Length density and numerical semigroups

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A *numerical semigroup* $S \subset \mathbb{Z}_{\geq 0}$: closed under **addition**.

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Factorization length

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$ and an element $n \in S$.
A *factorization* $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$ of n

$$n = a_1 n_1 + \cdots + a_k n_k$$

has *length*

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Extremal factorization length

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$, let
$$\mathsf{L}(n) = \{a_1 + \cdots + a_k : n = a_1 n_1 + \cdots + a_k n_k\}$$
denotes the *length set* of n, and
$$\mathsf{M}(n) = \max \mathsf{L}(n) \qquad \text{and}$$
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denote the *maximum* and *minimum* factorization lengths of n.

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- Max length factorization: lots of small generators
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$S = \langle 9, 10, 21 \rangle$:

$$\mathsf{M}(30) = 3 \quad \text{with} \quad 30 = 3(10)$$
$$\mathsf{M}(129) = 14 \quad \text{with} \quad 129 = 3(10) + 11(9)$$

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Theorem (BOP, 2014)

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n \gg 0$ (i.e., for n sufficiently large),

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Equivalently, $\mathsf{M}(n)$, $\mathsf{m}(n)$ *are eventually quasilinear:*

$$\mathsf{M}(n) = \tfrac{1}{n_1} n + a_0(n)$$
$$\mathsf{m}(n) = \tfrac{1}{n_k} n + b_0(n)$$

for periodic functions $a_0(n)$, $b_0(n)$.

$$\mathsf{M}(n) = \begin{cases} \frac{1}{n_1} n + \underline{\qquad} & \text{if } n \equiv 0 \bmod n_1 \\ \frac{1}{n_1} n + \underline{\qquad} & \text{if } n \equiv 1 \bmod n_1 \\ \ldots \end{cases}$$

Elasticity

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$,

$$\rho(n) = \mathsf{M}(n)/\mathsf{m}(n)$$

denotes the *elasticity* of n. Measures "spread" of factorization lengths.

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$$\mathsf{L}(104) = \{\ 7, \ldots, 15\} \qquad \rho = \tfrac{15}{7}$$

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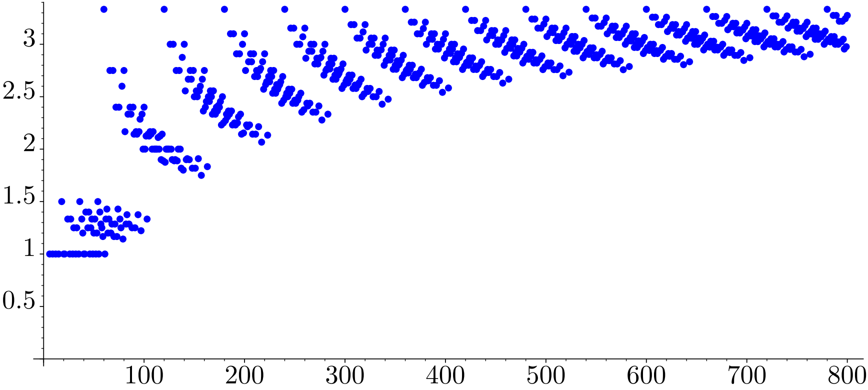

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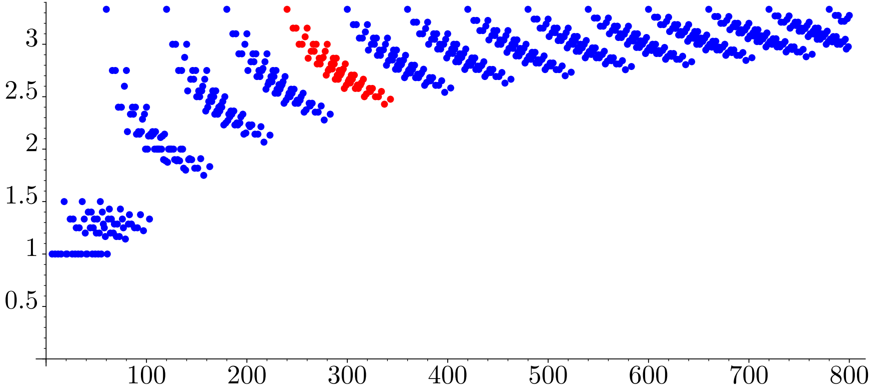

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Theorem (BOP, 2017)

For n large,

$$\rho(n + n_1 n_k) = \frac{\mathsf{M}(n) + n_k}{\mathsf{m}(n) + n_1}$$

A new invariant: length density

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$ with $|\mathsf{L}(n)| > 1$,

$$\mathsf{LD}(n) = \frac{|\mathsf{L}(n)| - 1}{\mathsf{M}(n) - \mathsf{m}(n)}$$

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Why this definition? Larger density $\implies$ "more filled-in" length set

- $0 \le \mathsf{LD}(n) \le 1$ (as density should be)
- $\mathsf{LD}(n) = 1 \quad \implies \quad \mathsf{L}(n) = [\mathsf{m}(n), \mathsf{M}(n)] \cap \mathbb{Z} \quad$ (an interval)

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Judgement calls:

- Why "-1" in the numerator instead of "$+1$" in the denominator?
- Why "inf" for $\mathsf{LD}(S)$?

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$$\mathsf{L}(142) = \{10, 11, 12, 14, 15, 16, 17, 18, 19\} \qquad \Delta(142) = \{1, 2\}$$

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$S = \langle 42, 86, 245, 285, 365, 463 \rangle$:

$$\mathsf{L}(3023) = \{7, 9, 11, 12, \ldots, 46, 47, 58, 62, 64\}, \quad \Delta(3023) = \{1, 2, 4, 9\}$$

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$$\frac{1}{\max \Delta(S)} \le \mathsf{LD}(n) \le \frac{1}{\min \Delta(S)}$$

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$$\mathsf{LD}(n) = \frac{|\mathsf{L}(n)| - 1}{\mathsf{M}(n) - \mathsf{m}(n)}$$

denotes the *length density* of n. Measures "density" of the length set.

Judgement calls revisited:

- Why "-1" in the numerator instead of "$+1$" in the denominator?
  Letting $\Delta(S) = \bigcup_{n \in S} \Delta(n)$, we have
  $$\frac{1}{\max \Delta(S)} \le \mathsf{LD}(n) \le \frac{1}{\min \Delta(S)}$$
- Why "inf" for $\mathsf{LD}(S)$?
  Most $\mathsf{L}(n)$ "almost" arithmetic sequence, step size $\min \Delta(S)$
  $$\text{“least well-behaved”} \iff \text{more missing lengths}$$
  Additionally,
  $$\mathsf{LD}(S) = 1/\min \Delta(S) \iff |\Delta(S)| = 1$$

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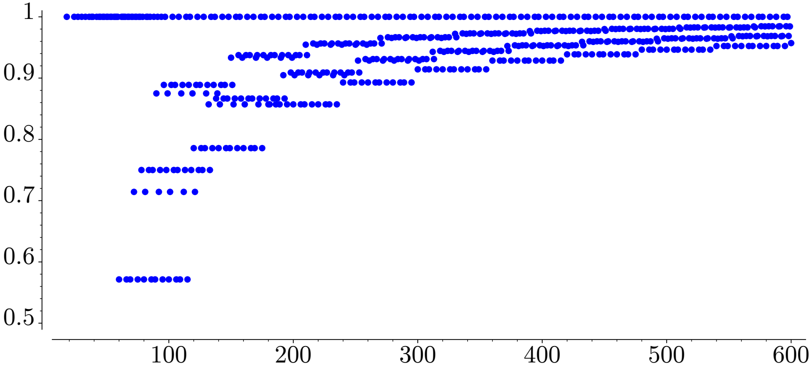

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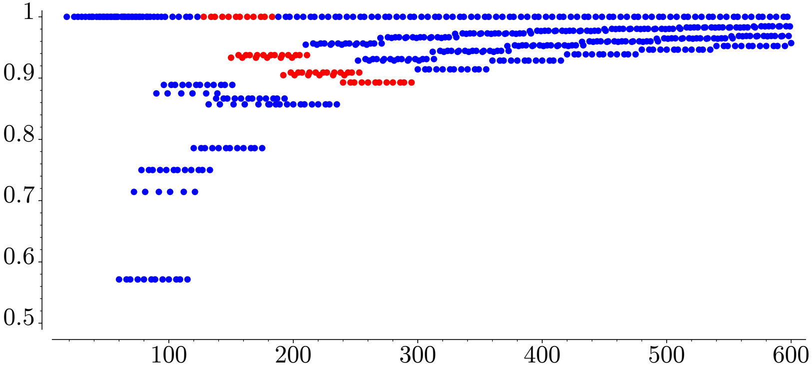

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Theorem (BCKMOPP, 2021)

- $\sup\{\mathsf{LD}(n) : n \in S\} = \frac{1}{\delta}$, *where* $\delta = \min \Delta(S)$
- $\frac{1}{\delta}$ *is the only limit point of* $\{\mathsf{LD}(n) : n \in S\}$
- *For large* $n \in S$,

$$\mathsf{LD}(n + n_1 n_k) = \frac{|\mathsf{L}(n)| - 1 + \frac{1}{\delta}(n_k - n_1)}{\mathsf{M}(n) - \mathsf{m}(n) + (n_k - n_1)}$$

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Example: $S = \langle 6, 9, 20 \rangle$

$$\mathsf{L}(142) = \{10, 11, 12, 13, 14, 15, \ldots, 18, 19\}$$
$$\mathsf{L}(142 + 180) = \{16, 17, 18, 19, 20, 21, \ldots, 38, 39\}$$

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  $\implies \mathsf{LD}(S) = \mathsf{LD}(n)$ for some $n \leq N_S$
  - Can we improve this bound?
  - Do certain elements (e.g., *higher Betti elements*) attain $\mathsf{LD}(S)$?

References

T. Barron, C. O'Neill, R. Pelayo (2017)
On the set of elasticities in numerical monoids
Semigroup Forum **94** (2017), no. 1, 37–50. (arXiv:1409.3425)

S. Chapman, C. O'Neill, V. Ponomarenko (2021)
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Thanks!

Length density and Betti elements

$\mathsf{LD}(S)$ need not be achieved at a Betti element.

Example: $S = \langle 20, 28, 42, 73 \rangle$
Betti elements: 84, 140, 146

$$\mathsf{L}(140) = \{4, 5, 6, 7\} \qquad \mathsf{LD}(140) = \tfrac{2}{3}$$

$$\mathsf{L}(146) = \{2, 3, 4, 5\} \qquad \mathsf{LD}(146) = \tfrac{2}{3}$$

$+28+28$ $\qquad$ $+20+42$

$$\mathsf{L}(202) = \{4, 5, 6, 7, 8, 9\} \qquad \mathsf{LD}(202) = \tfrac{3}{5}$$