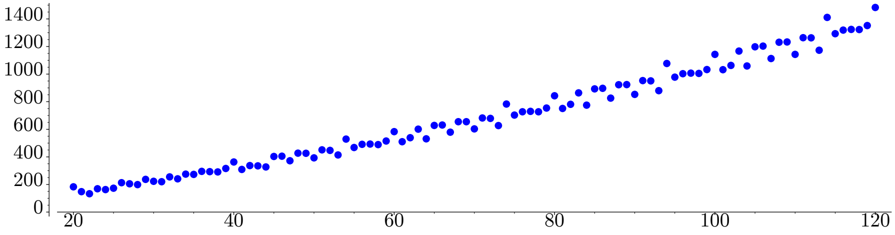

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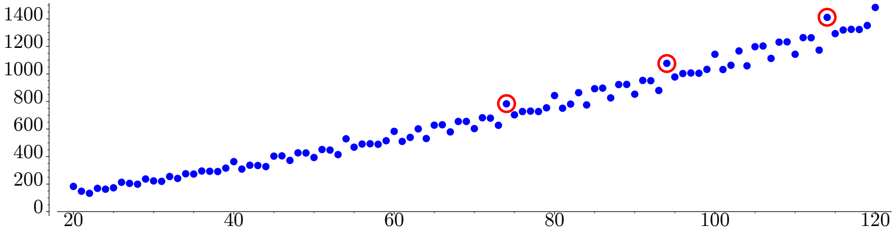

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Example: $n_3 = 4$

$$\begin{aligned}
\langle 2, 7 \rangle &= \{0, \quad 2, \quad 4, \quad 6, 7, 8, \ldots\} \\
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Not true for $m_f = \#$ of numerical semigroups with Frobenius number f

$$m_{11} = 51 \qquad m_{12} = 40 \qquad m_{13} = 106$$

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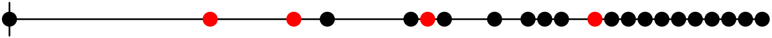

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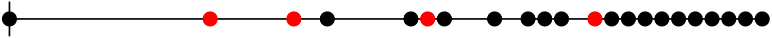

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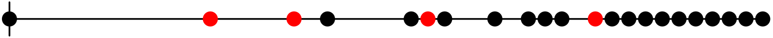

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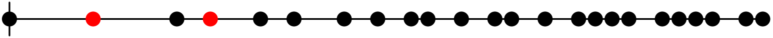

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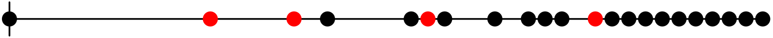

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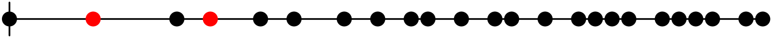

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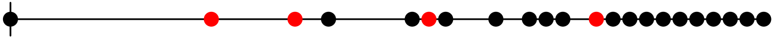

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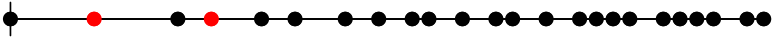

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Proved in many special cases, including $\mathrm{g}(S) \leq 60$.

So much more out there!

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- Error-correcting codes (*Arf* numerical semigroups)
- Music theory

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C. O'Neill and R. Pelayo,
How do you measure primality?,
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Factorization invariants in numerical monoids,
Contemporary Mathematics **685** (2017), 231–249. [arXiv:1508.00128]

S. Chapman and C. O'Neill,
Factoring in the Chicken McNugget monoid,
Mathematics Magazine **91** (2018), no. 5, 323–336. [arXiv:1709.01606]

S. Chapman, R. Garcia, and C. O'Neill,
Beyond coins, stamps, and Chicken McNuggets: an invitation to numerical semigroups,
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