On the atomic density of arithmetical congruence monoids

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Joint with Jacob Hartzer, Nils Olsson, Derek Rawling (undergraduates)

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Arithmetical congruence monoids (ACMs)

Definition

An *arithmetical congruence monoid* is a **multiplicative** set

$$M_{a,b} = \{1\} \cup \{a, a+b, a+2b, a+3b, \ldots\} \subseteq (\mathbb{Z}_{\geq 1}, \cdot)$$

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- $441 = 9 \cdot 49 = 21 \cdot 21$
  $= (3^2) \cdot (7^2) = (3 \cdot 7) \cdot (3 \cdot 7)$.

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What about the distriubtion of atoms in $M_{a,b}$?

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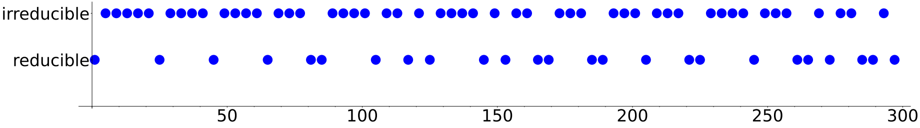

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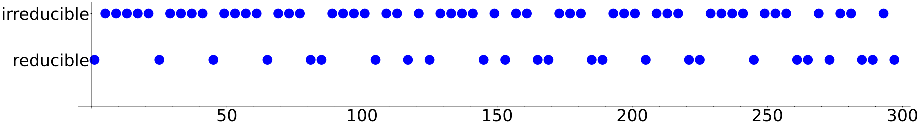

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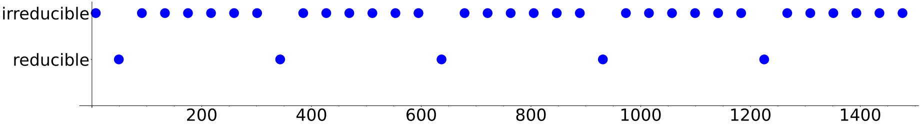

ACM software package

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sage: H.IrreduciblesUpToElement(10000001)
sage: H.IsIrreducible(999997)  # immediate
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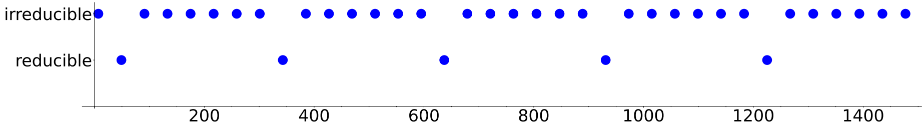

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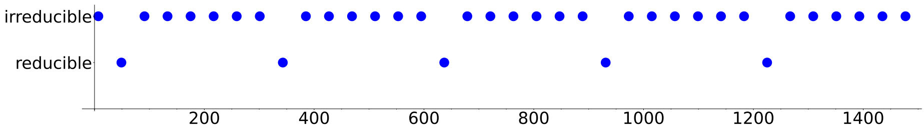

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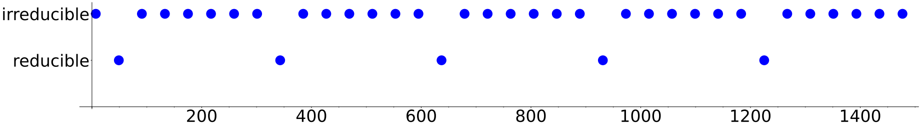

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$M_{1,4}$:	1,	25,	45,	65,	81,	85,	...
$M_{5,20}$:	25,	125,	225,	325,	425,	525,	...
$M_{7,42}$:	49,	343,	637,	931,	1225,	1519,	...
$M_{51,150}$:	2601,	10251,	17901,	25551,	33201,	40401,	...
$M_{25,200}$:	625,	5625,	10625,	15625,	20625,	25625,	...
$M_{341,620}$:	116281,	327701,	539121,	750541,	923521,	961961,	...

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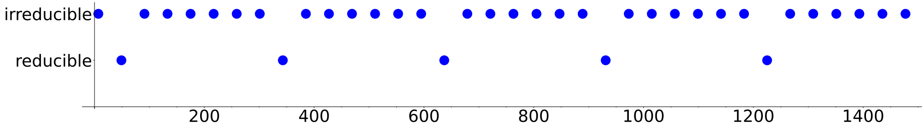

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Example

$M_{5,20} = \{1, 5, 25, 45, 65, 85, 105, 125, 145, 165, 185, 205, 225, 245, \ldots\}$
Reducible elements:

$$\begin{array}{lll}
25 = 5 \cdot 5 & 525 = 5 \cdot 105 & 1025 = 5 \cdot 205 \\
125 = 5 \cdot 25 & 625 = 5 \cdot 125 & 1125 = 5 \cdot 225 \\
225 = 5 \cdot 45 & 725 = 5 \cdot 145 & 1225 = 5 \cdot 245 \\
325 = 5 \cdot 65 & 825 = 5 \cdot 165 & 1325 = 5 \cdot 265 \\
425 = 5 \cdot 85 & 925 = 5 \cdot 185 & 1425 = 5 \cdot 285
\end{array}$$

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For $M_{1,4}$, a sequence of 6 consecutive reducible elements:

$$20884505 = 5 \cdot 4176901 \qquad 20884517 = 17 \cdot 1228501$$
$$20884509 = 9 \cdot 2320501 \qquad 20884521 = 21 \cdot 994501$$
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For $M_{9,12}$, a sequence of 4 evenly-spaced reducible elements:

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Idea

Look for (arbitrarily) long sequences of evenly-spaced reducible elements.

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Let $d = \gcd(a, b)$. *The elements*

$$d(a + jb) + (a + b - d)\prod_{i=1}^{k}(a + ib) \quad \textit{for} \quad j = 1, \ldots, k$$

are all reducible, with constant difference db.

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Theorem (Hartzer, O.)

$M_{a,b}$ has periodic atom set if and only if $a \mid b$ and $a > 1$.

The next natural question

Famous properties of the primes in $\mathbb{Z}$:

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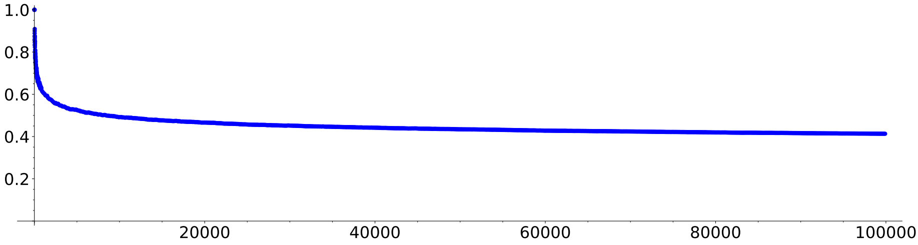

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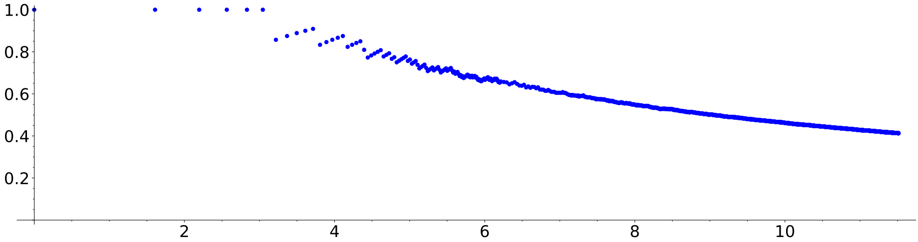

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If $a \in M_{1,4}$ is an atom, then either:

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Example

If $a \in M_{1,5}$ is an atom, then either a is prime, or:

- $a = p_1 p_2$ for $p_1 \equiv 2 \bmod 5$ and $p_2 \equiv 3 \bmod 5$;
- $a = p_1 p_2 p_3$ for $p_1, p_2 \equiv 2 \bmod 5$ and $p_3 \equiv 4 \bmod 5$; ...

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Minimal zero-sum sequences of $(\mathbb{Z}_5^*, \cdot)$:

$$[1], \quad [2]^4, \quad [3]^4, \quad [4]^2, \quad [2]^2[4], \quad [3]^2[4], \quad \text{and} \quad [2][3]$$

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But... not all is lost: if $p, m \equiv 5 \bmod 6$, then $4pm = (2p)(2m)$ is reducible

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Fix an ACM $M_{a,b}$ with $a > 1$, and let $d = \gcd(a, b)$.
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$a = 4 \cdot 7^k \in M_{4,6}$ is an atom for any odd $k \geq 1$.
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The key observation: one can always locate an equivalence class modulo b from which at most one prime can divide atoms divisible by d^2.

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The following set has density 0 whenever $\gcd(c, b) = 1$*:*

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Theorem

Any ACM $M_{a,b}$ *has atomic density* $\mathcal{D}(M_{a,b}) = 1 - \frac{1}{d}$, *where* $d = \gcd(a, b)$.

Conclusion

Famous properties of the primes in $\mathbb{Z}$:

- The primes are sporatic (no discernable pattern, highly non-periodic)
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In the land of ACMs:

Theorem [Hartzer–O, 2017]

An ACM $M_{a,b}$ has periodic atom set if and only if $a \mid b$ and $a > 1$.

Theorem [Olsson–O–Rawling, 2022]

Any ACM $M_{a,b}$ has atomic density $\mathcal{D}(M_{a,b}) = 1 - \frac{1}{d}$, where $d = \gcd(a, b)$.

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