Numerical semigroups: a sales pitch

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Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$ and an element $n \in S$.

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$$\mathsf{m}(82) = 5 \quad \text{with} \quad 82 = 3(16) + 2(17)$$
$$\mathsf{m}(462) = 25 \quad \text{with} \quad 462 = 3(16) + 2(17) + 20(19)$$

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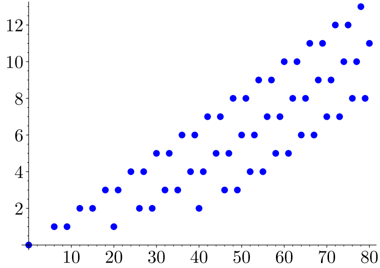

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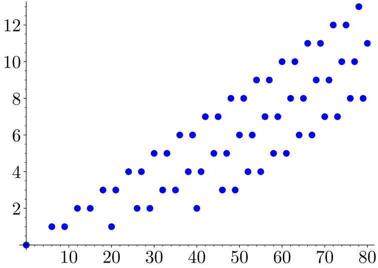

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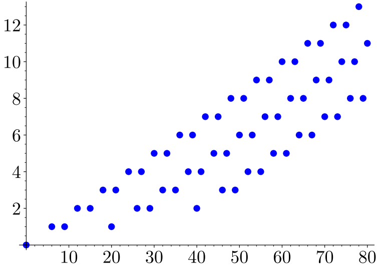

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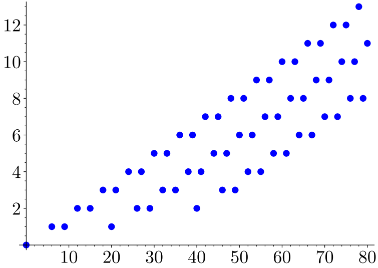

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$$\mathsf{M}(n) = \tfrac{1}{6}n + a_0(n) \qquad \text{for some periodic function } a_0(n)$$

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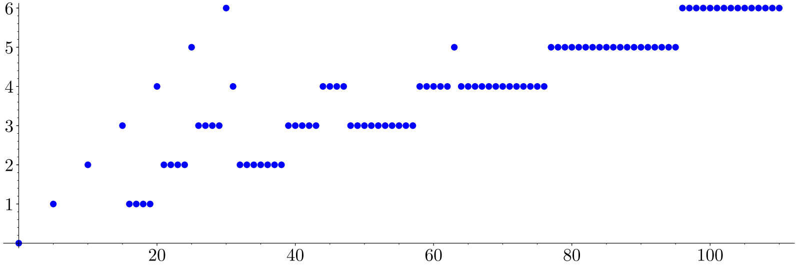

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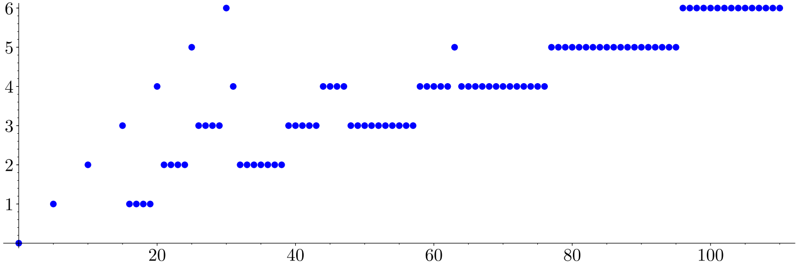

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Conclusion: $\mathsf{m}(n)$ is quasilinear for $n \geq 64$, with period 19.

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Let $S = \langle n_1, \ldots, n_k \rangle$. For $n \gg 0$ (i.e., for n sufficiently large),

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$M_j = \langle j, j + 6, j + 9, j + 20 \rangle$: $\qquad \mathsf{F}(M_j) = \frac{1}{20} j^2 + \cdots$ for $j \geq 44$.

Parametrized families of numerical semigroups

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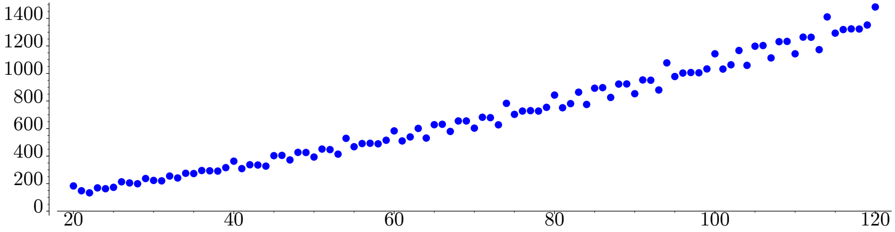

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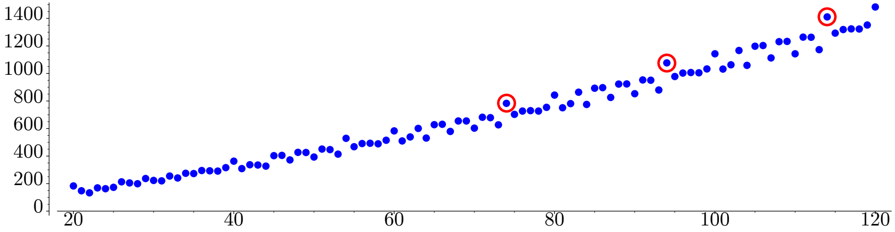

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Not true for $m_f = \#$ of numerical semigroups with Frobenius number f

$$m_{11} = 51 \qquad m_{12} = 40 \qquad m_{13} = 106$$

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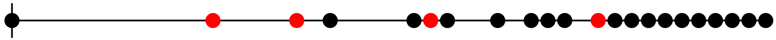

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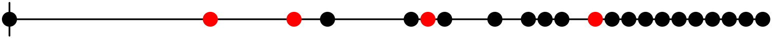

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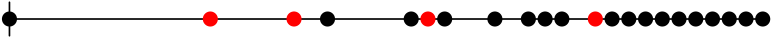

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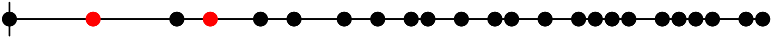

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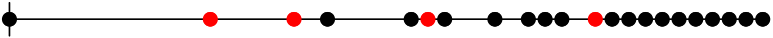

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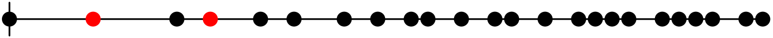

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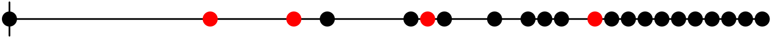

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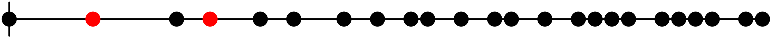

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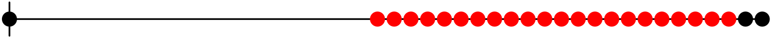

Proved in many special cases, including $\mathrm{g}(S) \leq 60$.

So much more out there!

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  An algorithm to compute ω-primality in a numerical monoid,
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Factoring in the Chicken McNugget monoid,
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