Numerical semigroups, minimal presentations, and posets

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$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

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Multiplicity: $\mathsf{m}(S) =$ smallest nonzero element

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

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The Apéry set is a "one stop shop" for computation.

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Theorem

If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup S with $\mathrm{Ap}(S) = A$ if and only if

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Big idea: the inequalities "$a_i + a_j \geq a_{i+j}$" to define a **cone** C_m.

Kunz cone

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The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

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$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

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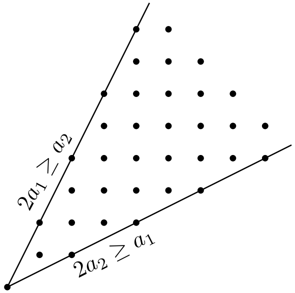

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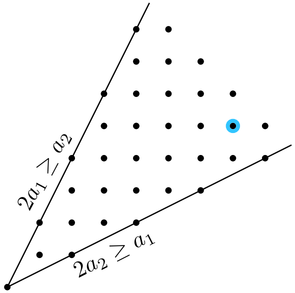

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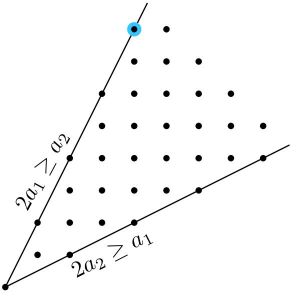

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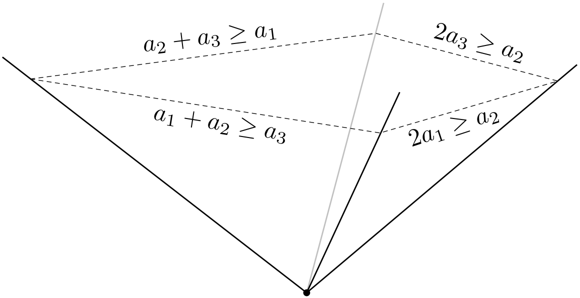

Faces of the Kunz cone

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When are numerical semigroups in (the relative interior of) the same face?

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Big picture: "moduli space" approach for studying XYZ's

- Define a space with XYZ's as points
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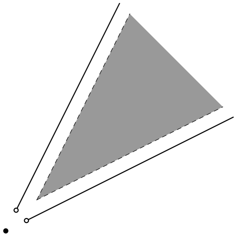
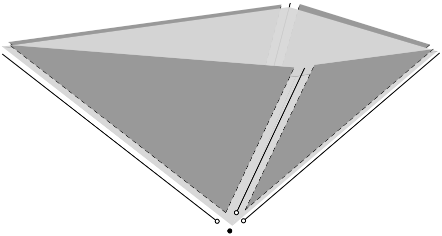
More interesting example: C_m





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Geometrically: "most" numerical semigroups with $\mathrm{m}(S) = m$ are MED

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Motivation: $S \in \mathrm{Int}(C_m)$ if and only if S has *max embedding dimension*

If $S = \langle n_1, \ldots, n_k \rangle$ with $\mathrm{e}(S) = k$, then

$$n_i \not\equiv n_j \bmod n_1 \quad \Longrightarrow \quad \mathrm{e}(S) \le \mathrm{m}(S)$$

If $\mathrm{e}(S) = \mathrm{m}(S)$, then S has *max embedding dimension*

$$S = \langle m, a_1, \ldots, a_{m-1} \rangle \quad \text{where} \quad \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\}$$

Geometrically: "most" numerical semigroups with $\mathrm{m}(S) = m$ are MED

What about the other faces?

Faces of the Kunz cone

Question

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Example: $S = \langle 4, 10, 11, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 11\}$$
$$a_1 = 13, \quad a_2 = 10, \quad a_3 = 11$$

$$2a_1 > a_2 \qquad a_1 + a_2 > a_3$$
$$2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

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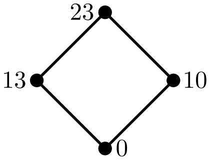
The *Apéry poset* of S: define $a \preceq a'$ whenever $a' - a \in S$.

$\mathrm{Ap}(S) = \{0, 13, 10, 23\}$



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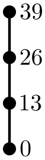

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$$S = \langle 6, 9, 20 \rangle \qquad S' = \langle 6, 26, 27 \rangle$$

$$\mathrm{Ap}(S) = \{0, 49, 20, \; 9, 40, 29\} \qquad \mathrm{Ap}(S') = \{0, 79, 26, 27, 52, 53\}$$

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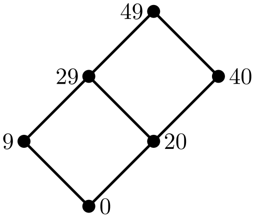

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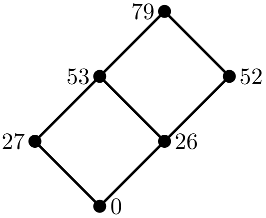

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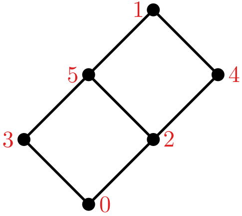

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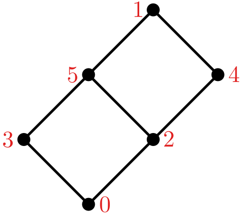

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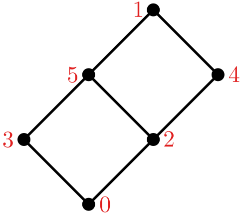

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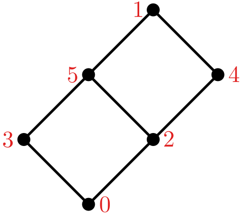

The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

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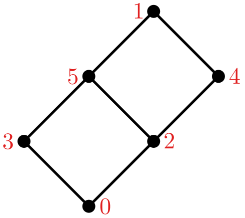

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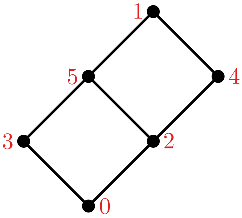

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Theorem (Bruns–García-Sánchez–O.–Wilburne)

Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

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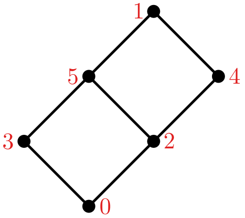

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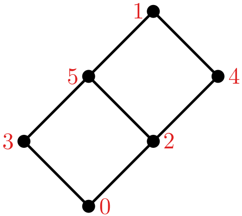

Defining facet equations:

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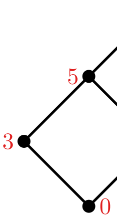

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	$3 \preceq 5$
$a_2 + a_5 = a_1$	$2 \preceq 1$
	$5 \preceq 1$
$a_3 + a_4 = a_1$	$3 \preceq 1$
	$4 \preceq 1$

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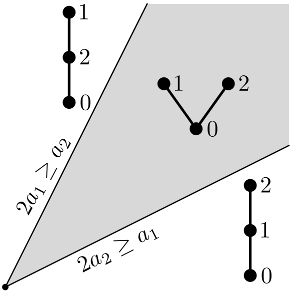
1
2
0
1
2
0
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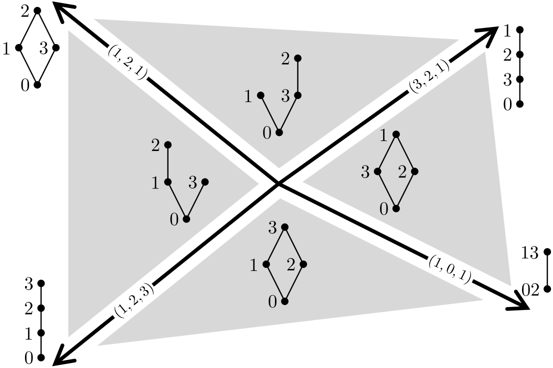
2
1
0


(1, 2, 1)
(3, 2, 1)
(1, 2, 3)
(1, 0, 1)
13
02

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What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

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  Cohen-Macaulay type of the *defining toric ideal* of S:
  $$I_S = \ker\left(\Bbbk[x_1, \ldots, x_k] \to \Bbbk[w^{n_1}, w^{n_2}, \ldots, w^{n_k}]\right)$$

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Spoiler

If S, S' have identical Kunz posets, then S and S' have the same number of minimal trades.

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$\mathsf{Z}(n) = \left\{ \mathbf{a} \in \mathbb{Z}_{\geq 0}^k : n = a_1 n_1 + \cdots + a_k n_k \right\}$$

is the *set of factorizations* of $n \in S$.

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$S = \langle 6, 9, 20 \rangle$:

$$\mathsf{Z}(60) = \{(10,0,0), (7,2,0), (4,4,0), (1,6,0), (0,0,3)\}$$

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$$\begin{aligned} \pi : \mathbb{Z}_{\geq 0}^k &\longrightarrow \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} &\longmapsto a_1 n_1 + \cdots + a_k n_k \end{aligned}$$

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The *kernel* $\ker \pi$ is the relation $\sim$ on $\mathbb{Z}_{\geq 0}^k$ with $\mathbf{a} \sim \mathbf{b}$ whenever

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that is closed under *translation*.

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

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Factorization homomorphism: Monomial map:

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$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k_{\geq 0}$$

Factorization homomorphism:

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$$\begin{array}{rcl} \varphi : \Bbbk[x_1, \ldots, x_k] & \longrightarrow & \Bbbk[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

Definition

The *kernel* $\ker \pi$ is the relation $\sim$ on $\mathbb{Z}^k_{\geq 0}$ with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b})$$

$\ker \pi$ is a *congruence*: an equivalence relation

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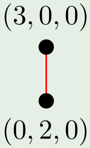

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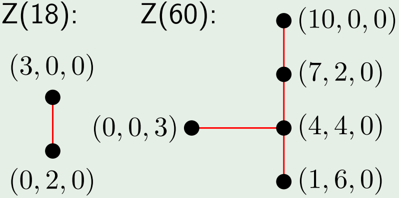

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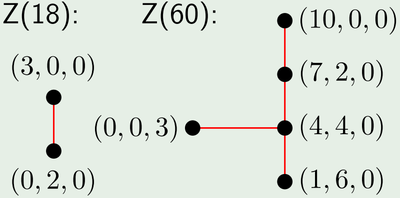

$$x^7y^2 - x^4y^4 = x^4y^2(x^3 - y^2)$$

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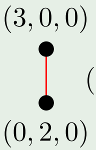

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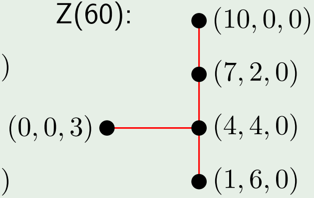

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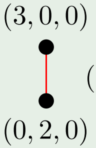

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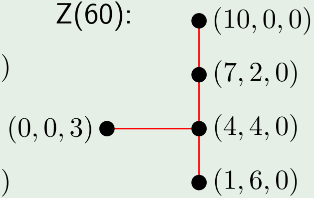

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Generating set for I_S $\Leftrightarrow$ Z(n) connected for all $n \in S$

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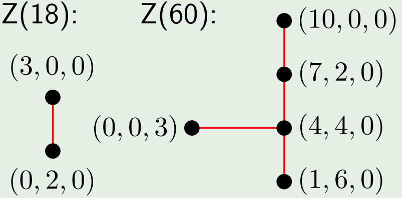

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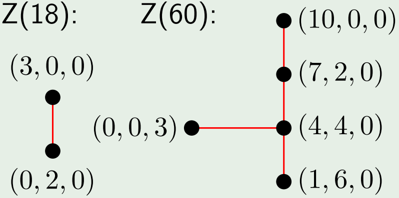

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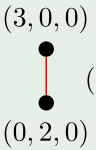

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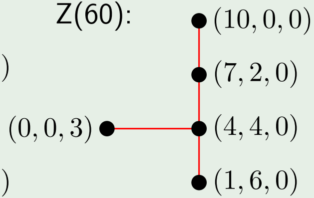

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$$\begin{aligned} I_S &= \langle x^3 - y^2, x^{10} - z^3 \rangle \\ &= \langle x^3 - y^2, x^7y^2 - z^3 \rangle \\ &= \langle x^3 - y^2, x^4y^4 - z^3 \rangle \\ &= \langle x^3 - y^2, x^6y - z^3 \rangle \end{aligned}$$

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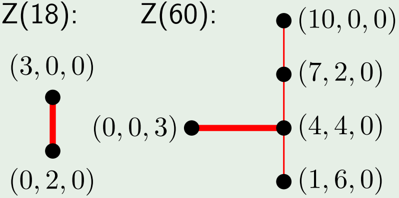

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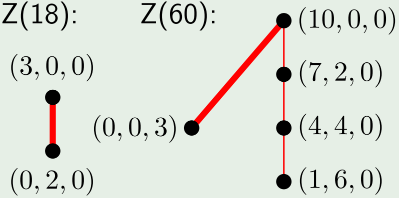

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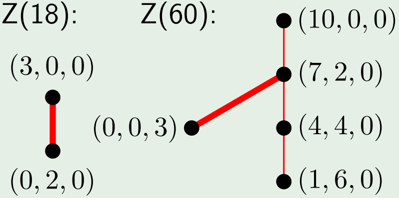

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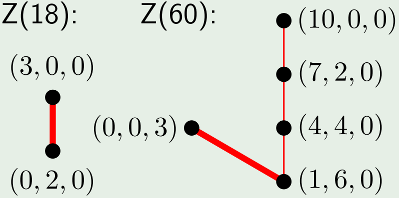

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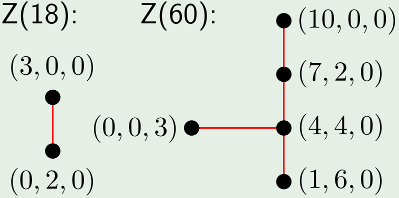

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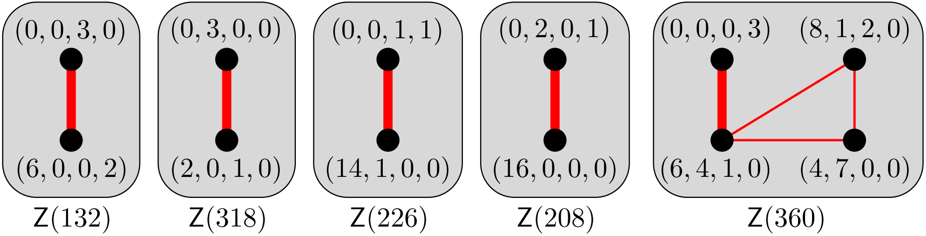

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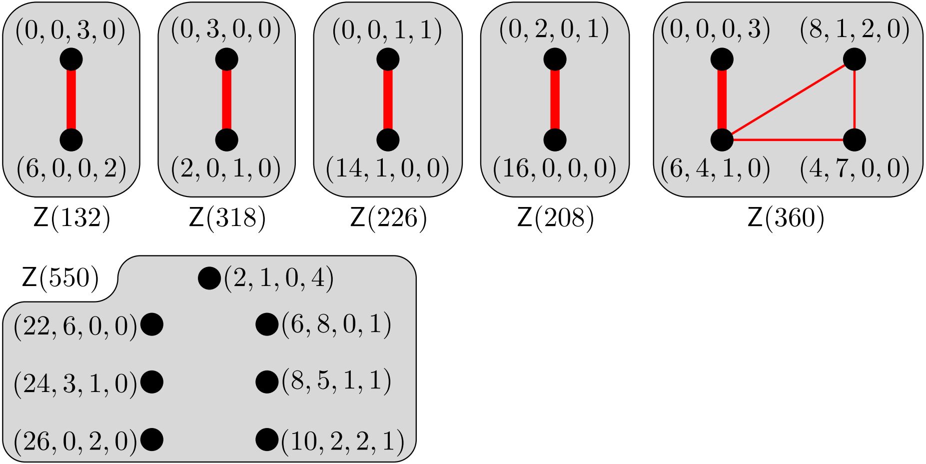

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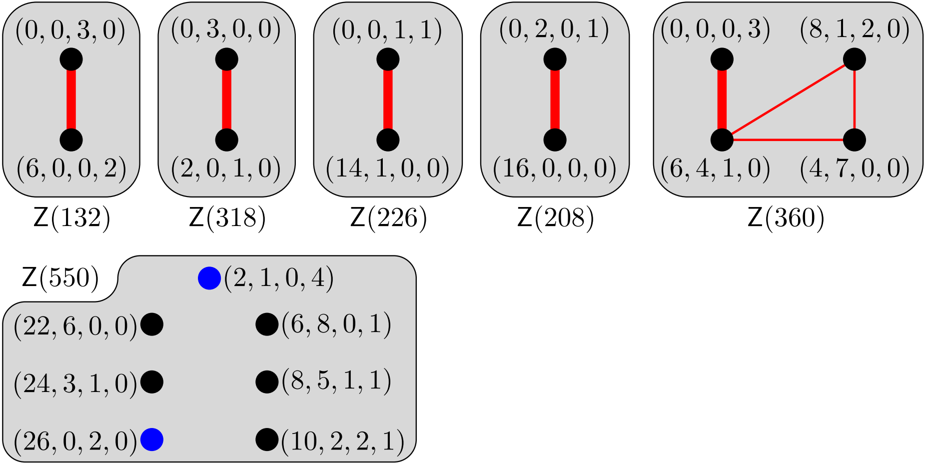

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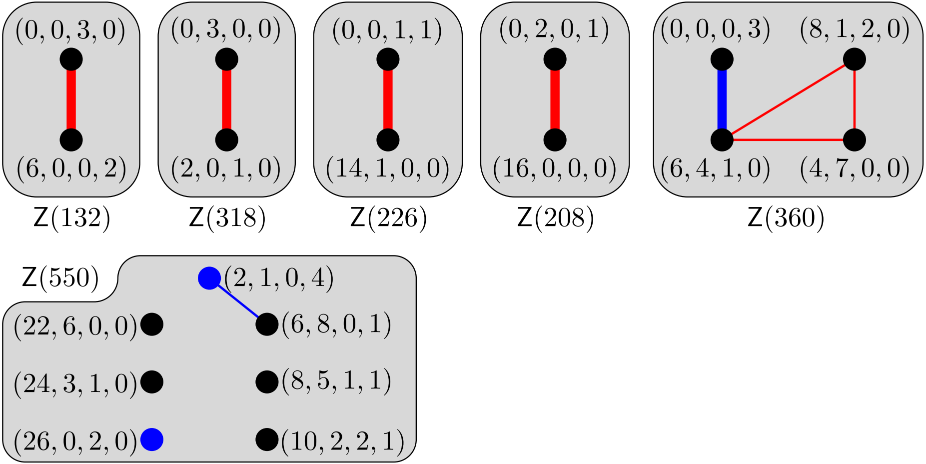

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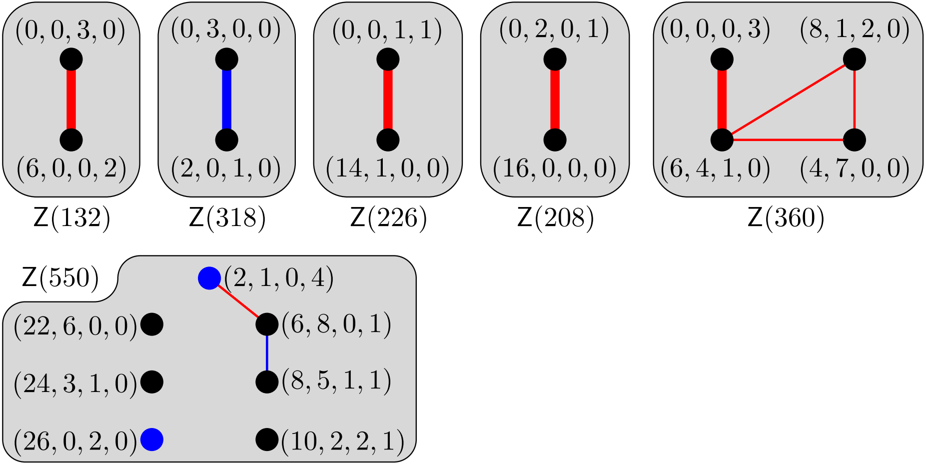

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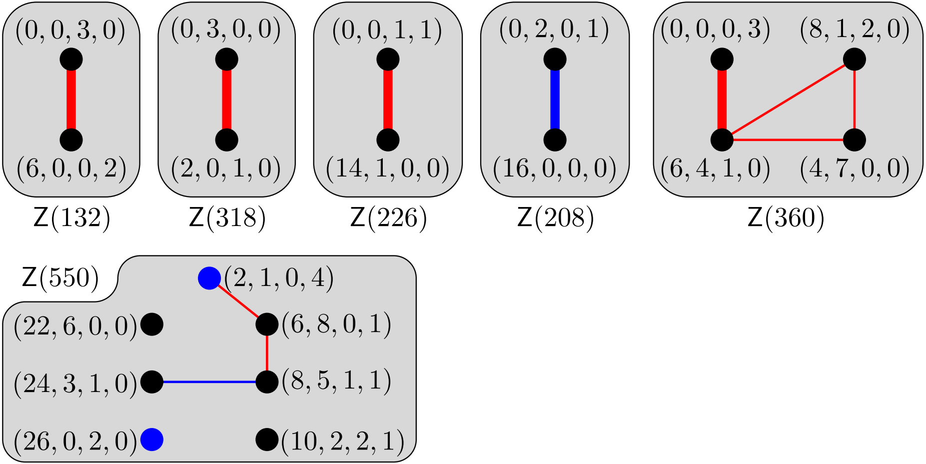

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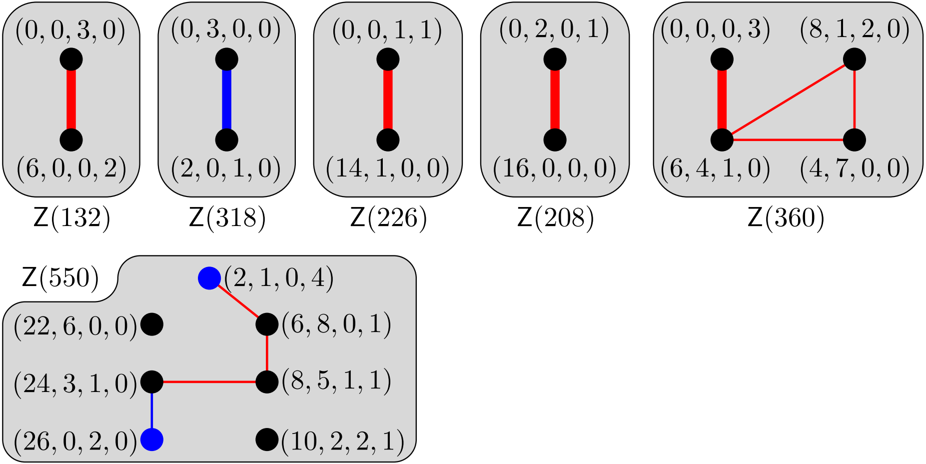

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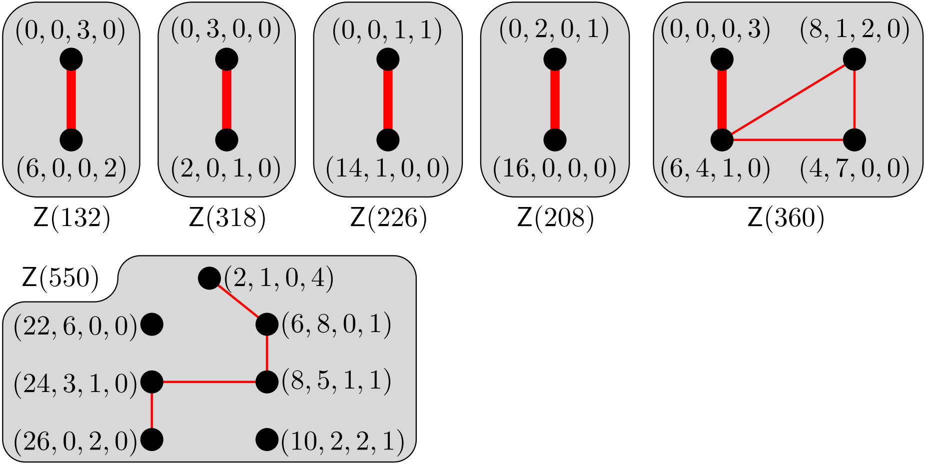

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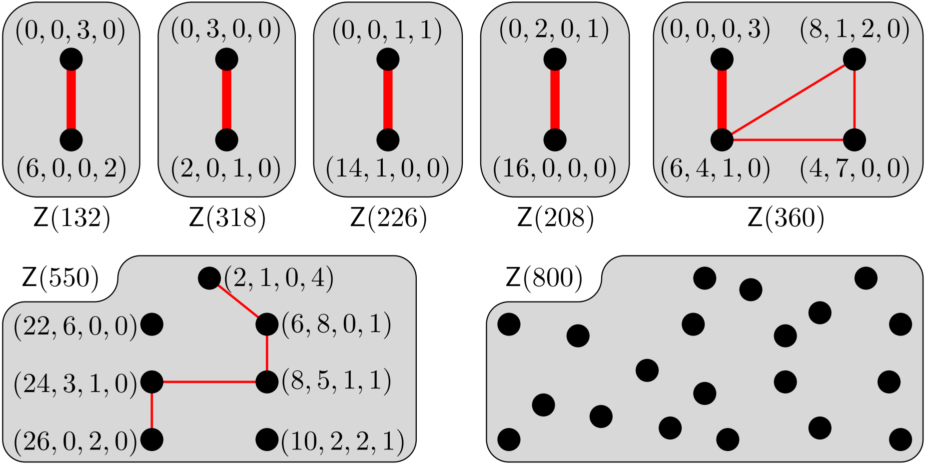

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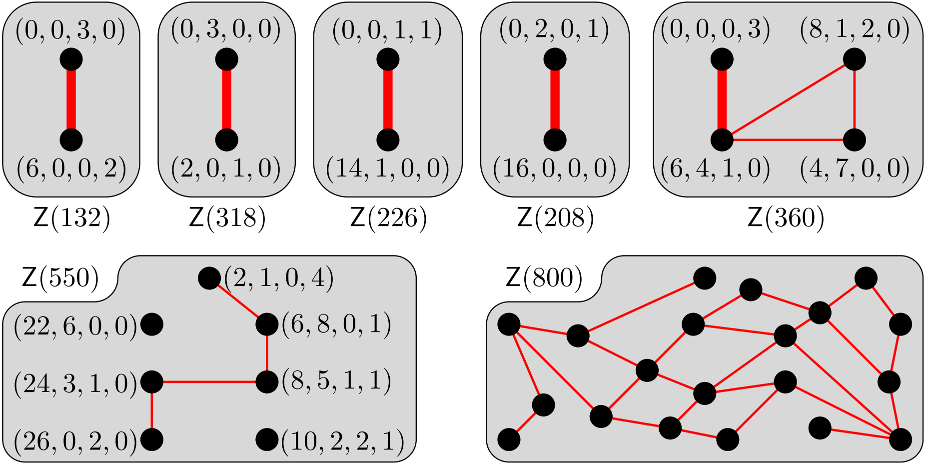

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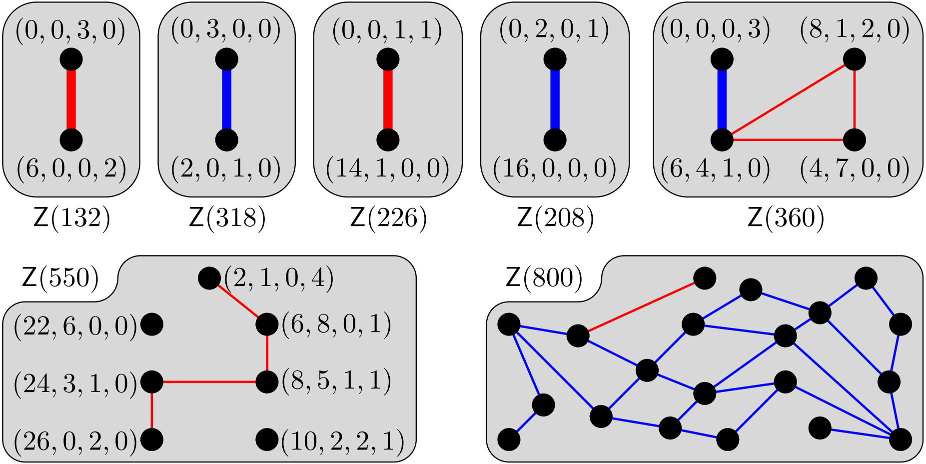

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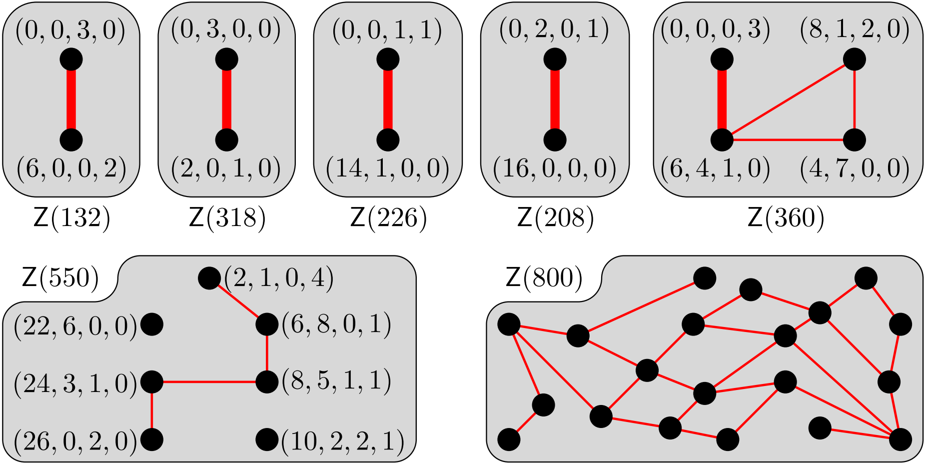

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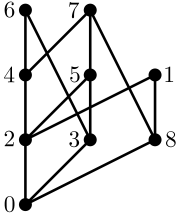

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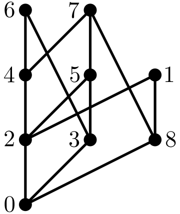

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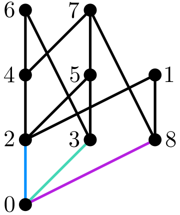

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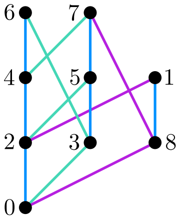

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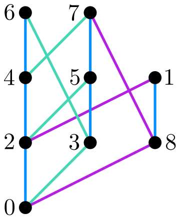

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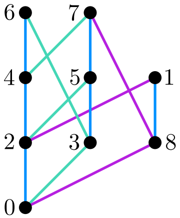

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$(0,3,0,0) \sim (0,0,2,0)$ (at a_6)
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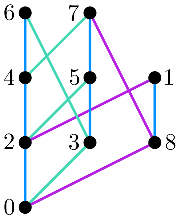

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Moral: can recover

- $\mathsf{Z}(a)$ for $a \in \text{Ap}(S)$
- (minimal) trades at $a \in \text{Ap}(S)$

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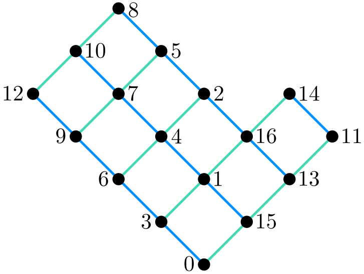

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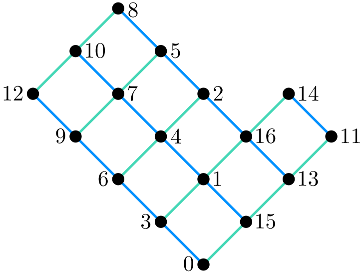

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3 minimal trades, none in $\mathrm{Ap}(S)$

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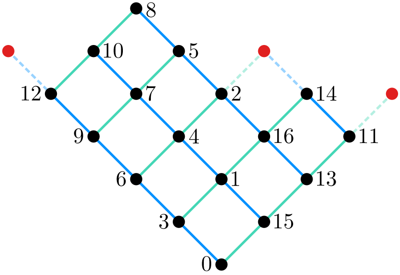

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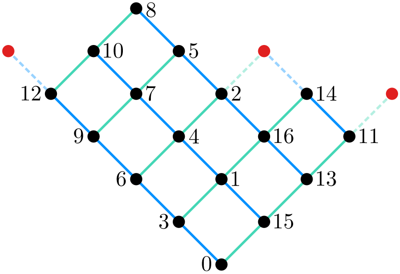
3 minimal trades, none in $\mathrm{Ap}(S)$

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

Key fact: each trade occurs at $a_i + n_j$ for some $a_i \in \mathrm{Ap}(S)$, generator n_j



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$a_{12} + a_3$:

$a_{11} + a_{15}$:

$a_2 + a_{15}$:

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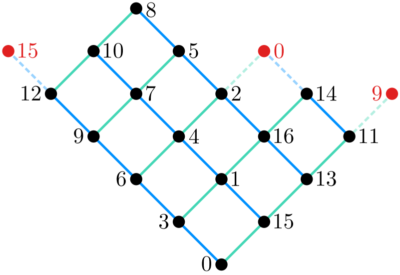

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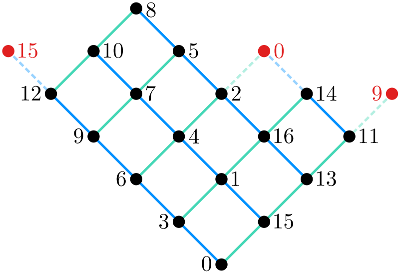

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$a_{12} + a_3$: $(0, 5, 0) \sim (*, 0, 1)$

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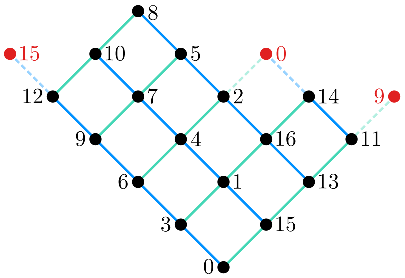

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If an *Apéry set of unique expression*,

- factorizations of $a \in \mathrm{Ap}(S)$ form monomial staircase
- one "outer" minimal trade for each monomial generator

Minimal trades and Kunz posets

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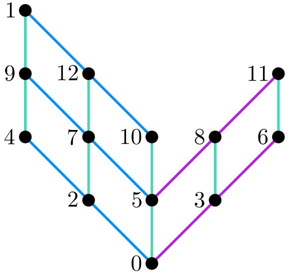

$$S = \langle 13, a_2, a_5, a_3 \rangle$$

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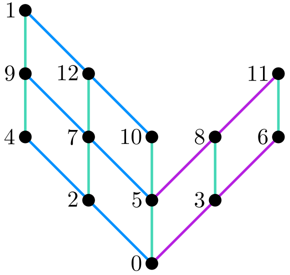

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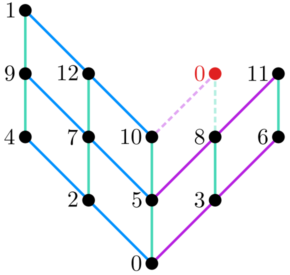

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Minimal trades and Kunz posets

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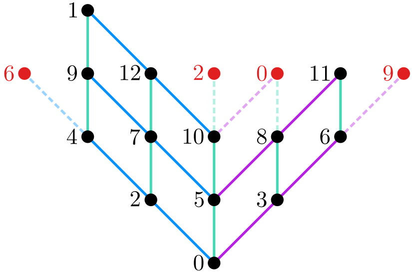

$S = \langle 13, a_2, a_5, a_3 \rangle$

5 minimal trades, none in $\mathrm{Ap}(S)$

0: $(0, 0, 2, 1)$

6: $(0, 3, 0, 0)$
2: $(0, 0, 3, 0)$
9: $(0, 0, 0, 3)$

Minimal trades and Kunz posets

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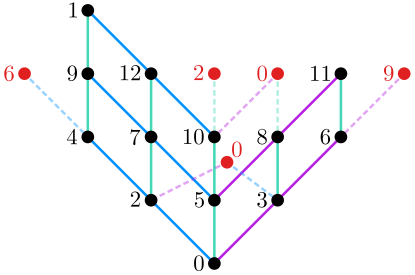

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0: $(0, 1, 0, 1)$

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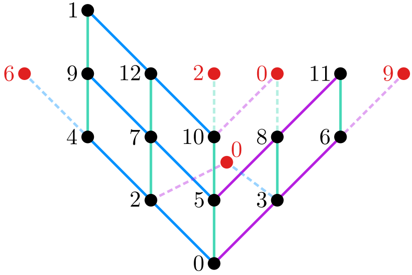

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Need: decrementing any coordinate lands in $\mathrm{Ap}(S)$

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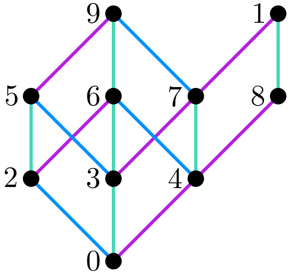

$$S = \langle 10, a_2, a_3, a_4 \rangle$$

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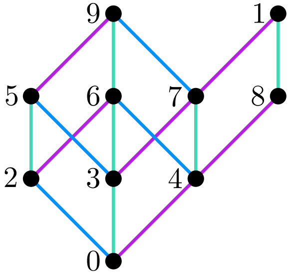

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"inner" trade at a_6:
$(0, 0, 2, 0) \sim (0, 1, 0, 1)$

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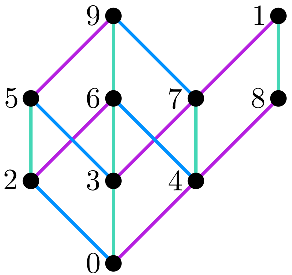

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$(0, 0, 0, 3)$, $(0, 2, 0, 0)$

Minimal trades and Kunz posets

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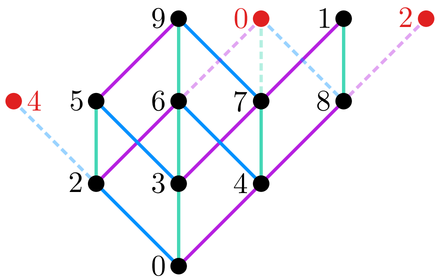

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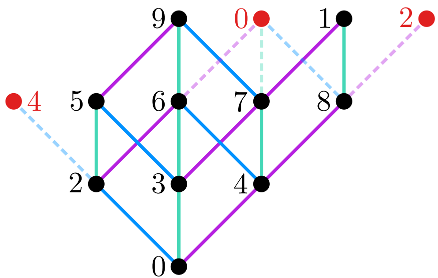

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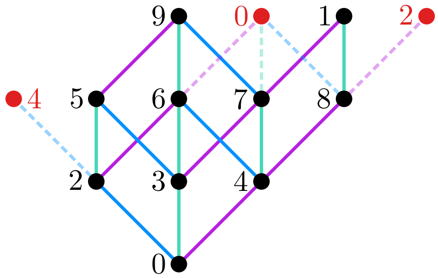

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Moral: use **sets** of factorizations, avoids overcounting minimal trades

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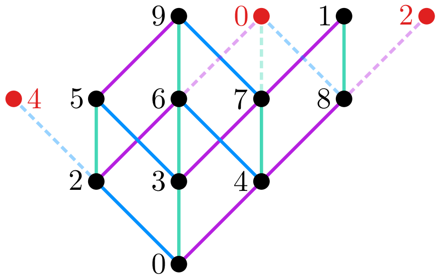

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$(0,0,2,0) \sim (0,1,0,1)$

Candidates for "outer" trades:
$(0,0,2,1)$, $(0,1,0,2)$,
$(0,0,0,3)$, $(0,2,0,0)$

Moral: use **sets** of factorizations, avoids overcounting minimal trades

0: $\{(0,0,2,1),(0,1,0,2)\}$
2: $\{(0,0,0,3)\}$, 4: $\{(0,2,0,0)\}$

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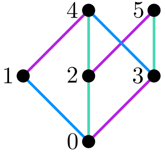
$$S = \langle 6, 7, 8, 9 \rangle$$



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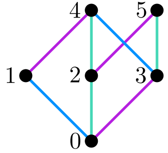

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"inner" trade at a_4:
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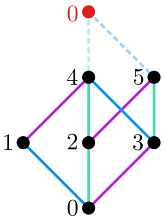

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"inner" trade at a_4:
$(0, 0, 2, 0) \sim (0, 1, 0, 1)$

candidate for "outer" trade:
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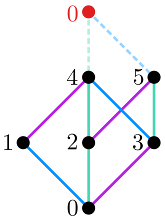

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No trades in $\mathsf{Z}(25)$:
$\{(0, 0, 2, 1), (0, 1, 0, 2), (3, 1, 0, 0)\}$

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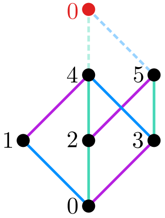

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The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \mathrm{supp}(B)$.

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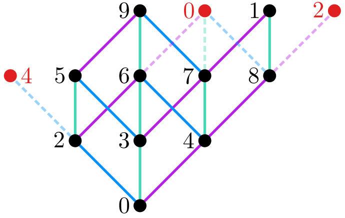

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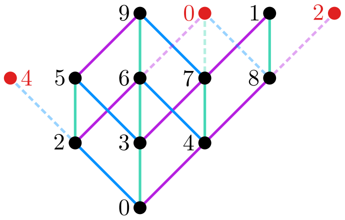

$B = \{(0,0,2,1),(0,1,0,2)\}$
$B - e_2 = \{(0,0,0,2)\} = \mathsf{Z}(a_8)$
$B - e_3 = \{(0,0,1,1)\} = \mathsf{Z}(a_7)$
$B - e_4 = \{(0,0,2,0),(0,1,0,1)\}$
$= \mathsf{Z}(a_6)$

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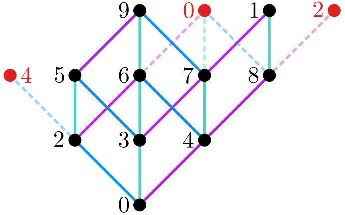

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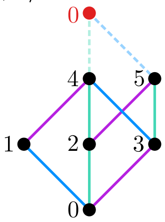

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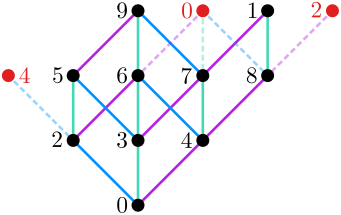

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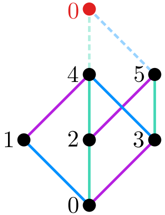

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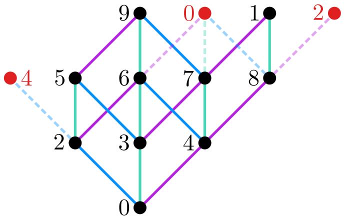

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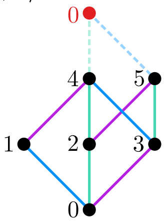

$B = \{(0,0,2,1)\}$?
$B - e_4 = \{(0,0,2,0)\} \subsetneq \mathsf{Z}(a_4)$

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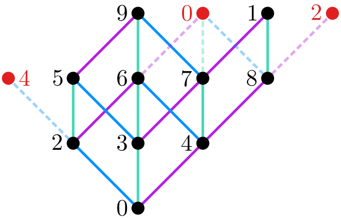

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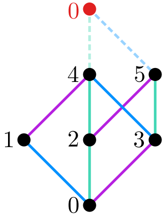
$B - e_2 = \{(0,0,0,2)\} = \mathsf{Z}(a_8)$
$B - e_3 = \{(0,0,1,1)\} = \mathsf{Z}(a_7)$
$B - e_4 = \{(0,0,2,0), (0,1,0,1)\}$
$= \mathsf{Z}(a_6)$

$S = \langle 6, 7, 8, 9 \rangle$



$B = \{(0,0,2,1)\}$?
$B - e_4 = \{(0,0,2,0)\} \subsetneq \mathsf{Z}(a_4)$

$B = \{(0,0,2,1), (0,1,0,2)\}$?

The main theorem

Definition

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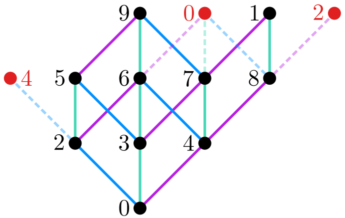
$S = \langle 10, a_2, a_3, a_4 \rangle$



$B = \{(0,0,2,1),(0,1,0,2)\}$

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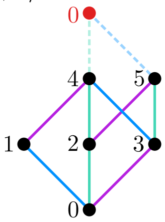

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Another subtlety: distinct outer Betti elements can **coincide** for some S

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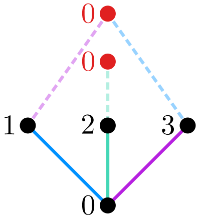

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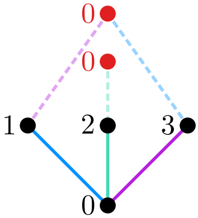

$S = \langle 4, 9, 14, 11 \rangle$

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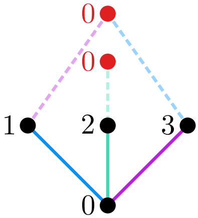

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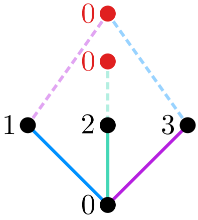

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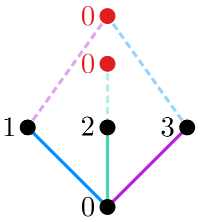

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If S has Kunz poset P, each minimal trade of S not ocurring in $\mathrm{Ap}(S)$ *contains a factorization from a distinct outer Betti element of P.*

In particular, if S, S' have identical Kunz poset, then S and S' have the same number of minimal trades.

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- Algorithm for computing minimal presentations
  Runtime depends **only** on m!

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  For $m = 4$, $\quad \#$ minimal trades $\in \{1, 2, 3, 6\}$

References

W. Bruns, P. García-Sánchez, C. O'Neill, D. Wilburne (2020)
Wilf's conjecture in fixed multiplicity
International Journal of Algebra and Computation **30** (2020), no. 4, 861–882. (arXiv:1903.04342)

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Numerical semigroups, polyhedra, and posets I: the group cone
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M. Delgado, P. García-Sánchez, J. Morais
GAP Numerical Semigroups Package
`http://www.gap-system.org/Packages/numericalsgps.html`.

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Thanks!