Numerical semigroups, minimal presentations, and posets

Christopher O'Neill

San Diego State University

cdoneill@sdsu.edu

Joint with *Tara Gomes and *Eduardo Torres Davila

* = undergraduate student

Slides available: `https://cdoneill.sdsu.edu/`

October 13, 2022

Numerical semigroups

Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Numerical semigroups

Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example:

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Numerical semigroups

Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example: "McNugget Semigroup"

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Numerical semigroups

Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example: "McNugget Semigroup"

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Example: $S = \langle 6, 9, 18, 20, 32 \rangle$

Numerical semigroups

Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example: "McNugget Semigroup"

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Example: $S = \langle 6, 9, \cancel{18}, 20, \cancel{32} \rangle$

Numerical semigroups

Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example: "McNugget Semigroup"

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Example: $S = \langle 6, 9, \cancel{18}, 20, \cancel{32} \rangle = McN$

Numerical semigroups

Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example: “McNugget Semigroup”

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Example: $S = \langle 6, 9, \cancel{18}, 20, \cancel{32} \rangle = McN$

Fact

Every numerical semigroup has a unique minimal generating set.

Numerical semigroups

Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example: "McNugget Semigroup"

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Example: $S = \langle 6, 9, \cancel{18}, 20, \cancel{32} \rangle = McN$

Fact

Every numerical semigroup has a unique minimal generating set.

Embedding dimension: $\mathsf{e}(S) = \#$ minimal generators

Numerical semigroups

Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example: "McNugget Semigroup"

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Example: $S = \langle 6, 9, \cancel{18}, 20, \cancel{32} \rangle = McN$

Fact

Every numerical semigroup has a unique minimal generating set.

Embedding dimension: $\mathsf{e}(S) = \#$ minimal generators
Multiplicity: $\mathsf{m}(S) =$ smallest nonzero element

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is
$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then
$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

Observations:

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

Observations:

- The elements of $\mathrm{Ap}(S)$ are distinct modulo m

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

Observations:

- The elements of $\mathrm{Ap}(S)$ are distinct modulo m
- $|\,\mathrm{Ap}(S)| = m$

Apéry sets

Fix a numerical semigroup S with $m = \mathrm{m}(S)$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

Apéry sets

Fix a numerical semigroup S with $m = \mathrm{m}(S)$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

Many things can be easily recovered from the Apéry set.

Apéry sets

Fix a numerical semigroup S with $m = \mathrm{m}(S)$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

Many things can be easily recovered from the Apéry set.

- Fast membership test:

$$n \in S \text{ if } n \geq a \text{ for } a \in \mathrm{Ap}(S) \text{ with } a \equiv n \bmod m$$

Apéry sets

Fix a numerical semigroup S with $m = \mathrm{m}(S)$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

Many things can be easily recovered from the Apéry set.

- Fast membership test:

$$n \in S \text{ if } n \geq a \text{ for } a \in \mathrm{Ap}(S) \text{ with } a \equiv n \bmod m$$

- Frobenius number: $\mathrm{F}(S) = \max(\mathrm{Ap}(S)) - m$

Apéry sets

Fix a numerical semigroup S with $m = \mathrm{m}(S)$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

Many things can be easily recovered from the Apéry set.

- Fast membership test:

$$n \in S \text{ if } n \geq a \text{ for } a \in \mathrm{Ap}(S) \text{ with } a \equiv n \bmod m$$

- Frobenius number: $\mathrm{F}(S) = \max(\mathrm{Ap}(S)) - m$
- Number of gaps (the *genus*):

$$\mathrm{g}(S) = |\mathbb{N} \setminus S| = \sum_{a \in \mathrm{Ap}(S)} \left\lfloor \frac{a}{m} \right\rfloor$$

Apéry sets

Fix a numerical semigroup S with $m = \mathrm{m}(S)$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

Many things can be easily recovered from the Apéry set.

- Fast membership test:

$$n \in S \text{ if } n \geq a \text{ for } a \in \mathrm{Ap}(S) \text{ with } a \equiv n \bmod m$$

- Frobenius number: $\mathrm{F}(S) = \max(\mathrm{Ap}(S)) - m$
- Number of gaps (the *genus*):

$$\mathrm{g}(S) = |\mathbb{N} \setminus S| = \sum_{a \in \mathrm{Ap}(S)} \left\lfloor \frac{a}{m} \right\rfloor$$

The Apéry set is a "one stop shop" for computation.

Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ equal the Apéry set of some numerical semigroup?

Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ equal the Apéry set of some numerical semigroup?

$$m = |A| = 5, \quad a_1 = 11, \ a_2 = 7, \ a_3 = 23, \ a_4 = 19$$

Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ equal the Apéry set of some numerical semigroup?

$$m = |A| = 5, \quad a_1 = 11,\ a_2 = 7,\ a_3 = 23,\ a_4 = 19$$

but $a_1 + a_2 \equiv 3 \bmod 5$ and $a_1 + a_2 < a_3$.

Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ equal the Apéry set of some numerical semigroup?

$$m = |A| = 5, \quad a_1 = 11,\ a_2 = 7,\ a_3 = 23,\ a_4 = 19$$

but $a_1 + a_2 \equiv 3 \bmod 5$ and $a_1 + a_2 < a_3$.

Is $\{0, 13, 14, 27, 10, 11\}$ equal the Apéry set of some numerical semigroup?

$$m = |A| = 6, \quad a_1 = 13,\ a_2 = 14,\ a_3 = 27,\ a_4 = 10,\ a_5 = 11$$

Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ equal the Apéry set of some numerical semigroup?

$$m = |A| = 5, \quad a_1 = 11,\ a_2 = 7,\ a_3 = 23,\ a_4 = 19$$

but $a_1 + a_2 \equiv 3 \bmod 5$ and $a_1 + a_2 < a_3$.

Is $\{0, 13, 14, 27, 10, 11\}$ equal the Apéry set of some numerical semigroup?

$$m = |A| = 6, \quad a_1 = 13,\ a_2 = 14,\ a_3 = 27,\ a_4 = 10,\ a_5 = 11$$

but $a_4 + a_5 \equiv 3 \bmod 6$ and $a_4 + a_5 < a_3$.

Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ equal the Apéry set of some numerical semigroup?

$$m = |A| = 5, \quad a_1 = 11,\ a_2 = 7,\ a_3 = 23,\ a_4 = 19$$

but $a_1 + a_2 \equiv 3 \bmod 5$ and $a_1 + a_2 < a_3$.

Is $\{0, 13, 14, 27, 10, 11\}$ equal the Apéry set of some numerical semigroup?

$$m = |A| = 6, \quad a_1 = 13,\ a_2 = 14,\ a_3 = 27,\ a_4 = 10,\ a_5 = 11$$

but $a_4 + a_5 \equiv 3 \bmod 6$ and $a_4 + a_5 < a_3$.

Theorem

If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup S with $\mathrm{Ap}(S) = A$ if and only if

$$a_i + a_j \geq a_{i+j} \qquad \textit{whenever} \qquad i + j \neq 0.$$

Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ equal the Apéry set of some numerical semigroup?

$$m = |A| = 5, \quad a_1 = 11,\ a_2 = 7,\ a_3 = 23,\ a_4 = 19$$

but $a_1 + a_2 \equiv 3 \bmod 5$ and $a_1 + a_2 < a_3$.

Is $\{0, 13, 14, 27, 10, 11\}$ equal the Apéry set of some numerical semigroup?

$$m = |A| = 6, \quad a_1 = 13,\ a_2 = 14,\ a_3 = 27,\ a_4 = 10,\ a_5 = 11$$

but $a_4 + a_5 \equiv 3 \bmod 6$ and $a_4 + a_5 < a_3$.

Theorem

If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup S with $\mathrm{Ap}(S) = A$ if and only if

$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

Big idea: the inequalities "$a_i + a_j \geq a_{i+j}$" to define a **cone** C_m.

Kunz cone

Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

Kunz cone

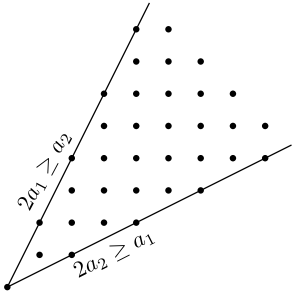
Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

Example: C_3



Kunz cone

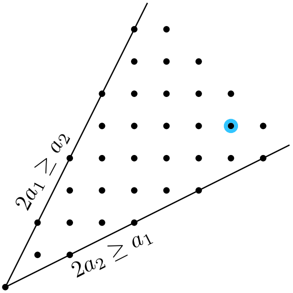
Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

Example: C_3



$S = \langle 3, 5, 7 \rangle$
$\mathrm{Ap}(S) = \{0, 7, 5\}$

Kunz cone

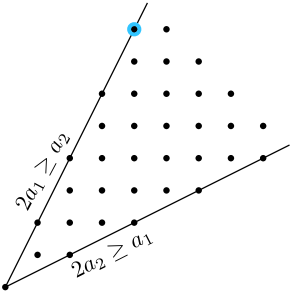
Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

Example: C_3



$S = \langle 3, 5, 7 \rangle$
$\mathrm{Ap}(S) = \{0, 7, 5\}$

$S = \langle 3, 4 \rangle$
$\mathrm{Ap}(S) = \{0, 4, 8\}$

Kunz cone

Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

Kunz cone

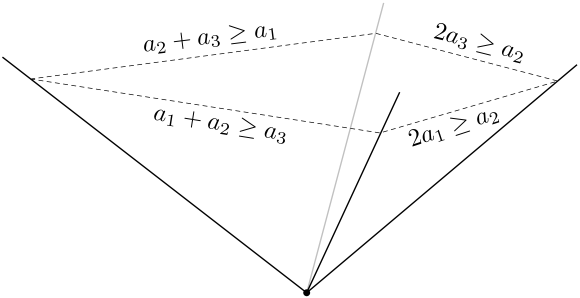
Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

Example: C_4



Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Big picture: "moduli space" approach for studying XYZ's

- Define a space with XYZ's as points
  Small changes to an XYZ $\rightsquigarrow$ small movements in space
- Let geometric/topological structure inform study of XYZ's

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Big picture: "moduli space" approach for studying XYZ's

- Define a space with XYZ's as points
  Small changes to an XYZ $\rightsquigarrow$ small movements in space
- Let geometric/topological structure inform study of XYZ's

Basic example: $\mathrm{GL}_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$

Faces of the Kunz cone

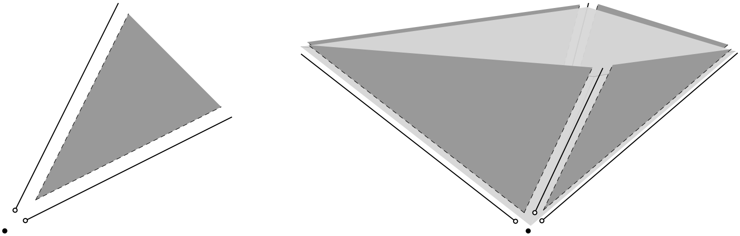
Question

When are numerical semigroups in (the relative interior of) the same face?

Big picture: "moduli space" approach for studying XYZ's

- Define a space with XYZ's as points
  Small changes to an XYZ $\rightsquigarrow$ small movements in space
- Let geometric/topological structure inform study of XYZ's

Basic example: $\mathrm{GL}_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$
More interesting example: C_m



Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Motivation: $S \in \mathrm{Int}(C_m)$ if and only if S has *max embedding dimension*

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Motivation: $S \in \mathrm{Int}(C_m)$ if and only if S has *max embedding dimension*

If $S = \langle n_1, \ldots, n_k \rangle$ with $\mathrm{e}(S) = k$, then

$$n_i \not\equiv n_j \bmod n_1 \quad \Longrightarrow \quad \mathrm{e}(S) \le \mathrm{m}(S)$$

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Motivation: $S \in \mathrm{Int}(C_m)$ if and only if S has *max embedding dimension*

If $S = \langle n_1, \ldots, n_k \rangle$ with $\mathrm{e}(S) = k$, then

$$n_i \not\equiv n_j \bmod n_1 \quad \Longrightarrow \quad \mathrm{e}(S) \leq \mathrm{m}(S)$$

If $\mathrm{e}(S) = \mathrm{m}(S)$, then S has *max embedding dimension*

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Motivation: $S \in \mathrm{Int}(C_m)$ if and only if S has *max embedding dimension*

If $S = \langle n_1, \ldots, n_k \rangle$ with $\mathrm{e}(S) = k$, then

$$n_i \not\equiv n_j \bmod n_1 \quad \implies \quad \mathrm{e}(S) \le \mathrm{m}(S)$$

If $\mathrm{e}(S) = \mathrm{m}(S)$, then S has *max embedding dimension*

$$S = \langle m, a_1, \ldots, a_{m-1} \rangle \quad \text{where} \quad \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\}$$

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Motivation: $S \in \mathrm{Int}(C_m)$ if and only if S has *max embedding dimension*

If $S = \langle n_1, \ldots, n_k \rangle$ with $\mathrm{e}(S) = k$, then

$$n_i \not\equiv n_j \bmod n_1 \quad \implies \quad \mathrm{e}(S) \le \mathrm{m}(S)$$

If $\mathrm{e}(S) = \mathrm{m}(S)$, then S has *max embedding dimension*

$$S = \langle m, a_1, \ldots, a_{m-1} \rangle \quad \text{where} \quad \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\}$$

Geometrically: "most" numerical semigroups with $\mathrm{m}(S) = m$ are MED

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Motivation: $S \in \mathrm{Int}(C_m)$ if and only if S has *max embedding dimension*

If $S = \langle n_1, \ldots, n_k \rangle$ with $\mathrm{e}(S) = k$, then

$$n_i \not\equiv n_j \bmod n_1 \quad \Longrightarrow \quad \mathrm{e}(S) \le \mathrm{m}(S)$$

If $\mathrm{e}(S) = \mathrm{m}(S)$, then S has *max embedding dimension*

$$S = \langle m, a_1, \ldots, a_{m-1} \rangle \quad \text{where} \quad \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\}$$

Geometrically: "most" numerical semigroups with $\mathrm{m}(S) = m$ are MED

What about the other faces?

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Example: $S = \langle 4, 10, 11, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 11\}$$
$$a_1 = 13, \quad a_2 = 10, \quad a_3 = 11$$

$$2a_1 > a_2 \qquad a_1 + a_2 > a_3$$
$$2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Example: $S = \langle 4, 10, 11, 13 \rangle$

$$\text{Ap}(S) = \{0, 13, 10, 11\} \qquad 2a_1 > a_2 \qquad a_1 + a_2 > a_3$$
$$a_1 = 13, \quad a_2 = 10, \quad a_3 = 11 \qquad 2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

Example: $S = \langle 4, 10, 13 \rangle$

$$\text{Ap}(S) = \{0, 13, 10, 23\} \qquad 2a_1 > a_2 \qquad a_1 + a_2 = a_3$$
$$a_1 = 13, \quad a_2 = 10, \quad a_3 = 23 \qquad 2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Example: $S = \langle 4, 10, 11, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 11\} \qquad a_1 = 13, \quad a_2 = 10, \quad a_3 = 11$$

$$2a_1 > a_2 \qquad a_1 + a_2 > a_3$$
$$2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

Example: $S = \langle 4, 10, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 23\} \qquad a_1 = 13, \quad a_2 = 10, \quad a_3 = 23$$

$$2a_1 > a_2 \qquad a_1 + a_2 = a_3$$
$$2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

Example: $S = \langle 4, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 26, 39\} \qquad a_1 = 13, \quad a_2 = 26, \quad a_3 = 39$$

$$2a_1 = a_2 \qquad a_1 + a_2 = a_3$$
$$2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Faces of the Kunz cone

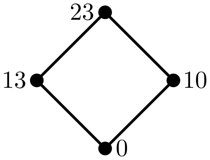
Question

When are numerical semigroups in (the relative interior of) the same face?

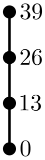
Definition

The *Apéry poset* of S: define $a \preceq a'$ whenever $a' - a \in S$.

$\mathrm{Ap}(S) = \{0, 13, 10, 23\}$



$\mathrm{Ap}(S) = \{0, 13, 26, 39\}$



Faces of the Kunz polyhedron

Question

When are numerical semigroups in (the relative interior of) the same face?

Faces of the Kunz polyhedron

Question

When are numerical semigroups in (the relative interior of) the same face?

$$S = \langle 6, 9, 20 \rangle$$
$$\mathrm{Ap}(S) = \{0, 49, 20, \; 9, 40, 29\}$$

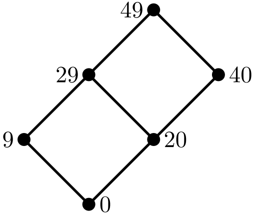
$$S' = \langle 6, 26, 27 \rangle$$
$$\mathrm{Ap}(S') = \{0, 79, 26, 27, 52, 53\}$$

Faces of the Kunz polyhedron

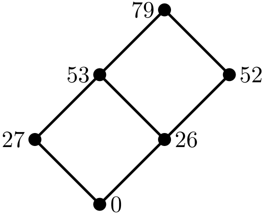
Question

When are numerical semigroups in (the relative interior of) the same face?

$S = \langle 6, 9, 20 \rangle$
$\mathrm{Ap}(S) = \{0, 49, 20, \; 9, 40, 29\}$



$S' = \langle 6, 26, 27 \rangle$
$\mathrm{Ap}(S') = \{0, 79, 26, 27, 52, 53\}$



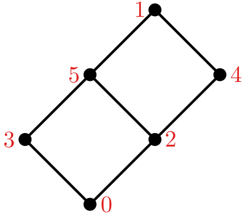
Faces of the Kunz polyhedron

Question

When are numerical semigroups in (the relative interior of) the same face?

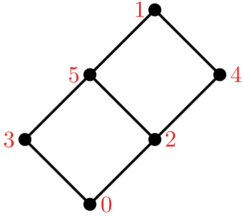
$S = \langle 6, 9, 20 \rangle$

$\mathrm{Ap}(S) = \{0, \underset{1}{49}, \underset{2}{20}, \underset{3}{9}, \underset{4}{40}, \underset{5}{29}\}$



$S' = \langle 6, 26, 27 \rangle$

$\mathrm{Ap}(S') = \{0, \underset{1}{79}, \underset{2}{26}, \underset{3}{27}, \underset{4}{52}, \underset{5}{53}\}$



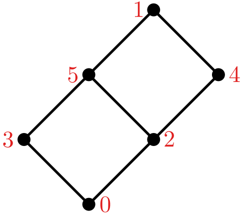
Faces of the Kunz polyhedron

Question

When are numerical semigroups in (the relative interior of) the same face?

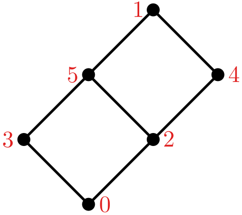
$S = \langle 6, 9, 20 \rangle$

$\mathrm{Ap}(S) = \{0, \underset{1}{49}, \underset{2}{20}, \underset{3}{9}, \underset{4}{40}, \underset{5}{29}\}$



$S' = \langle 6, 26, 27 \rangle$

$\mathrm{Ap}(S') = \{0, \underset{1}{79}, \underset{2}{26}, \underset{3}{27}, \underset{4}{52}, \underset{5}{53}\}$



The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

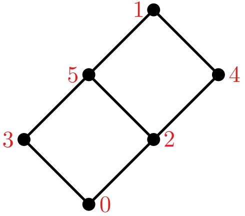
Faces of the Kunz polyhedron

Question

When are numerical semigroups in (the relative interior of) the same face?

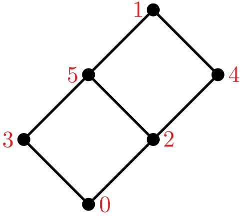
$S = \langle 6, 9, 20 \rangle$

$\mathrm{Ap}(S) = \{0, \underset{1}{49}, \underset{2}{20}, \underset{3}{9}, \underset{4}{40}, \underset{5}{29}\}$



$S' = \langle 6, 26, 27 \rangle$

$\mathrm{Ap}(S') = \{0, \underset{1}{79}, \underset{2}{26}, \underset{3}{27}, \underset{4}{52}, \underset{5}{53}\}$



The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

Theorem (Bruns–García-Sánchez–O.–Wilburne)

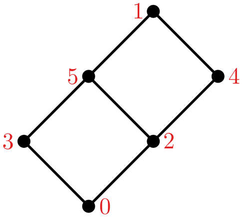
Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

Faces of the Kunz polyhedron

Question

When are numerical semigroups in (the relative interior of) the same face?

$$S = \langle 6, 9, 20 \rangle$$
$$\mathrm{Ap}(S) = \{0, \underset{1}{49}, \underset{2}{20}, \underset{3}{9}, \underset{4}{40}, \underset{5}{29}\}$$



The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

Theorem (Bruns–García-Sánchez–O.–Wilburne)

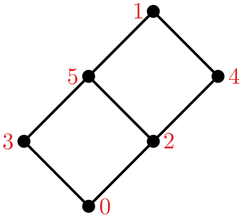
Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

Faces of the Kunz polyhedron

Question

When are numerical semigroups in (the relative interior of) the same face?

$$S = \langle 6, 9, 20 \rangle$$
$$\mathrm{Ap}(S) = \{0, \underset{1}{49}, \underset{2}{20}, \underset{3}{9}, \underset{4}{40}, \underset{5}{29}\}$$



Defining facet equations:

$$2a_2 = a_4$$
$$a_2 + a_3 = a_5$$
$$a_2 + a_5 = a_1$$
$$a_3 + a_4 = a_1$$

The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

Theorem (Bruns–García-Sánchez–O.–Wilburne)

Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

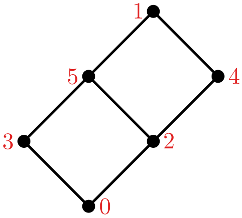
Faces of the Kunz polyhedron

Question

When are numerical semigroups in (the relative interior of) the same face?

$$S = \langle 6, 9, 20 \rangle$$

$$\mathrm{Ap}(S) = \{0, \underset{1}{49}, \underset{2}{20}, \underset{3}{9}, \underset{4}{40}, \underset{5}{29}\}$$



Defining facet equations:

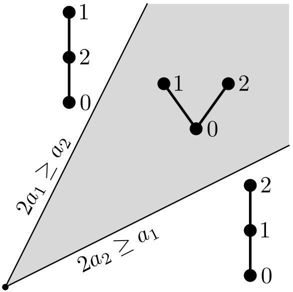
$2a_2 = a_4$	$2 \preceq 4$
$a_2 + a_3 = a_5$	$2 \preceq 5$
	$3 \preceq 5$
$a_2 + a_5 = a_1$	$2 \preceq 1$
	$5 \preceq 1$
$a_3 + a_4 = a_1$	$3 \preceq 1$
	$4 \preceq 1$

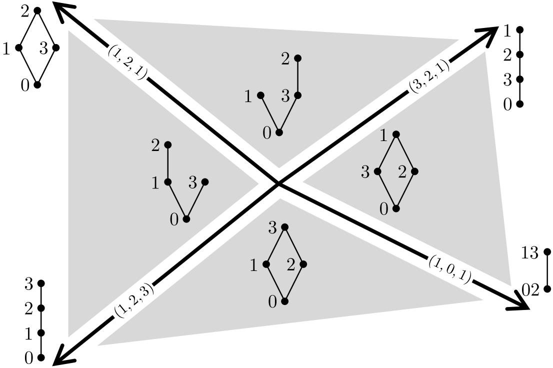
The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

Theorem (Bruns–García-Sánchez–O.–Wilburne)

Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

C_3 and C_4





Faces of the Kunz cone

What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

Faces of the Kunz cone

What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

- $\mathsf{m}(S) = |P|$

Faces of the Kunz cone

What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

- $\mathrm{m}(S) = |P|$
- $\mathrm{e}(S) = 1 + \#$ atoms of P

Faces of the Kunz cone

What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

- $\mathrm{m}(S) = |P|$
- $\mathrm{e}(S) = 1 + \#$ atoms of P
- $\mathrm{t}(S) = \#$ maximal elements of P
  Cohen-Macaulay type of the *defining toric ideal* of S:
  $$I_S = \ker\left(\mathbb{k}[x_1, \ldots, x_k] \to \mathbb{k}[w^{n_1}, w^{n_2}, \ldots, w^{n_k}]\right)$$

Faces of the Kunz cone

What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

- $\mathrm{m}(S) = |P|$
- $\mathrm{e}(S) = 1 + \#$ atoms of P
- $\mathrm{t}(S) = \#$ maximal elements of P

  Cohen-Macaulay type of the *defining toric ideal* of S:

$$I_S = \ker\left(\Bbbk[x_1, \ldots, x_k] \to \Bbbk[w^{n_1}, w^{n_2}, \ldots, w^{n_k}]\right)$$

Spoiler

If S, S' have identical Kunz posets, then I_S and $I_{S'}$ have the same number of minimal generators.

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$\mathsf{Z}(n) = \left\{ \mathbf{a} \in \mathbb{Z}_{\geq 0}^k : n = a_1 n_1 + \cdots + a_k n_k \right\}$$

is the *set of factorizations* of $n \in S$.

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$\mathsf{Z}(n) = \left\{ \mathbf{a} \in \mathbb{Z}_{\geq 0}^k : n = a_1 n_1 + \cdots + a_k n_k \right\}$$

is the *set of factorizations* of $n \in S$.

Example

$S = \langle 6, 9, 20 \rangle$:

$$\mathsf{Z}(60) = \{(10,0,0), (7,2,0), (4,4,0), (1,6,0), (0,0,3)\}$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$\mathsf{Z}(n) = \left\{ \mathbf{a} \in \mathbb{Z}_{\geq 0}^k : n = a_1 n_1 + \cdots + a_k n_k \right\}$$

is the *set of factorizations* of $n \in S$.

Example

$S = \langle 6, 9, 20 \rangle$:

$$\mathsf{Z}(60) = \{(10,0,0), (7,2,0), (4,4,0), (1,6,0), (0,0,3)\}$$

$$\mathsf{Z}(1000001) =$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$\mathsf{Z}(n) = \left\{ \mathbf{a} \in \mathbb{Z}_{\geq 0}^k : n = a_1 n_1 + \cdots + a_k n_k \right\}$$

is the *set of factorizations* of $n \in S$.

Example

$S = \langle 6, 9, 20 \rangle$:

$$\mathsf{Z}(60) = \{(10,0,0), (7,2,0), (4,4,0), (1,6,0), (0,0,3)\}$$

$$\mathsf{Z}(1000001) = \{ \underbrace{\qquad\qquad}_{\text{shortest}}, \quad \ldots, \quad \underbrace{\qquad\qquad}_{\text{longest}} \}$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$\mathsf{Z}(n) = \left\{ \mathbf{a} \in \mathbb{Z}_{\geq 0}^k : n = a_1 n_1 + \cdots + a_k n_k \right\}$$

is the *set of factorizations* of $n \in S$.

Example

$S = \langle 6, 9, 20 \rangle$:

$$\mathsf{Z}(60) = \{(10,0,0), (7,2,0), (4,4,0), (1,6,0), (0,0,3)\}$$

$$\mathsf{Z}(1000001) = \{ \underbrace{(2,1,49999)}_{\text{shortest}}, \quad \ldots, \quad \underbrace{}_{\text{longest}} \}$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$\mathsf{Z}(n) = \left\{ \mathbf{a} \in \mathbb{Z}_{\geq 0}^k : n = a_1 n_1 + \cdots + a_k n_k \right\}$$

is the *set of factorizations* of $n \in S$.

Example

$S = \langle 6, 9, 20 \rangle$:

$$\mathsf{Z}(60) = \{(10,0,0), (7,2,0), (4,4,0), (1,6,0), (0,0,3)\}$$

$$\mathsf{Z}(1000001) = \{ \underbrace{(2,1,49999)}_{\text{shortest}}, \quad \ldots, \quad \underbrace{(166662,1,1)}_{\text{longest}} \}$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{Z}_{\geq 0}^k & \longrightarrow & \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array}$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{Z}_{\geq 0}^k & \longrightarrow & \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array}$$

Definition

The *kernel* $\ker \pi$ is the relation $\sim$ on $\mathbb{Z}_{\geq 0}^k$ with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b})$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k_{\geq 0}$$

Factorization homomorphism:

$$\begin{aligned} \pi : \mathbb{Z}^k_{\geq 0} &\longrightarrow \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} &\longmapsto a_1 n_1 + \cdots + a_k n_k \end{aligned}$$

Definition

The *kernel* $\ker \pi$ is the relation $\sim$ on $\mathbb{Z}^k_{\geq 0}$ with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b})$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism:

$$\begin{aligned} \pi : \mathbb{Z}_{\geq 0}^k &\longrightarrow \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} &\longmapsto a_1 n_1 + \cdots + a_k n_k \end{aligned}$$

Definition

The *kernel* $\ker \pi$ is the relation $\sim$ on $\mathbb{Z}_{\geq 0}^k$ with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b})$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

that is closed under *translation*.

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism:

$$\begin{aligned} \pi : \mathbb{Z}_{\geq 0}^k &\longrightarrow \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} &\longmapsto a_1 n_1 + \cdots + a_k n_k \end{aligned}$$

Definition

The *kernel* $\ker \pi$ is the relation $\sim$ on $\mathbb{Z}_{\geq 0}^k$ with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b})$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

that is closed under *translation*

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism: Monomial map:

$$\begin{aligned} \pi : \mathbb{Z}_{\geq 0}^k &\longrightarrow \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} &\longmapsto a_1 n_1 + \cdots + a_k n_k \end{aligned}$$

Definition

The *kernel* $\ker \pi$ is the relation $\sim$ on $\mathbb{Z}_{\geq 0}^k$ with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b})$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

that is closed under *translation*

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism:

$$\begin{aligned} \pi : \mathbb{Z}_{\geq 0}^k &\longrightarrow \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} &\longmapsto a_1 n_1 + \cdots + a_k n_k \end{aligned}$$

Monomial map:

$$\begin{aligned} \varphi : \Bbbk[x_1, \ldots, x_k] &\longrightarrow \Bbbk[w] \\ x_i &\longmapsto w^{n_i} \end{aligned}$$

Definition

The *kernel* $\ker \pi$ is the relation $\sim$ on $\mathbb{Z}_{\geq 0}^k$ with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b})$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

that is closed under *translation*

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k_{\geq 0}$$

Factorization homomorphism:

$$\begin{aligned} \pi : \mathbb{Z}^k_{\geq 0} &\longrightarrow \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} &\longmapsto a_1 n_1 + \cdots + a_k n_k \end{aligned}$$

Monomial map:

$$\begin{aligned} \varphi : \Bbbk[x_1, \ldots, x_k] &\longrightarrow \Bbbk[w] \\ x_i &\longmapsto w^{n_i} \end{aligned}$$

Definition

The *kernel* $\ker \pi$ is the relation $\sim$ on $\mathbb{Z}^k_{\geq 0}$ with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b}) \qquad x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S = \ker \varphi$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

that is closed under *translation*

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k_{\geq 0}$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{Z}^k_{\geq 0} & \longrightarrow & \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array}$$

Monomial map:

$$\begin{array}{rcl} \varphi : \Bbbk[x_1, \ldots, x_k] & \longrightarrow & \Bbbk[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

Definition

The *kernel* $\ker \pi$ is the relation $\sim$ on $\mathbb{Z}^k_{\geq 0}$ with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b}) \qquad x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S = \ker \varphi$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a} \qquad x^{\mathbf{a}} - x^{\mathbf{a}} = 0 \in I_S$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

that is closed under *translation*

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k_{\geq 0}$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{Z}^k_{\geq 0} & \longrightarrow & \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array}$$

Monomial map:

$$\begin{array}{rcl} \varphi : \Bbbk[x_1, \ldots, x_k] & \longrightarrow & \Bbbk[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

Definition

The *kernel* $\ker \pi$ is the relation $\sim$ on $\mathbb{Z}^k_{\geq 0}$ with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b}) \qquad\qquad x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S = \ker \varphi$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a} \qquad\qquad x^{\mathbf{a}} - x^{\mathbf{a}} = 0 \in I_S$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a} \qquad\qquad x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S \Rightarrow x^{\mathbf{b}} - x^{\mathbf{a}} \in I_S$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

that is closed under *translation*

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k_{\geq 0}$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{Z}^k_{\geq 0} & \longrightarrow & \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array}$$

Monomial map:

$$\begin{array}{rcl} \varphi : \Bbbk[x_1, \ldots, x_k] & \longrightarrow & \Bbbk[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

Definition

The *kernel* $\ker \pi$ is the relation $\sim$ on $\mathbb{Z}^k_{\geq 0}$ with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b}) \qquad\qquad x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S = \ker \varphi$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a} \qquad\qquad x^{\mathbf{a}} - x^{\mathbf{a}} = 0 \in I_S$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a} \qquad\qquad x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S \Rightarrow x^{\mathbf{b}} - x^{\mathbf{a}} \in I_S$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c} \qquad\qquad (x^{\mathbf{a}} - x^{\mathbf{b}}) + (x^{\mathbf{b}} - x^{\mathbf{c}}) = x^{\mathbf{a}} - x^{\mathbf{c}}$$

that is closed under *translation*

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{Z}_{\geq 0}^k & \longrightarrow & \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array}$$

Monomial map:

$$\begin{array}{rcl} \varphi : \Bbbk[x_1, \ldots, x_k] & \longrightarrow & \Bbbk[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

Definition

The *kernel* $\ker \pi$ is the relation $\sim$ on $\mathbb{Z}_{\geq 0}^k$ with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b}) \qquad x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S = \ker \varphi$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a} \qquad x^{\mathbf{a}} - x^{\mathbf{a}} = 0 \in I_S$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a} \qquad x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S \Rightarrow x^{\mathbf{b}} - x^{\mathbf{a}} \in I_S$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c} \qquad (x^{\mathbf{a}} - x^{\mathbf{b}}) + (x^{\mathbf{b}} - x^{\mathbf{c}}) = x^{\mathbf{a}} - x^{\mathbf{c}}$$

that is closed under *translation*

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c} \qquad x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S \Rightarrow x^{\mathbf{c}}(x^{\mathbf{a}} - x^{\mathbf{b}}) \in I_S$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{Z}_{\geq 0}^k & \longrightarrow & \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array}$$

Monomial map:

$$\begin{array}{rcl} \varphi : \Bbbk[x_1, \ldots, x_k] & \longrightarrow & \Bbbk[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

Example

$S = \langle 6, 9, 20 \rangle$:

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism:

$$\begin{aligned} \pi : \mathbb{Z}_{\geq 0}^k &\longrightarrow \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} &\longmapsto a_1 n_1 + \cdots + a_k n_k \end{aligned}$$

Monomial map:

$$\begin{aligned} \varphi : \Bbbk[x_1, \ldots, x_k] &\longrightarrow \Bbbk[w] \\ x_i &\longmapsto w^{n_i} \end{aligned}$$

Example

$S = \langle 6, 9, 20 \rangle$: $\quad I_S = \langle x^3 - y^2, x^4 y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{Z}_{\geq 0}^k & \longrightarrow & \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array}$$

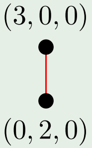
Monomial map:

$$\begin{array}{rcl} \varphi : \Bbbk[x_1, \ldots, x_k] & \longrightarrow & \Bbbk[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

Example

$S = \langle 6, 9, 20 \rangle$: $\quad I_S = \langle x^3 - y^2, x^4 y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$

Z(18):



Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k_{\geq 0}$$

Factorization homomorphism:

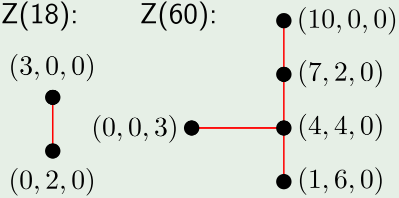
$$\begin{aligned} \pi : \mathbb{Z}^k_{\geq 0} &\longrightarrow \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} &\longmapsto a_1 n_1 + \cdots + a_k n_k \end{aligned}$$

Monomial map:

$$\begin{aligned} \varphi : \mathbb{k}[x_1, \ldots, x_k] &\longrightarrow \mathbb{k}[w] \\ x_i &\longmapsto w^{n_i} \end{aligned}$$

Example

$S = \langle 6, 9, 20 \rangle$: $\quad I_S = \langle x^3 - y^2, \, x^4 y^4 - z^3 \rangle \subseteq \mathbb{k}[x, y, z]$



Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism:

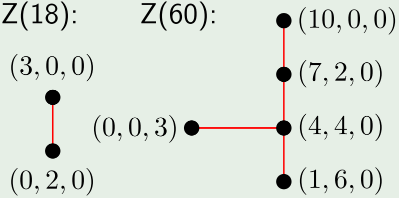
$$\begin{aligned} \pi : \mathbb{Z}_{\geq 0}^k &\longrightarrow \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} &\longmapsto a_1 n_1 + \cdots + a_k n_k \end{aligned}$$

Monomial map:

$$\begin{aligned} \varphi : \Bbbk[x_1, \ldots, x_k] &\longrightarrow \Bbbk[w] \\ x_i &\longmapsto w^{n_i} \end{aligned}$$

Example

$S = \langle 6, 9, 20 \rangle$: $\quad I_S = \langle x^3 - y^2, x^4 y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$



$$x^7 y^2 - x^4 y^4 = x^4 y^2 (x^3 - y^2)$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{Z}_{\geq 0}^k & \longrightarrow & \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array}$$

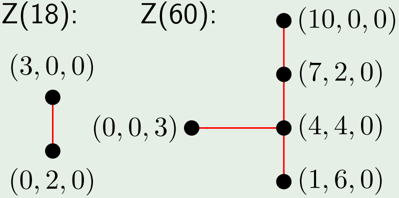
Monomial map:

$$\begin{array}{rcl} \varphi : \Bbbk[x_1, \ldots, x_k] & \longrightarrow & \Bbbk[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

Example

$S = \langle 6, 9, 20 \rangle$: $\quad I_S = \langle x^3 - y^2, x^4y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$

Z(18): Z(60):



$$x^7y^2 - x^4y^4 = x^4y^2(x^3 - y^2)$$

$$x^7y^2 - z^3 = (x^7y^2 - x^4y^4) + (x^4y^4 - z^3)$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{Z}_{\geq 0}^k & \longrightarrow & \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array}$$

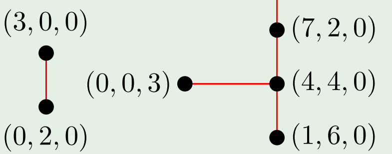
Monomial map:

$$\begin{array}{rcl} \varphi : \Bbbk[x_1, \ldots, x_k] & \longrightarrow & \Bbbk[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

Example

$S = \langle 6, 9, 20 \rangle$: $\quad I_S = \langle x^3 - y^2, x^4y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$

Z(18): Z(60):



$$x^7y^2 - x^4y^4 = x^4y^2(x^3 - y^2)$$

$$x^7y^2 - z^3 = (x^7y^2 - x^4y^4) + (x^4y^4 - z^3)$$

Generating set for I_S $\Leftrightarrow$ Z(n) connected for all $n \in S$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism:

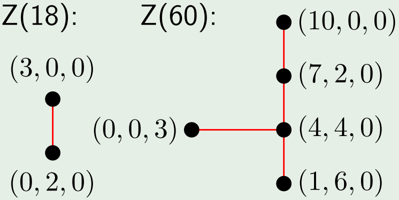
$$\begin{aligned} \pi : \mathbb{Z}_{\geq 0}^k &\longrightarrow \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} &\longmapsto a_1 n_1 + \cdots + a_k n_k \end{aligned}$$

Monomial map:

$$\begin{aligned} \varphi : \mathbb{k}[x_1, \ldots, x_k] &\longrightarrow \mathbb{k}[w] \\ x_i &\longmapsto w^{n_i} \end{aligned}$$

Example

$S = \langle 6, 9, 20 \rangle$: $\quad I_S = \langle x^3 - y^2, x^4 y^4 - z^3 \rangle \subseteq \mathbb{k}[x, y, z]$



Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k_{\geq 0}$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{Z}^k_{\geq 0} & \longrightarrow & \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array}$$

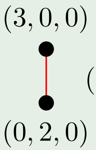
Monomial map:

$$\begin{array}{rcl} \varphi : \Bbbk[x_1, \ldots, x_k] & \longrightarrow & \Bbbk[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

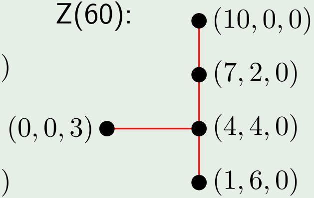
Example

$S = \langle 6, 9, 20 \rangle$: $\quad I_S = \langle x^3 - y^2, x^4 y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$

Z(18):



Z(60):



All minimal generating sets:

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k_{\geq 0}$$

Factorization homomorphism:

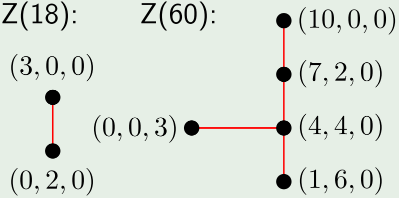
$$\begin{array}{rcl} \pi : \mathbb{Z}^k_{\geq 0} & \longrightarrow & \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array}$$

Monomial map:

$$\begin{array}{rcl} \varphi : \Bbbk[x_1, \ldots, x_k] & \longrightarrow & \Bbbk[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

Example

$S = \langle 6, 9, 20 \rangle$: $\quad I_S = \langle x^3 - y^2, \, x^4 y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$



All minimal generating sets:

$$\begin{aligned} I_S &= \langle x^3 - y^2, \, x^{10} - z^3 \rangle \\ &= \langle x^3 - y^2, \, x^7 y^2 - z^3 \rangle \\ &= \langle x^3 - y^2, \, x^4 y^4 - z^3 \rangle \\ &= \langle x^3 - y^2, \, x^6 y - z^3 \rangle \end{aligned}$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k_{\geq 0}$$

Factorization homomorphism:

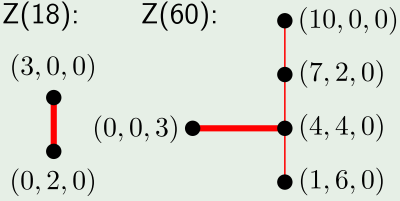
$$\begin{array}{rcl} \pi : \mathbb{Z}^k_{\geq 0} & \longrightarrow & \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array}$$

Monomial map:

$$\begin{array}{rcl} \varphi : \Bbbk[x_1, \ldots, x_k] & \longrightarrow & \Bbbk[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

Example

$S = \langle 6, 9, 20 \rangle$: $\quad I_S = \langle x^3 - y^2, x^4y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$



All minimal generating sets:

$$\begin{aligned} I_S &= \langle x^3 - y^2, x^{10} - z^3 \rangle \\ &= \langle x^3 - y^2, x^7y^2 - z^3 \rangle \\ &= \langle x^3 - y^2, x^4y^4 - z^3 \rangle \\ &= \langle x^3 - y^2, x^6y - z^3 \rangle \end{aligned}$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{Z}_{\geq 0}^k & \longrightarrow & \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array}$$

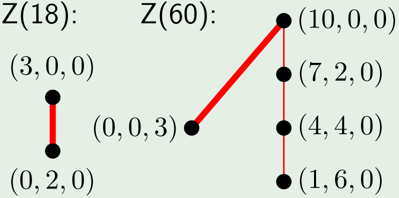
Monomial map:

$$\begin{array}{rcl} \varphi : \Bbbk[x_1, \ldots, x_k] & \longrightarrow & \Bbbk[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

Example

$S = \langle 6, 9, 20 \rangle$: $\quad I_S = \langle x^3 - y^2,\ x^4y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$

Z(18): Z(60):



All minimal generating sets:

$$\begin{aligned} I_S &= \langle x^3 - y^2,\ x^{10} - z^3 \rangle \\ &= \langle x^3 - y^2,\ x^7y^2 - z^3 \rangle \\ &= \langle x^3 - y^2,\ x^4y^4 - z^3 \rangle \\ &= \langle x^3 - y^2,\ x^6y - z^3 \rangle \end{aligned}$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k_{\geq 0}$$

Factorization homomorphism:

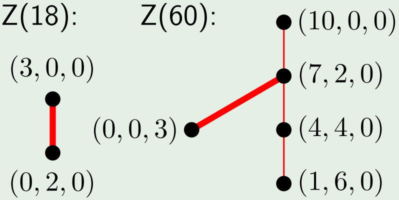
$$\begin{aligned} \pi : \mathbb{Z}^k_{\geq 0} &\longrightarrow \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} &\longmapsto a_1 n_1 + \cdots + a_k n_k \end{aligned}$$

Monomial map:

$$\begin{aligned} \varphi : \Bbbk[x_1, \ldots, x_k] &\longrightarrow \Bbbk[w] \\ x_i &\longmapsto w^{n_i} \end{aligned}$$

Example

$S = \langle 6, 9, 20 \rangle$: $\quad I_S = \langle x^3 - y^2,\ x^4y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$



All minimal generating sets:

$$\begin{aligned} I_S &= \langle x^3 - y^2,\ x^{10} - z^3 \rangle \\ &= \langle x^3 - y^2,\ x^7y^2 - z^3 \rangle \\ &= \langle x^3 - y^2,\ x^4y^4 - z^3 \rangle \\ &= \langle x^3 - y^2,\ x^6y - z^3 \rangle \end{aligned}$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{Z}_{\geq 0}^k & \longrightarrow & \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array}$$

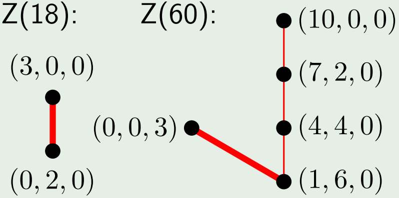
Monomial map:

$$\begin{array}{rcl} \varphi : \Bbbk[x_1, \ldots, x_k] & \longrightarrow & \Bbbk[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

Example

$S = \langle 6, 9, 20 \rangle$: $\quad I_S = \langle x^3 - y^2, x^4y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$

Z(18): Z(60):



All minimal generating sets:

$$\begin{aligned} I_S &= \langle x^3 - y^2, x^{10} - z^3 \rangle \\ &= \langle x^3 - y^2, x^7y^2 - z^3 \rangle \\ &= \langle x^3 - y^2, x^4y^4 - z^3 \rangle \\ &= \langle x^3 - y^2, x^6y - z^3 \rangle \end{aligned}$$

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{Z}_{\geq 0}^k & \longrightarrow & \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array}$$

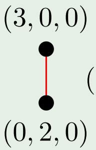
Monomial map:

$$\begin{array}{rcl} \varphi : \Bbbk[x_1, \ldots, x_k] & \longrightarrow & \Bbbk[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

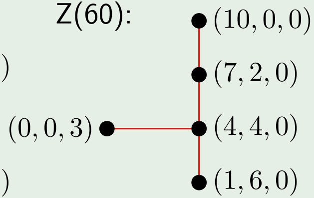
Example

$S = \langle 6, 9, 20 \rangle$: $\quad I_S = \langle x^3 - y^2, x^4y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$

Z(18):



Z(60):



All minimal generating sets:

$$\begin{aligned} I_S &= \langle x^3 - y^2, x^{10} - z^3 \rangle \\ &= \langle x^3 - y^2, x^7y^2 - z^3 \rangle \\ &= \langle x^3 - y^2, x^4y^4 - z^3 \rangle \\ &= \langle x^3 - y^2, x^6y - z^3 \rangle \end{aligned}$$

Minimal presentations and Betti elements

$$S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0} \qquad \pi : \mathbb{Z}_{\geq 0}^k \longrightarrow S$$

Minimal presentations and Betti elements

$$S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0} \qquad \pi : \mathbb{Z}_{\geq 0}^k \longrightarrow S$$

A larger example: $S = \langle 13, 44, 106, 120 \rangle$

Minimal presentations and Betti elements

$$S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0} \qquad \pi : \mathbb{Z}_{\geq 0}^k \longrightarrow S$$

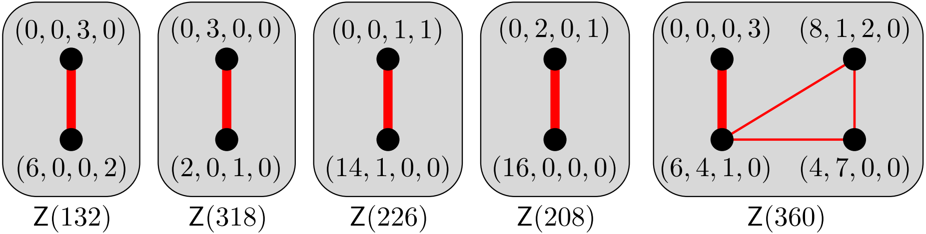
A larger example: $S = \langle 13, 44, 106, 120 \rangle$

$I_S = \langle x_1^6 x_4^2 - x_3^3,\ x_1^2 x_3 - x_2^3,\ x_1^{14} x_2 - x_3 x_4,\ x_1^{16} - x_2^2 x_4,\ x_1^6 x_2^4 x_3 - x_4^3 \rangle$

$$S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0} \qquad \pi : \mathbb{Z}_{\geq 0}^k \longrightarrow S$$

A larger example: $S = \langle 13, 44, 106, 120 \rangle$

$$I_S = \langle x_1^6 x_4^2 - x_3^3,\ x_1^2 x_3 - x_2^3,\ x_1^{14} x_2 - x_3 x_4,\ x_1^{16} - x_2^2 x_4,\ x_1^6 x_2^4 x_3 - x_4^3 \rangle$$

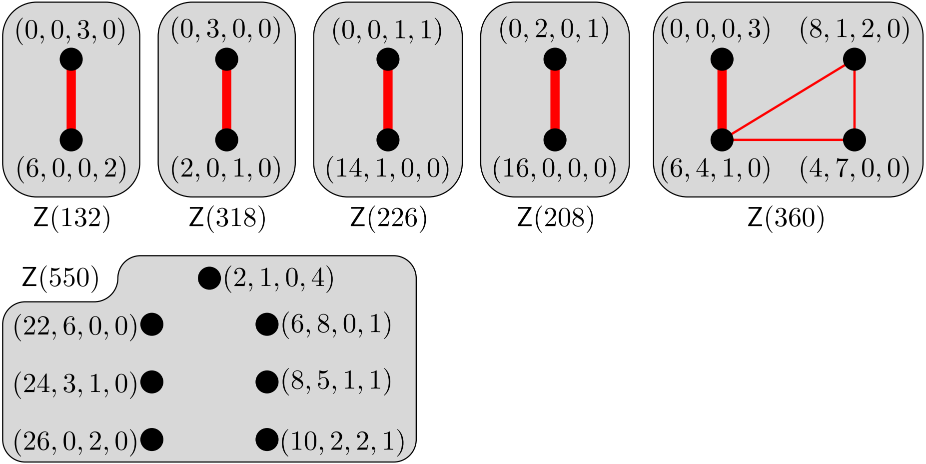


Minimal presentations and Betti elements

$$S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0} \qquad \pi : \mathbb{Z}^k_{\geq 0} \longrightarrow S$$

A larger example: $S = \langle 13, 44, 106, 120 \rangle$

$I_S = \langle x_1^6 x_4^2 - x_3^3,\ x_1^2 x_3 - x_2^3,\ x_1^{14} x_2 - x_3 x_4,\ x_1^{16} - x_2^2 x_4,\ x_1^6 x_2^4 x_3 - x_4^3 \rangle$

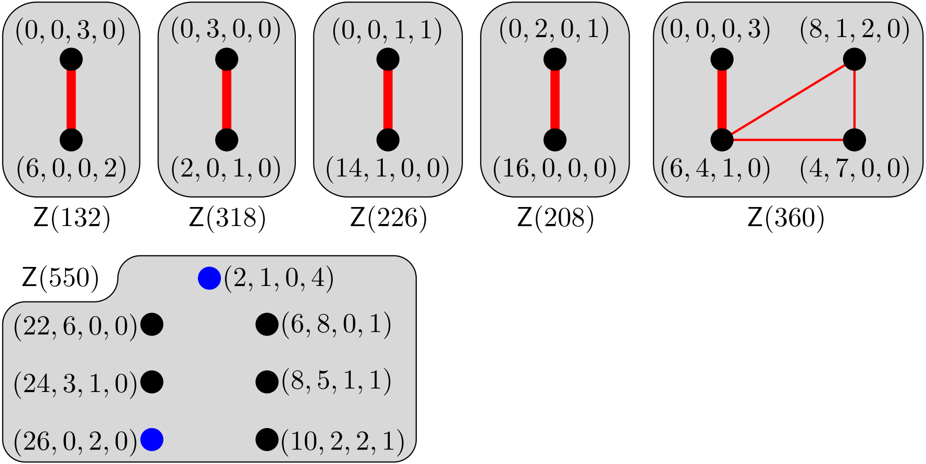


Minimal presentations and Betti elements

$$S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0} \qquad \pi : \mathbb{Z}_{\geq 0}^k \longrightarrow S$$

A larger example: $S = \langle 13, 44, 106, 120 \rangle$

$$I_S = \langle x_1^6 x_4^2 - x_3^3,\ x_1^2 x_3 - x_2^3,\ x_1^{14} x_2 - x_3 x_4,\ x_1^{16} - x_2^2 x_4,\ x_1^6 x_2^4 x_3 - x_4^3 \rangle$$

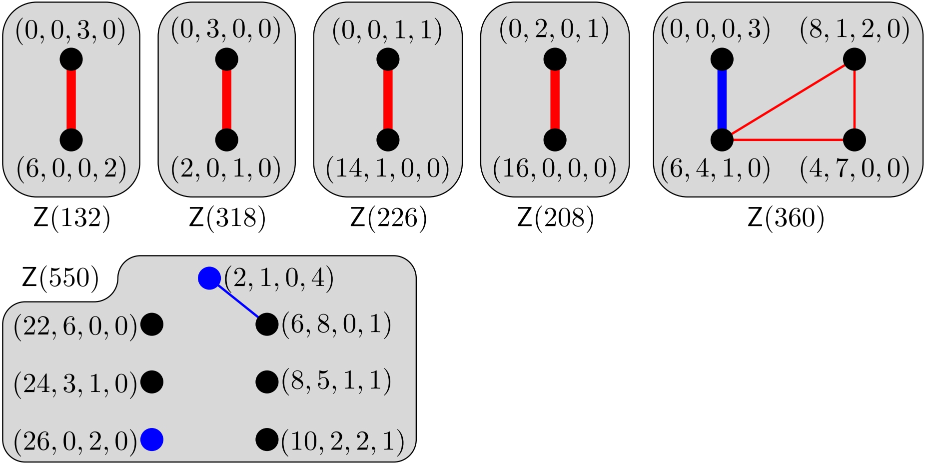


Minimal presentations and Betti elements

$$S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0} \qquad \pi : \mathbb{Z}_{\geq 0}^k \longrightarrow S$$

A larger example: $S = \langle 13, 44, 106, 120 \rangle$

$$I_S = \langle x_1^6 x_4^2 - x_3^3,\ x_1^2 x_3 - x_2^3,\ x_1^{14} x_2 - x_3 x_4,\ x_1^{16} - x_2^2 x_4,\ x_1^6 x_2^4 x_3 - x_4^3 \rangle$$

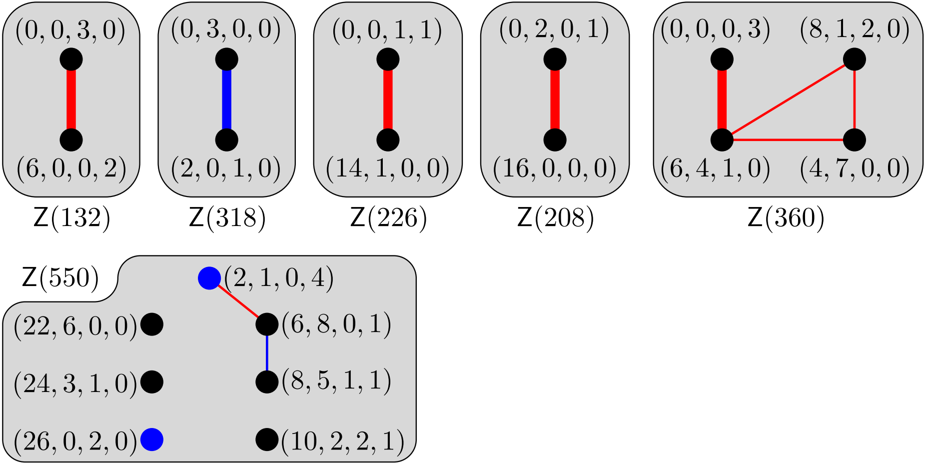


Minimal presentations and Betti elements

$$S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0} \qquad \pi : \mathbb{Z}_{\geq 0}^k \longrightarrow S$$

A larger example: $S = \langle 13, 44, 106, 120 \rangle$

$$I_S = \langle x_1^6 x_4^2 - x_3^3,\ x_1^2 x_3 - x_2^3,\ x_1^{14} x_2 - x_3 x_4,\ x_1^{16} - x_2^2 x_4,\ x_1^6 x_2^4 x_3 - x_4^3 \rangle$$

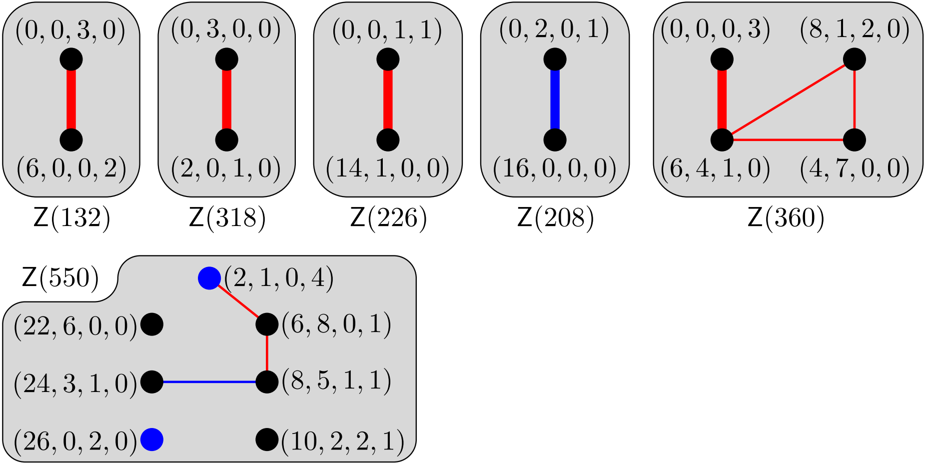


Minimal presentations and Betti elements

$$S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0} \qquad \pi : \mathbb{Z}_{\geq 0}^k \longrightarrow S$$

A larger example: $S = \langle 13, 44, 106, 120 \rangle$

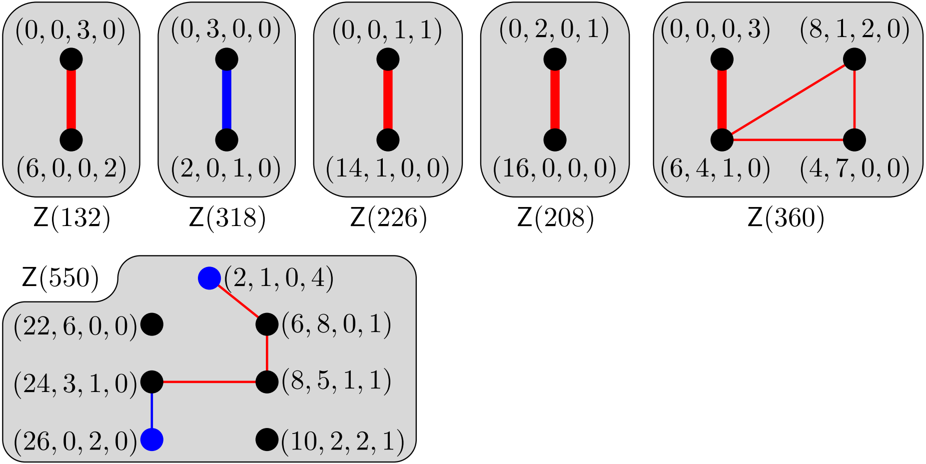
$$I_S = \langle x_1^6 x_4^2 - x_3^3,\; x_1^2 x_3 - x_2^3,\; x_1^{14} x_2 - x_3 x_4,\; x_1^{16} - x_2^2 x_4,\; x_1^6 x_2^4 x_3 - x_4^3 \rangle$$



$$S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0} \qquad \pi : \mathbb{Z}^k_{\geq 0} \longrightarrow S$$

A larger example: $S = \langle 13, 44, 106, 120 \rangle$

$$I_S = \langle x_1^6 x_4^2 - x_3^3,\ x_1^2 x_3 - x_2^3,\ x_1^{14} x_2 - x_3 x_4,\ x_1^{16} - x_2^2 x_4,\ x_1^6 x_2^4 x_3 - x_4^3 \rangle$$

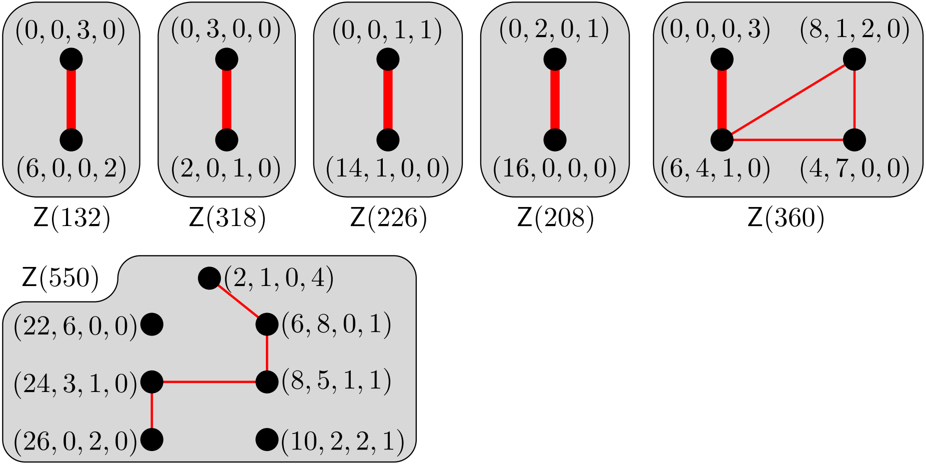


Minimal presentations and Betti elements

$$S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0} \qquad \pi : \mathbb{Z}^k_{\geq 0} \longrightarrow S$$

A larger example: $S = \langle 13, 44, 106, 120 \rangle$

$$I_S = \langle x_1^6 x_4^2 - x_3^3,\ x_1^2 x_3 - x_2^3,\ x_1^{14} x_2 - x_3 x_4,\ x_1^{16} - x_2^2 x_4,\ x_1^6 x_2^4 x_3 - x_4^3 \rangle$$

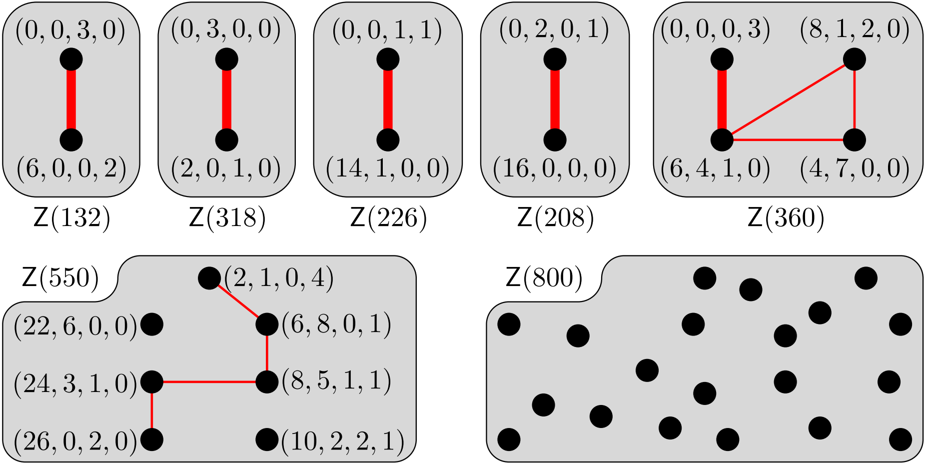


Minimal presentations and Betti elements

$$S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0} \qquad \pi : \mathbb{Z}_{\geq 0}^k \longrightarrow S$$

A larger example: $S = \langle 13, 44, 106, 120 \rangle$

$$I_S = \langle x_1^6 x_4^2 - x_3^3,\ x_1^2 x_3 - x_2^3,\ x_1^{14} x_2 - x_3 x_4,\ x_1^{16} - x_2^2 x_4,\ x_1^6 x_2^4 x_3 - x_4^3 \rangle$$

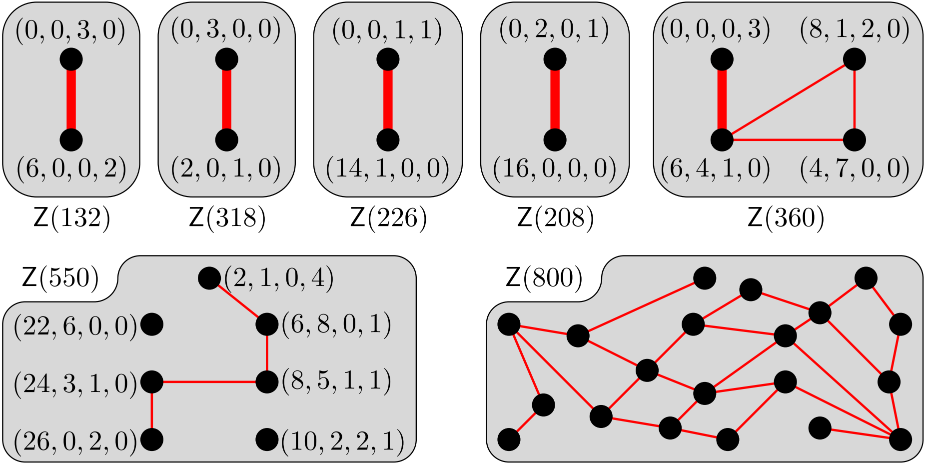


Minimal presentations and Betti elements

$$S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0} \qquad \pi : \mathbb{Z}_{\geq 0}^k \longrightarrow S$$

A larger example: $S = \langle 13, 44, 106, 120 \rangle$

$I_S = \langle x_1^6 x_4^2 - x_3^3,\ x_1^2 x_3 - x_2^3,\ x_1^{14} x_2 - x_3 x_4,\ x_1^{16} - x_2^2 x_4,\ x_1^6 x_2^4 x_3 - x_4^3 \rangle$

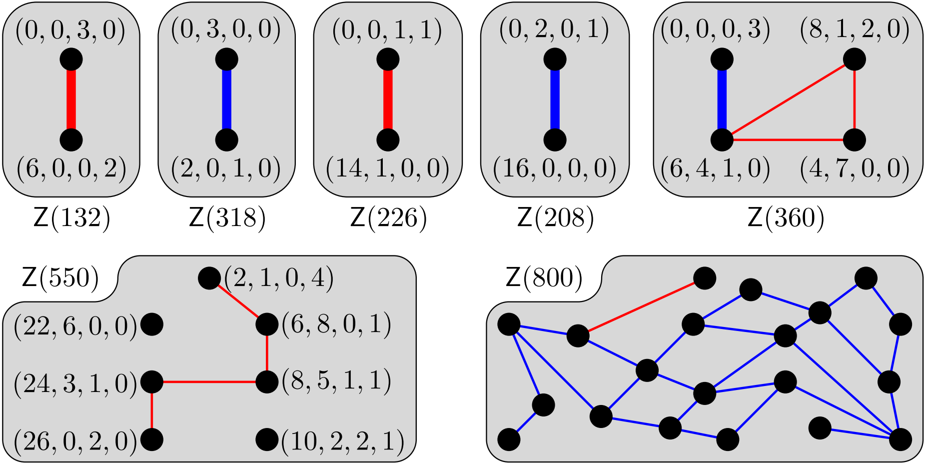


Minimal presentations and Betti elements

$$S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0} \qquad \pi : \mathbb{Z}_{\geq 0}^k \longrightarrow S$$

A larger example: $S = \langle 13, 44, 106, 120 \rangle$

$$I_S = \langle x_1^6 x_4^2 - x_3^3,\ x_1^2 x_3 - x_2^3,\ x_1^{14} x_2 - x_3 x_4,\ x_1^{16} - x_2^2 x_4,\ x_1^6 x_2^4 x_3 - x_4^3 \rangle$$

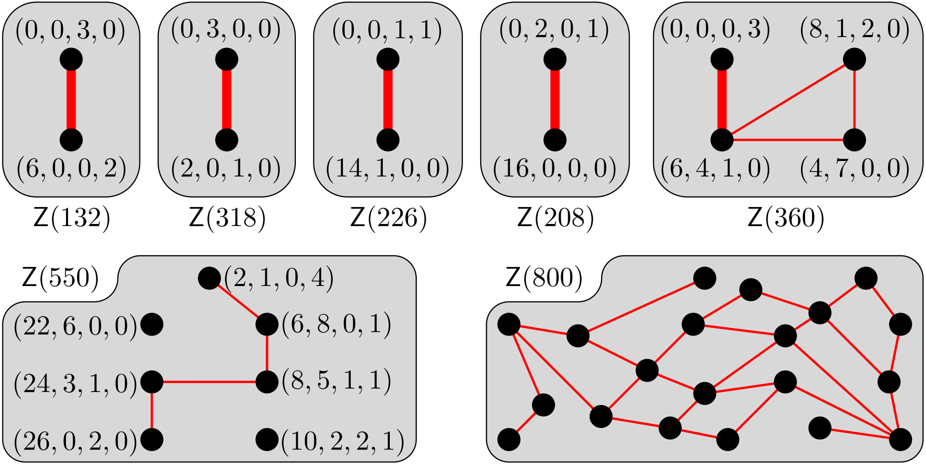


Minimal presentations and Betti elements

$$S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0} \qquad \pi : \mathbb{Z}^k_{\geq 0} \longrightarrow S$$

A larger example: $S = \langle 13, 44, 106, 120 \rangle$

$$I_S = \langle x_1^6 x_4^2 - x_3^3,\ x_1^2 x_3 - x_2^3,\ x_1^{14} x_2 - x_3 x_4,\ x_1^{16} - x_2^2 x_4,\ x_1^6 x_2^4 x_3 - x_4^3 \rangle$$



Minimal trades and Kunz posets

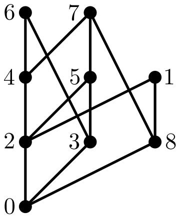
Question

How can one recover minimal trade structure from the Kunz poset?

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

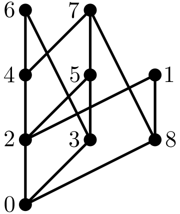


Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

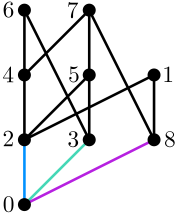
$$\mathrm{Ap}(S) = \{0, a_1, a_2, \ldots, a_8\}$$



Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?



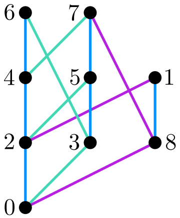
$$\mathrm{Ap}(S) = \{0, a_1, a_2, \ldots, a_8\}$$

$$S = \langle 9, a_2, a_3, a_8 \rangle$$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?



$\mathrm{Ap}(S) = \{0, a_1, a_2, \ldots, a_8\}$

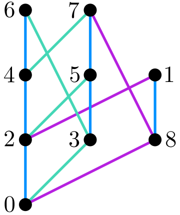
$S = \langle 9, a_2, a_3, a_8 \rangle$

Cover relations: add a generator

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?



$\mathrm{Ap}(S) = \{0, a_1, a_2, \ldots, a_8\}$

$S = \langle 9, a_2, a_3, a_8 \rangle$

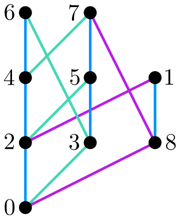
Cover relations: add a generator

$\mathsf{Z}(a_6) = \{(0, 3, 0, 0), (0, 0, 2, 0)\}$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?



$\mathrm{Ap}(S) = \{0, a_1, a_2, \ldots, a_8\}$

$S = \langle 9, a_2, a_3, a_8 \rangle$

Cover relations: add a generator

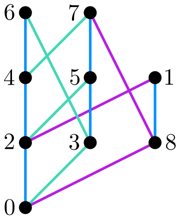
$\mathsf{Z}(a_6) = \{(0,3,0,0), (0,0,2,0)\}$

2 "inner" minimal trades:
$(0,3,0,0) \sim (0,0,2,0)$ (at a_6)
$(0,2,1,0) \sim (0,0,0,2)$ (at a_7)

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?



$\mathrm{Ap}(S) = \{0, a_1, a_2, \ldots, a_8\}$

$S = \langle 9, a_2, a_3, a_8 \rangle$

Cover relations: add a generator

$\mathsf{Z}(a_6) = \{(0,3,0,0), (0,0,2,0)\}$

2 "inner" minimal trades:
$(0,3,0,0) \sim (0,0,2,0)$ (at a_6)
$(0,2,1,0) \sim (0,0,0,2)$ (at a_7)

Moral: can recover

- $\mathsf{Z}(a)$ for $a \in \mathrm{Ap}(S)$
- (minimal) trades at $a \in \mathrm{Ap}(S)$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

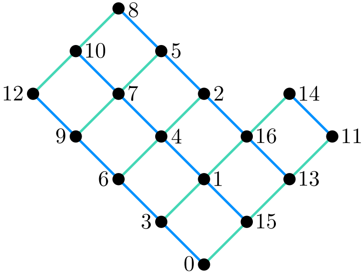
Key fact: each Betti element b has the form $a_i + n_j$ for some i, j

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



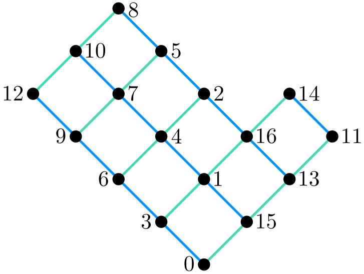
$$S = \langle 17, a_3, a_{15} \rangle$$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 17, a_3, a_{15} \rangle$

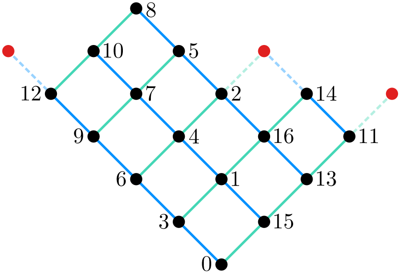
3 minimal trades, none in $\mathrm{Ap}(S)$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 17, a_3, a_{15} \rangle$

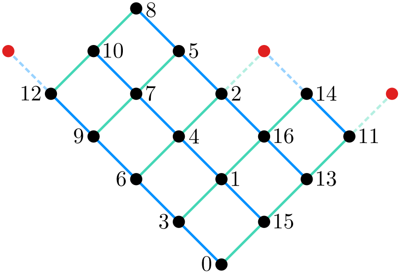
3 minimal trades, none in $\mathrm{Ap}(S)$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 17, a_3, a_{15} \rangle$

3 minimal trades, none in $\mathrm{Ap}(S)$

$a_{12} + a_3$:

$a_{11} + a_{15}$:

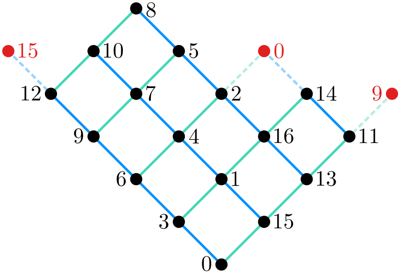
$a_2 + a_{15}$:

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 17, a_3, a_{15} \rangle$

3 minimal trades, none in Ap(S)

$a_{12} + a_3$:

$a_{11} + a_{15}$:

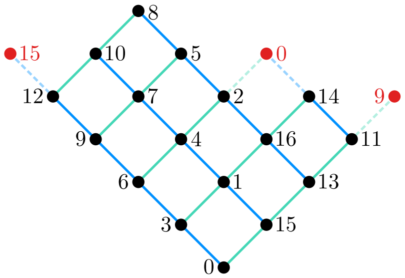
$a_2 + a_{15}$:

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 17, a_3, a_{15} \rangle$

3 minimal trades, none in $\mathrm{Ap}(S)$

$a_{12} + a_3$: $(0, 5, 0) \sim (*, 0, 1)$

$a_{11} + a_{15}$:

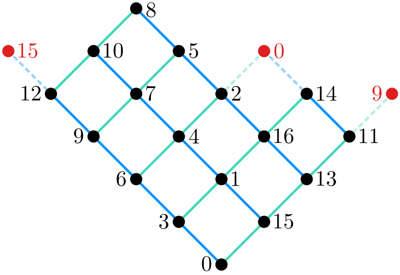
$a_2 + a_{15}$:

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 17, a_3, a_{15} \rangle$

3 minimal trades, none in Ap(S)

$a_{12} + a_3$: $(0, 5, 0) \sim (*, 0, 1)$

$a_{11} + a_{15}$: $(0, 0, 4) \sim (*, 3, 0)$

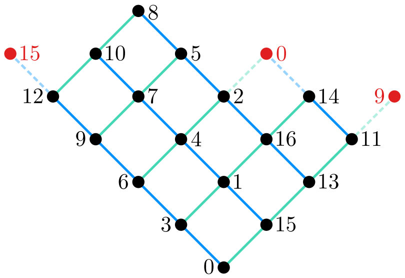
$a_2 + a_{15}$:

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 17, a_3, a_{15} \rangle$

3 minimal trades, none in $\mathrm{Ap}(S)$

$a_{12} + a_3$: $(0, 5, 0) \sim (*, 0, 1)$

$a_{11} + a_{15}$: $(0, 0, 4) \sim (*, 3, 0)$

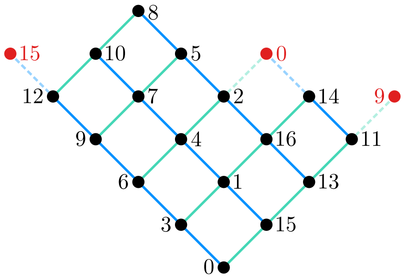
$a_2 + a_{15}$: $(0, 2, 3) \sim (*, 0, 0)$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 17, a_3, a_{15} \rangle$

3 minimal trades, none in $\mathrm{Ap}(S)$

$a_{12} + a_3$: $(0, 5, 0) \sim (*, 0, 1)$

$a_{11} + a_{15}$: $(0, 0, 4) \sim (*, 3, 0)$

$a_2 + a_{15}$: $(0, 2, 3) \sim (*, 0, 0)$

If an *Apéry set of unique expression*,

- factorizations of $a \in \mathrm{Ap}(S)$ form monomial staircase
- one "outer" minimal trade for each monomial generator

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

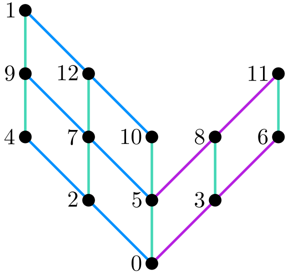
Key fact: each Betti element b has the form $a_i + n_j$ for some i, j

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



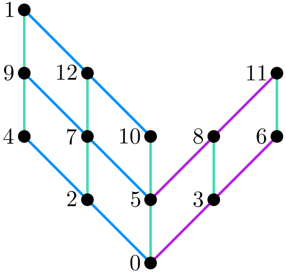
$$S = \langle 13, a_2, a_5, a_3 \rangle$$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 13, a_2, a_5, a_3 \rangle$

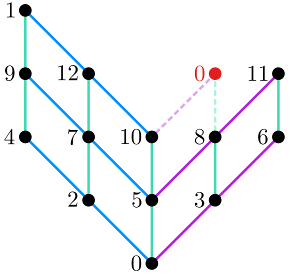
5 minimal trades, none in Ap(S)

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 13, a_2, a_5, a_3 \rangle$

5 minimal trades, none in $\mathrm{Ap}(S)$

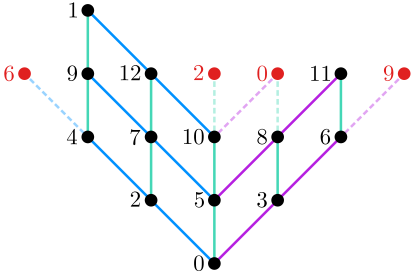
0: $(0, 0, 2, 1)$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 13, a_2, a_5, a_3 \rangle$

5 minimal trades, none in Ap(S)

0: $(0, 0, 2, 1)$

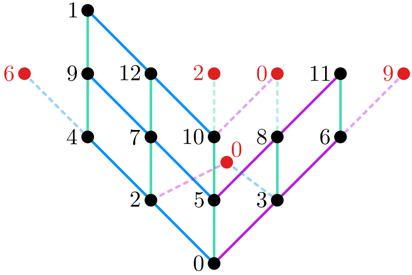
6: $(0, 3, 0, 0)$
2: $(0, 0, 3, 0)$
9: $(0, 0, 0, 3)$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 13, a_2, a_5, a_3 \rangle$

5 minimal trades, none in $\mathrm{Ap}(S)$

0: $(0, 0, 2, 1)$

6: $(0, 3, 0, 0)$
2: $(0, 0, 3, 0)$
9: $(0, 0, 0, 3)$

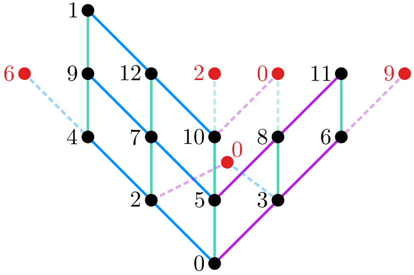
0: $(0, 1, 0, 1)$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 13, a_2, a_5, a_3 \rangle$

5 minimal trades, none in Ap(S)

0: $(0, 0, 2, 1)$

6: $(0, 3, 0, 0)$
2: $(0, 0, 3, 0)$
9: $(0, 0, 0, 3)$

0: $(0, 1, 0, 1)$

Need: decrementing any coordinate lands in Ap(S)

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

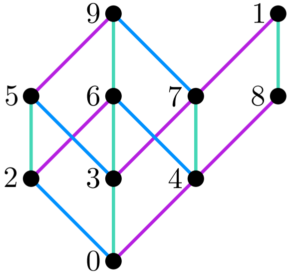
Key fact: each Betti element b has the form $a_i + n_j$ for some i, j

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



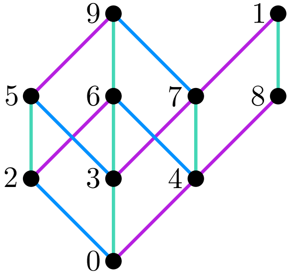
$$S = \langle 10, a_2, a_3, a_4 \rangle$$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 10, a_2, a_3, a_4 \rangle$

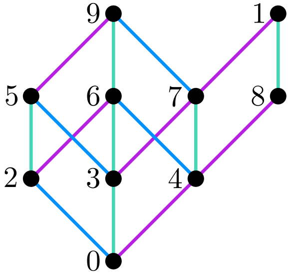
"inner" trade at a_6:
$(0, 0, 2, 0) \sim (0, 1, 0, 1)$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 10, a_2, a_3, a_4 \rangle$

"inner" trade at a_6:
$(0,0,2,0) \sim (0,1,0,1)$

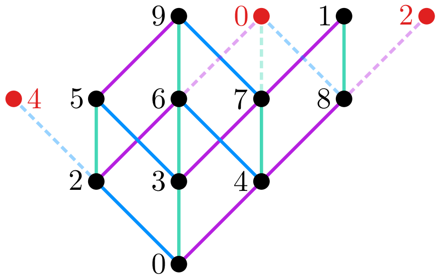
Candidates for "outer" trades:
$(0,0,2,1)$, $(0,1,0,2)$,
$(0,0,0,3)$, $(0,2,0,0)$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 10, a_2, a_3, a_4 \rangle$

"inner" trade at a_6:
$(0,0,2,0) \sim (0,1,0,1)$

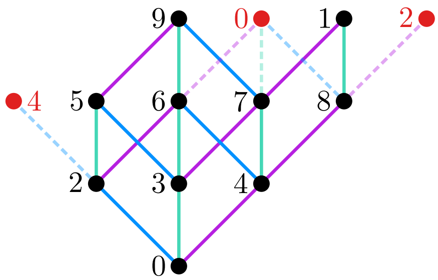
Candidates for "outer" trades:
$(0,0,2,1)$, $(0,1,0,2)$,
$(0,0,0,3)$, $(0,2,0,0)$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 10, a_2, a_3, a_4 \rangle$

"inner" trade at a_6:
$(0,0,2,0) \sim (0,1,0,1)$

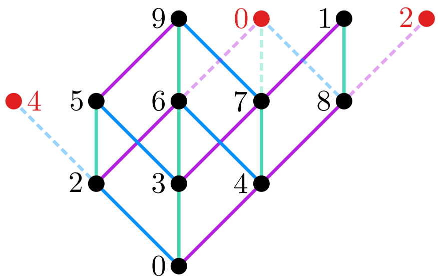
Candidates for "outer" trades:
$(0,0,2,1)$, $(0,1,0,2)$,
$(0,0,0,3)$, $(0,2,0,0)$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 10, a_2, a_3, a_4 \rangle$

"inner" trade at a_6:
$(0, 0, 2, 0) \sim (0, 1, 0, 1)$

Candidates for "outer" trades:
$(0, 0, 2, 1)$, $(0, 1, 0, 2)$,
$(0, 0, 0, 3)$, $(0, 2, 0, 0)$

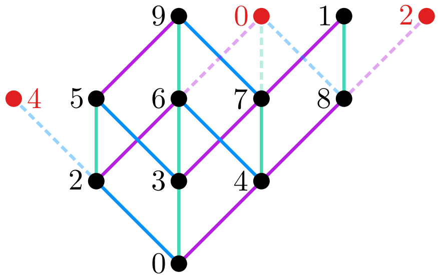
Moral: use **sets** of factorizations, avoids overcounting minimal trades

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 10, a_2, a_3, a_4 \rangle$

"inner" trade at a_6:
$(0,0,2,0) \sim (0,1,0,1)$

Candidates for "outer" trades:
$(0,0,2,1)$, $(0,1,0,2)$,
$(0,0,0,3)$, $(0,2,0,0)$

Moral: use **sets** of factorizations, avoids overcounting minimal trades

0: $\{(0,0,2,1),(0,1,0,2)\}$
2: $\{(0,0,0,3)\}$, 4: $\{(0,2,0,0)\}$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j

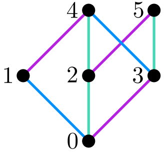
Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j

$$S = \langle 6, 7, 8, 9 \rangle$$

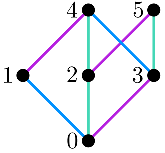


Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 6, 7, 8, 9 \rangle$

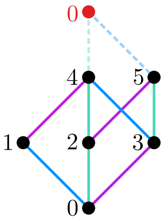
"inner" trade at a_4:
$(0, 0, 2, 0) \sim (0, 1, 0, 1)$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 6, 7, 8, 9 \rangle$

"inner" trade at a_4:
$(0, 0, 2, 0) \sim (0, 1, 0, 1)$

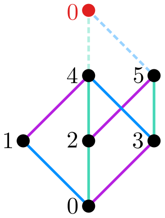
candidate for "outer" trade:
$(0, 0, 2, 1) \in \mathsf{Z}(25)$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 6, 7, 8, 9 \rangle$

"inner" trade at a_4:
$(0, 0, 2, 0) \sim (0, 1, 0, 1)$

candidate for "outer" trade:
$(0, 0, 2, 1) \in \mathsf{Z}(25)$

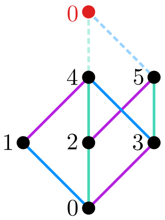
No trades in $\mathsf{Z}(25)$:
$\{(0, 0, 2, 1), (0, 1, 0, 2), (3, 1, 0, 0)\}$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each Betti element b has the form $a_i + n_j$ for some i, j



$S = \langle 6, 7, 8, 9 \rangle$

"inner" trade at a_4:
$(0, 0, 2, 0) \sim (0, 1, 0, 1)$

candidate for "outer" trade:
$(0, 0, 2, 1) \in \mathsf{Z}(25)$

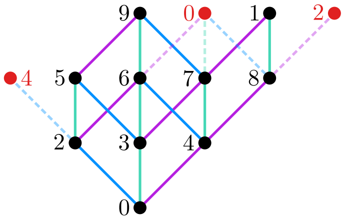
No trades in $\mathsf{Z}(25)$:
$\{(0, 0, 2, 1), (0, 1, 0, 2), (3, 1, 0, 0)\}$

The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \mathrm{supp}(B)$.

$S = \langle 10, a_2, a_3, a_4 \rangle$

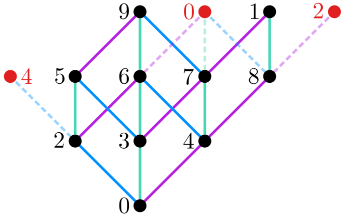


The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \mathrm{supp}(B)$.

$S = \langle 10, a_2, a_3, a_4 \rangle$



$B = \{(0, 0, 2, 1), (0, 1, 0, 2)\}$

$B - e_2 = \{(0, 0, 0, 2)\} = \mathsf{Z}(a_8)$

$B - e_3 = \{(0, 0, 1, 1)\} = \mathsf{Z}(a_7)$

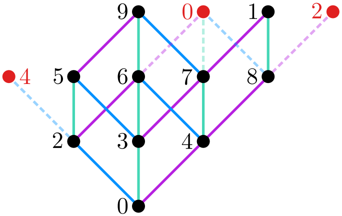
$B - e_4 = \{(0, 0, 2, 0), (0, 1, 0, 1)\}$
$= \mathsf{Z}(a_6)$

The main theorem

Definition

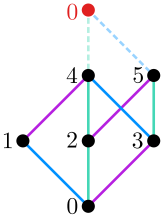
An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \operatorname{supp}(B)$.

$S = \langle 10, a_2, a_3, a_4 \rangle$



$B = \{(0,0,2,1),(0,1,0,2)\}$
$B - e_2 = \{(0,0,0,2)\} = \mathsf{Z}(a_8)$
$B - e_3 = \{(0,0,1,1)\} = \mathsf{Z}(a_7)$
$B - e_4 = \{(0,0,2,0),(0,1,0,1)\}$
$= \mathsf{Z}(a_6)$

$S = \langle 6, 7, 8, 9 \rangle$

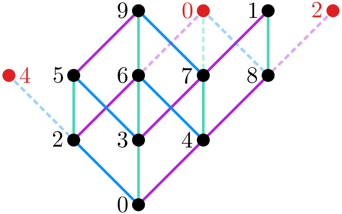


The main theorem

Definition

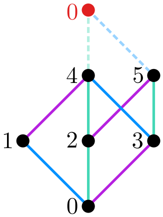
An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \operatorname{supp}(B)$.

$S = \langle 10, a_2, a_3, a_4 \rangle$



$B = \{(0,0,2,1),(0,1,0,2)\}$
$B - e_2 = \{(0,0,0,2)\} = \mathsf{Z}(a_8)$
$B - e_3 = \{(0,0,1,1)\} = \mathsf{Z}(a_7)$
$B - e_4 = \{(0,0,2,0),(0,1,0,1)\}$
$= \mathsf{Z}(a_6)$

$S = \langle 6, 7, 8, 9 \rangle$



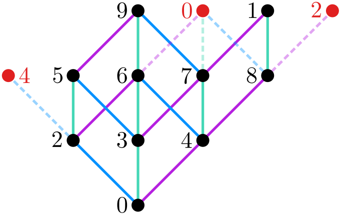
$B = \{(0,0,2,1)\}$?

The main theorem

Definition

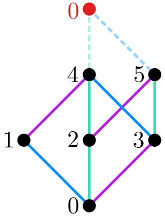
An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \mathrm{supp}(B)$.

$S = \langle 10, a_2, a_3, a_4 \rangle$



$B = \{(0,0,2,1), (0,1,0,2)\}$
$B - e_2 = \{(0,0,0,2)\} = \mathsf{Z}(a_8)$
$B - e_3 = \{(0,0,1,1)\} = \mathsf{Z}(a_7)$
$B - e_4 = \{(0,0,2,0), (0,1,0,1)\}$
$= \mathsf{Z}(a_6)$

$S = \langle 6, 7, 8, 9 \rangle$



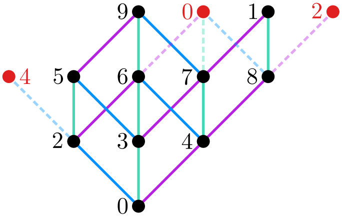
$B = \{(0,0,2,1)\}$?
$B - e_4 = \{(0,0,2,0)\} \subsetneq \mathsf{Z}(a_4)$

The main theorem

Definition

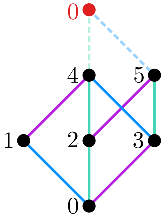
An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \mathrm{supp}(B)$.

$S = \langle 10, a_2, a_3, a_4 \rangle$



$B = \{(0,0,2,1),(0,1,0,2)\}$
$B - e_2 = \{(0,0,0,2)\} = \mathsf{Z}(a_8)$
$B - e_3 = \{(0,0,1,1)\} = \mathsf{Z}(a_7)$
$B - e_4 = \{(0,0,2,0),(0,1,0,1)\}$
$= \mathsf{Z}(a_6)$

$S = \langle 6, 7, 8, 9 \rangle$



$B = \{(0,0,2,1)\}$?
$B - e_4 = \{(0,0,2,0)\} \subsetneq \mathsf{Z}(a_4)$

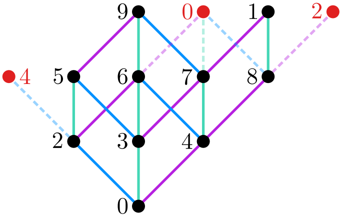
$B = \{(0,0,2,1),(0,1,0,2)\}$?

The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \mathrm{supp}(B)$.

$S = \langle 10, a_2, a_3, a_4 \rangle$



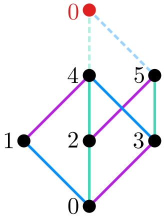
$B = \{(0,0,2,1),(0,1,0,2)\}$

$B - e_2 = \{(0,0,0,2)\} = \mathsf{Z}(a_8)$

$B - e_3 = \{(0,0,1,1)\} = \mathsf{Z}(a_7)$

$B - e_4 = \{(0,0,2,0),(0,1,0,1)\}$
$= \mathsf{Z}(a_6)$

$S = \langle 6, 7, 8, 9 \rangle$



$B = \{(0,0,2,1)\}$?

$B - e_4 = \{(0,0,2,0)\} \subsetneq \mathsf{Z}(a_4)$

$B = \{(0,0,2,1),(0,1,0,2)\}$?

$B - e_3 = \{(0,0,1,1)\} \not\subseteq \mathsf{Z}(a_i)$

The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \operatorname{supp}(B)$.

The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \mathrm{supp}(B)$.

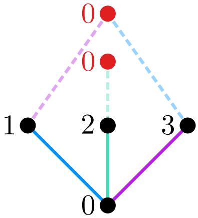
Another subtlety: distinct outer Betti elements can **coincide** for some S

The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \mathrm{supp}(B)$.

Another subtlety: distinct outer Betti elements can **coincide** for some S



$B_1 = \{(0, 0, 2, 0)\}$

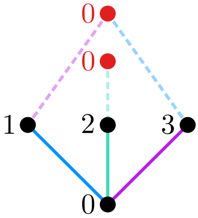
$B_2 = \{(0, 1, 0, 1)\}$

The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \mathrm{supp}(B)$.

Another subtlety: distinct outer Betti elements can **coincide** for some S



$S = \langle 4, 9, 14, 11 \rangle$

$B_1 = \{(0, 0, 2, 0)\}$

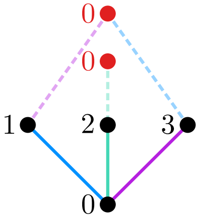
$B_2 = \{(0, 1, 0, 1)\}$

The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \operatorname{supp}(B)$.

Another subtlety: distinct outer Betti elements can **coincide** for some S



$S = \langle 4, 9, 14, 11 \rangle$

20: $(0,1,0,1), (5,0,0,0)$
28: $(0,0,2,0), (2,1,0,0), (5,0,0,0)$

$B_1 = \{(0,0,2,0)\}$

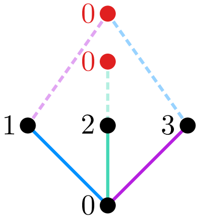
$B_2 = \{(0,1,0,1)\}$

The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \operatorname{supp}(B)$.

Another subtlety: distinct outer Betti elements can **coincide** for some S



$S = \langle 4, 9, 14, 11 \rangle$

20: $(0,1,0,1), (5,0,0,0)$
28: $(0,0,2,0), (2,1,0,0), (5,0,0,0)$

$S = \langle 4, 9, 10, 11 \rangle$

$B_1 = \{(0,0,2,0)\}$

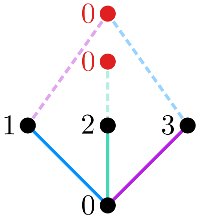
$B_2 = \{(0,1,0,1)\}$

The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \operatorname{supp}(B)$.

Another subtlety: distinct outer Betti elements can **coincide** for some S



$B_1 = \{(0,0,2,0)\}$

$B_2 = \{(0,1,0,1)\}$

$S = \langle 4, 9, 14, 11 \rangle$

20: $(0,1,0,1), (5,0,0,0)$
28: $(0,0,2,0), (2,1,0,0), (5,0,0,0)$

$S = \langle 4, 9, 10, 11 \rangle$

20: $(0,0,2,0), (0,1,0,1), (5,0,0,0)$

The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \mathrm{supp}(B)$.

The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \mathrm{supp}(B)$.

Theorem (Gomes–O.–Torres Davila)

If S has Kunz poset P, each minimal trade of S not ocurring in $\mathrm{Ap}(S)$ contains a factorization from a distinct outer Betti element of P.

In particular, if S, S' have identical Kunz poset, then I_S and $I_{S'}$ have the same number of minimal generators.

The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \mathrm{supp}(B)$.

Theorem (Gomes–O.–Torres Davila)

If S has Kunz poset P, each minimal trade of S not ocurring in $\mathrm{Ap}(S)$ *contains a factorization from a distinct outer Betti element of P.*

In particular, if S, S' have identical Kunz poset, then I_S and $I_{S'}$ have the same number of minimal generators.

Bonus:

- Algorithm for computing minimal presentations
  Runtime depends **only** on m!

The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \mathrm{supp}(B)$.

Theorem (Gomes–O.–Torres Davila)

If S has Kunz poset P, each minimal trade of S not ocurring in $\mathrm{Ap}(S)$ *contains a factorization from a distinct outer Betti element of P.*

In particular, if S, S' have identical Kunz poset, then I_S and $I_{S'}$ have the same number of minimal generators.

Bonus:

- Algorithm for computing minimal presentations
  Runtime depends **only** on m!
- New tool for classifying minimal presentations

The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \operatorname{supp}(B)$.

Theorem (Gomes–O.–Torres Davila)

If S has Kunz poset P, each minimal trade of S not ocurring in $\operatorname{Ap}(S)$ *contains a factorization from a distinct outer Betti element of P.*

In particular, if S, S' have identical Kunz poset, then I_S and $I_{S'}$ have the same number of minimal generators.

Bonus:

- Algorithm for computing minimal presentations
  Runtime depends **only** on m!
- New tool for classifying minimal presentations
  $\#$ minimal trades $\le \binom{m}{2}$

The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \mathrm{supp}(B)$.

Theorem (Gomes–O.–Torres Davila)

If S has Kunz poset P, each minimal trade of S not ocurring in $\mathrm{Ap}(S)$ *contains a factorization from a distinct outer Betti element of P.*

In particular, if S, S' have identical Kunz poset, then I_S and $I_{S'}$ have the same number of minimal generators.

Bonus:

- Algorithm for computing minimal presentations
  Runtime depends **only** on m!
- New tool for classifying minimal presentations
  $\#$ minimal trades $\le \binom{m}{2}$
  For $m = 4$, $\quad \#$ minimal trades $\in \{1, 2, 3, 6\}$

References

W. Bruns, P. García-Sánchez, C. O'Neill, D. Wilburne (2020)
Wilf's conjecture in fixed multiplicity
International Journal of Algebra and Computation **30** (2020), no. 4, 861–882. (arXiv:1903.04342)

N. Kaplan, C. O'Neill, (2021)
Numerical semigroups, polyhedra, and posets I: the group cone
Combinatorial Theory **1** (2021), #19. (arXiv:1912.03741)

T. Gomes, C. O'Neill, E. Torres Davila (2022)
Numerical semigroups, polyhedra, and posets III: minimal presentations and face dimension.
under review. (arXiv:2009.05921)

M. Delgado, P. García-Sánchez, J. Morais
GAP Numerical Semigroups Package
http://www.gap-system.org/Packages/numericalsgps.html.

References

W. Bruns, P. García-Sánchez, C. O'Neill, D. Wilburne (2020)
Wilf's conjecture in fixed multiplicity
International Journal of Algebra and Computation **30** (2020), no. 4, 861–882. (arXiv:1903.04342)

N. Kaplan, C. O'Neill, (2021)
Numerical semigroups, polyhedra, and posets I: the group cone
Combinatorial Theory **1** (2021), #19. (arXiv:1912.03741)

T. Gomes, C. O'Neill, E. Torres Davila (2022)
Numerical semigroups, polyhedra, and posets III: minimal presentations and face dimension.
under review. (arXiv:2009.05921)

M. Delgado, P. García-Sánchez, J. Morais
GAP Numerical Semigroups Package
http://www.gap-system.org/Packages/numericalsgps.html.

Thanks!