Numerical semigroups, minimal presentations, and posets

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$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

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Multiplicity: $\mathrm{m}(S) =$ smallest nonzero element

Minimal presentations

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$\mathsf{Z}(n) = \left\{ \mathbf{a} \in \mathbb{Z}_{\geq 0}^k : n = a_1 n_1 + \cdots + a_k n_k \right\}$$

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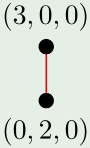

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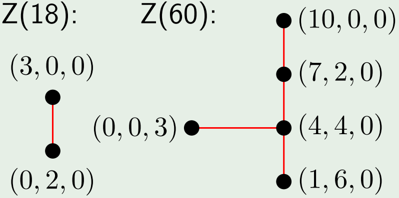

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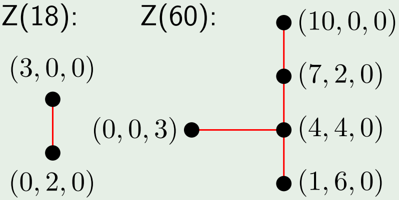

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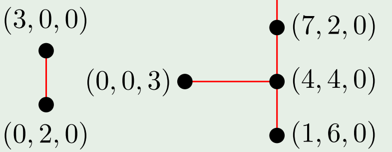

$$\begin{aligned} x^7y^2 - x^4y^4 &= x^4y^2(x^3 - y^2) \\ x^7y^2 - z^3 &= (x^7y^2 - x^4y^4) \\ &\quad + (x^4y^4 - z^3) \end{aligned}$$

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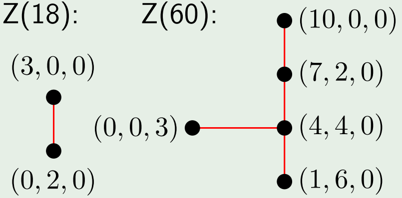

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Generating set for I_S $\Leftrightarrow$ $Z(n)$ connected for all $n \in S$

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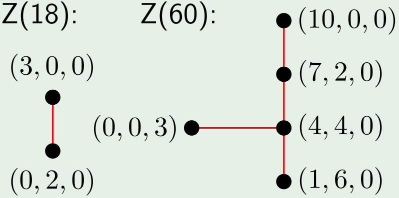

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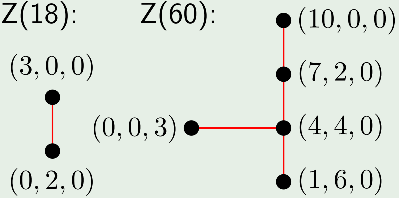

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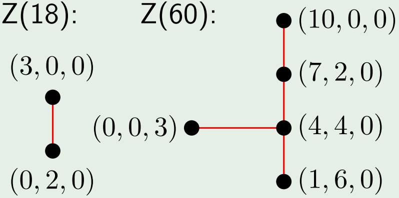

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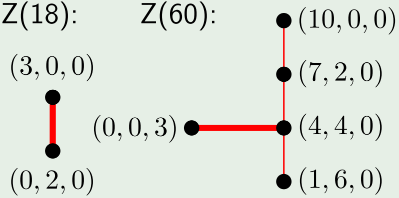

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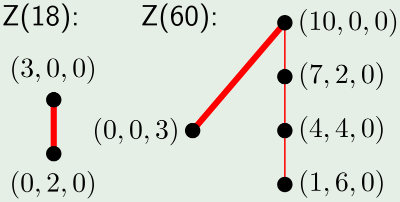

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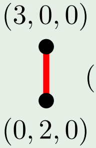

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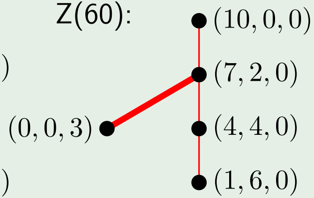

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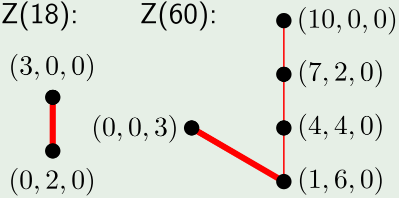

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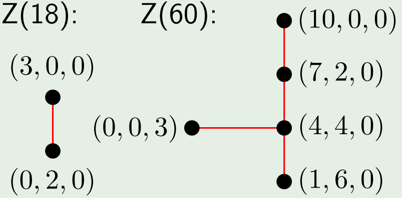

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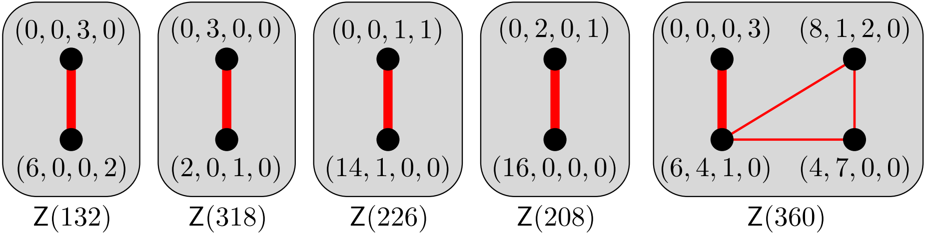

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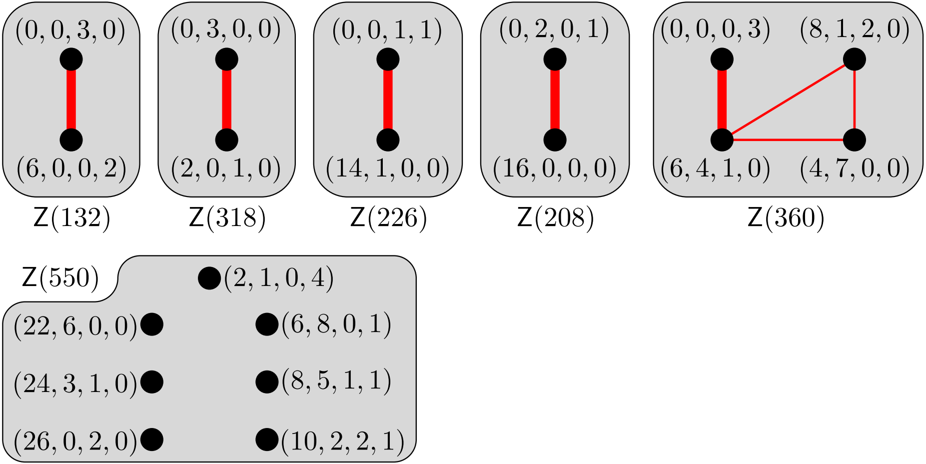

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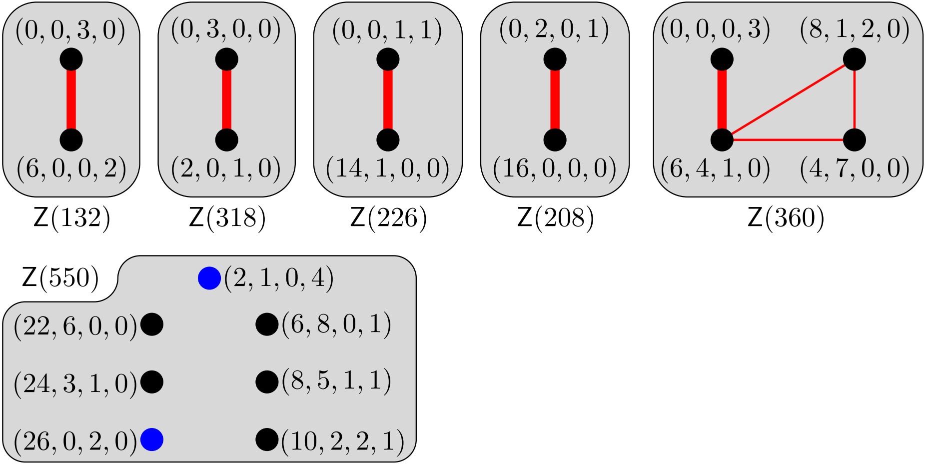

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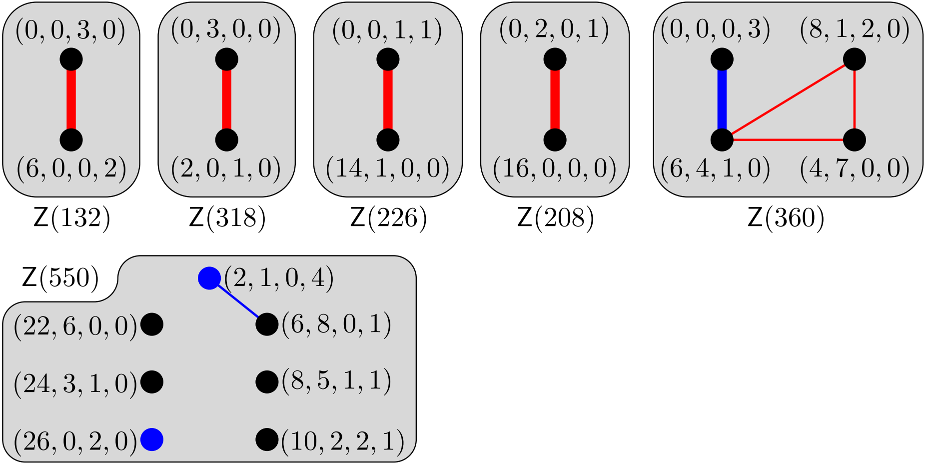

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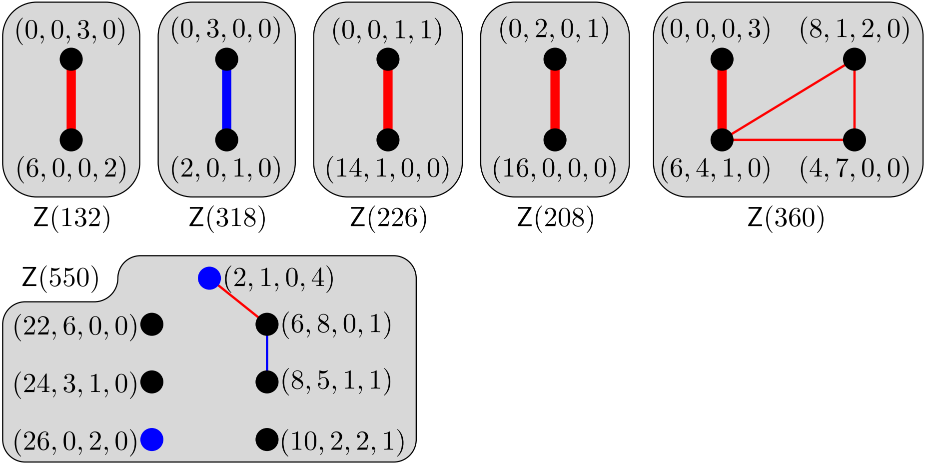

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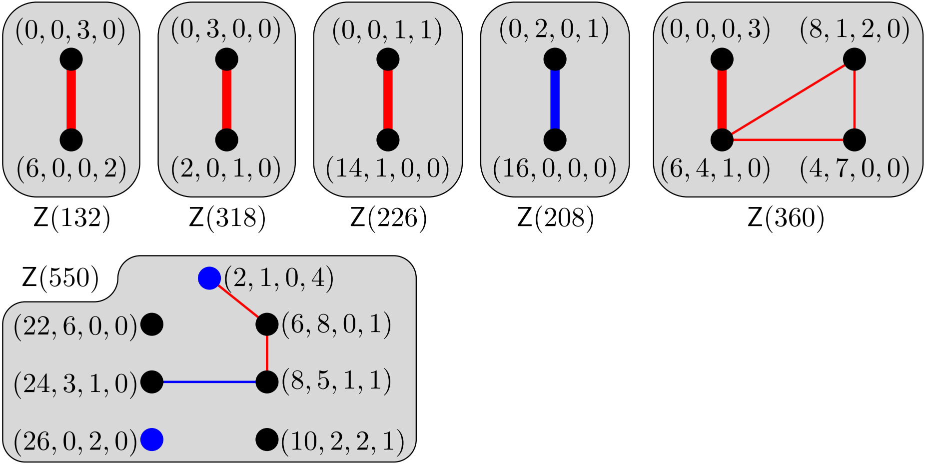

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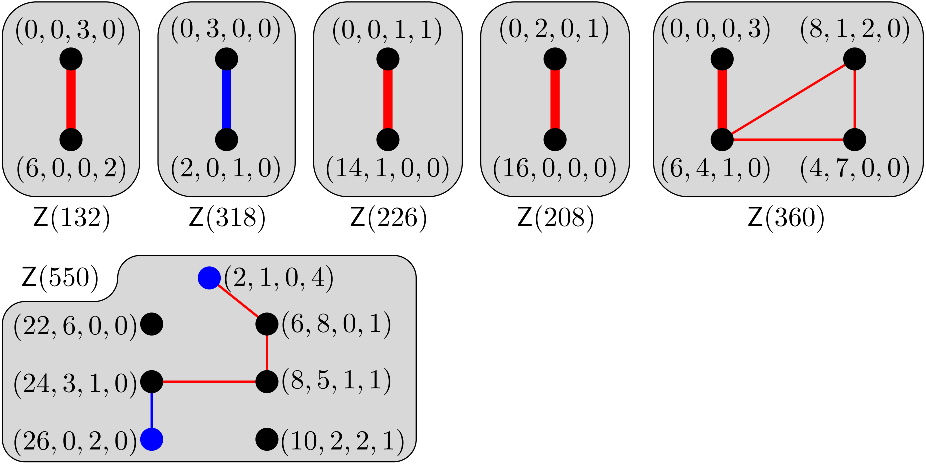

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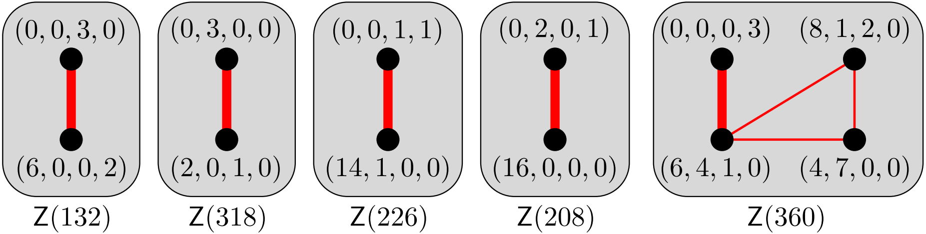

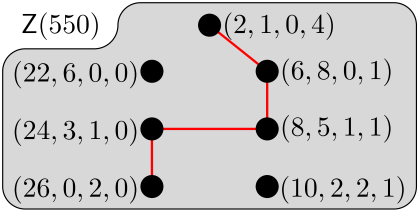

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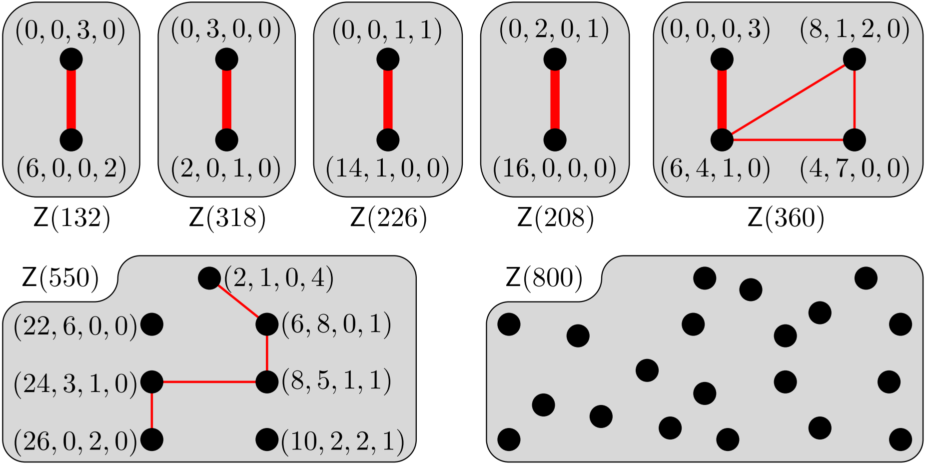

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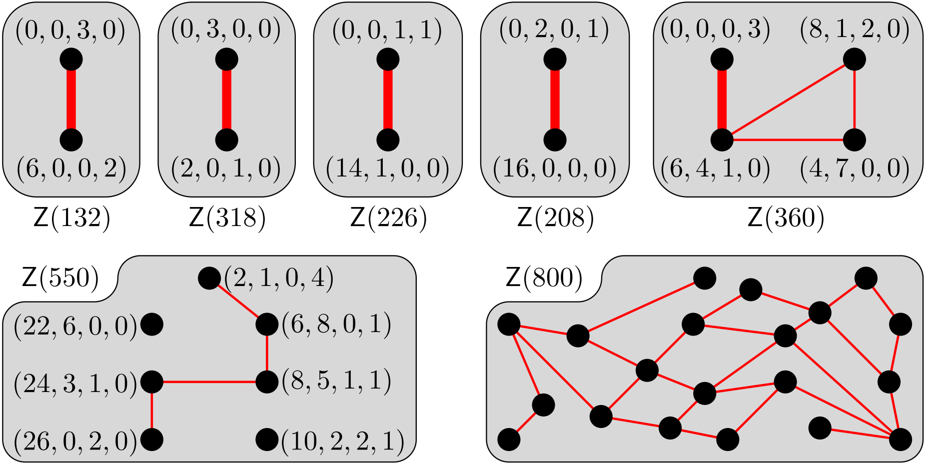

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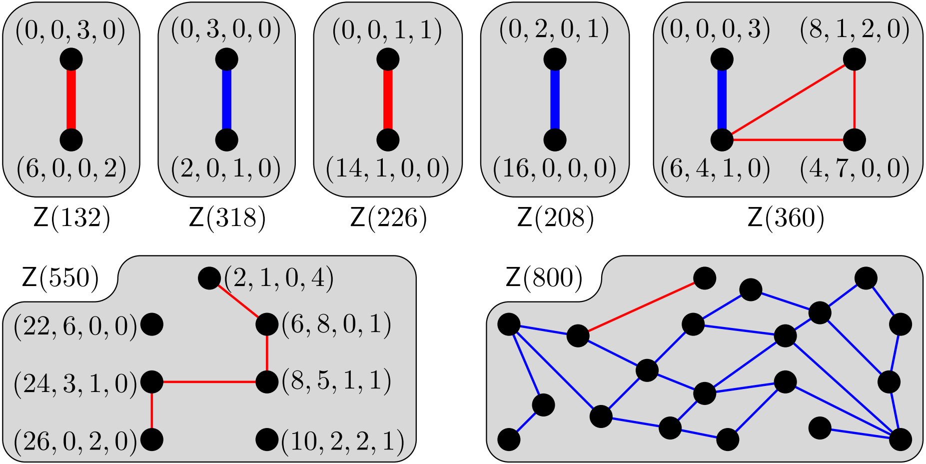

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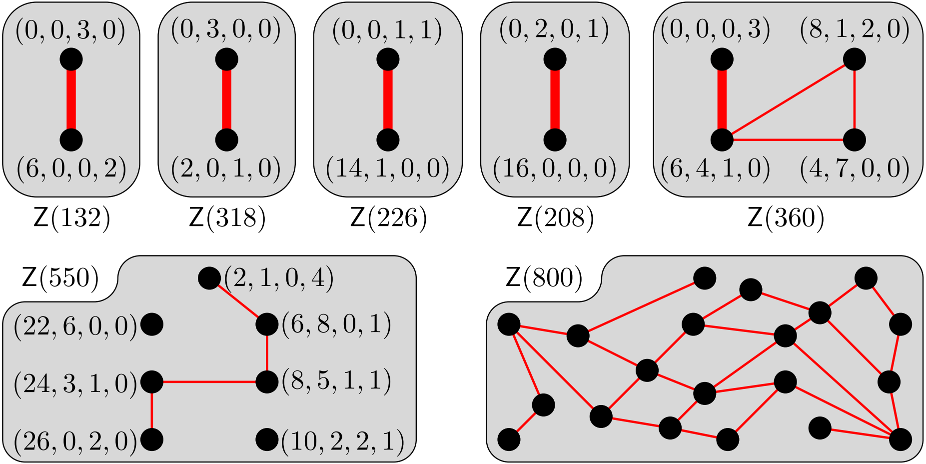

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

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Apéry posets and Kunz posets

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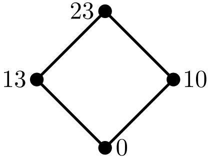

$S = \langle 4, 13 \rangle$

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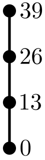

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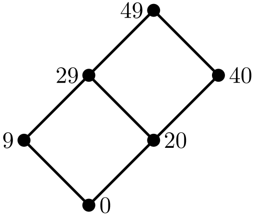

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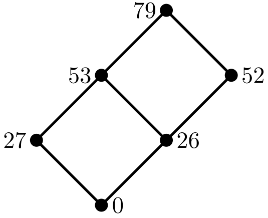

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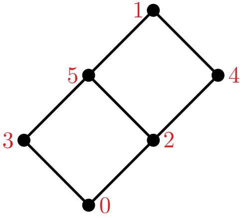

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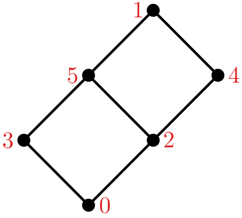

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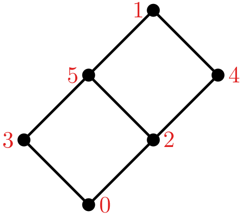

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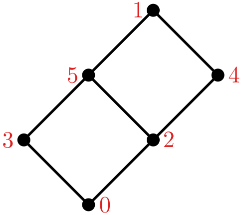

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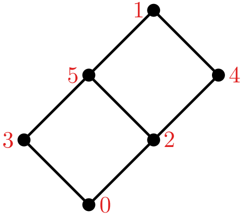

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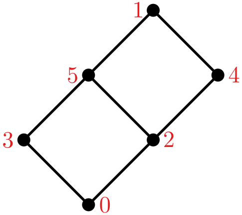

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Theorem (Gomes–O.–Torres Davila)

If S, S' have identical Kunz poset, then S and S' have the same number of minimal trades.

Minimal trades and Kunz posets

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How can one recover minimal trade structure from the Kunz poset?

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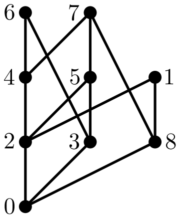

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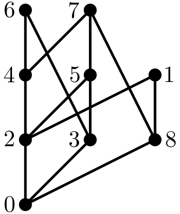

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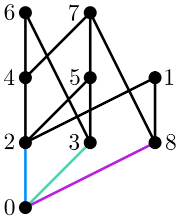

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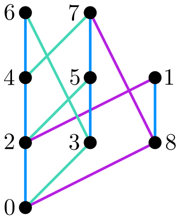

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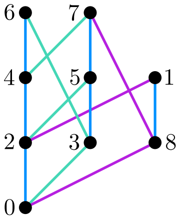

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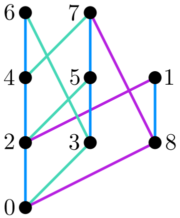

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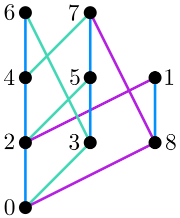

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Moral: can recover

- $\mathsf{Z}(a)$ for $a \in \mathrm{Ap}(S)$
- (minimal) trades at $a \in \mathrm{Ap}(S)$

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Minimal trades and Kunz posets

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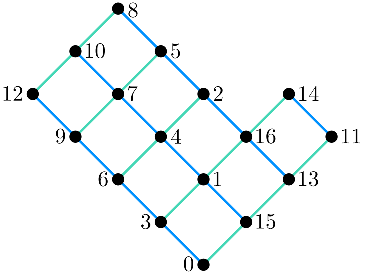
$$S = \langle 17, a_3, a_{15} \rangle$$



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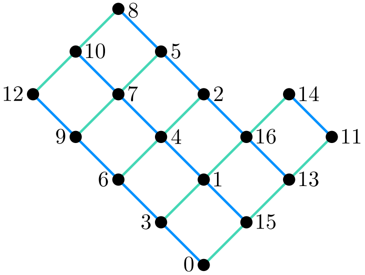

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3 minimal trades, none in $\mathrm{Ap}(S)$

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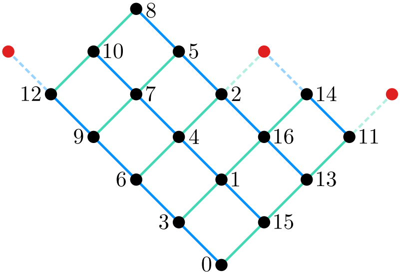

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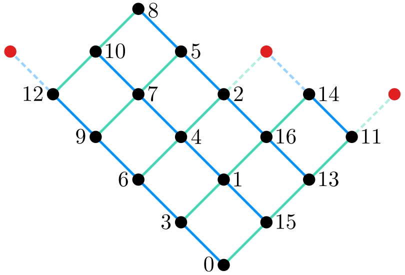

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$a_{12} + a_3$:

$a_{11} + a_{15}$:

$a_2 + a_{15}$:

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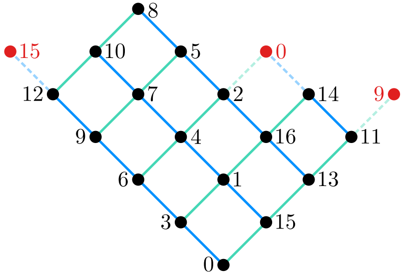

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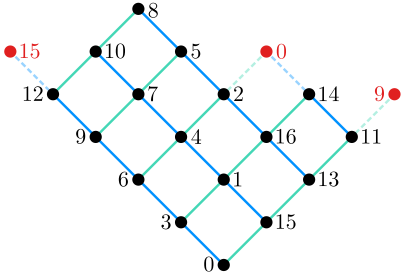

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$a_{12} + a_3$: $(0, 5, 0) \sim (*, 0, 1)$

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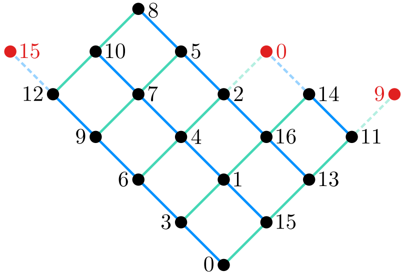

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If S has an *Apéry set of unique expression*:

- factorizations of $a \in \mathrm{Ap}(S)$ form monomial staircase
- one "outer" minimal trade for each monomial generator

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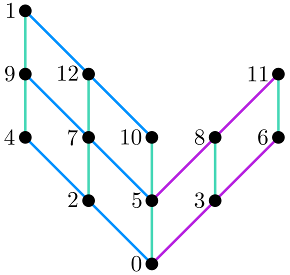

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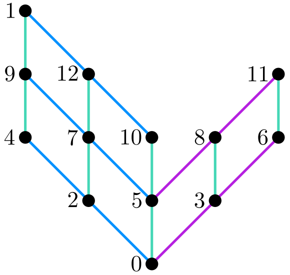

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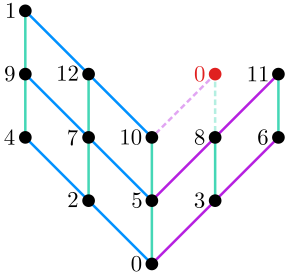

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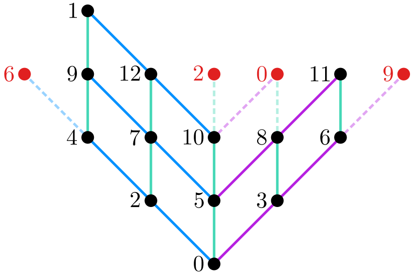

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0: $(0, 0, 2, 1)$

6: $(0, 3, 0, 0)$
2: $(0, 0, 3, 0)$
9: $(0, 0, 0, 3)$

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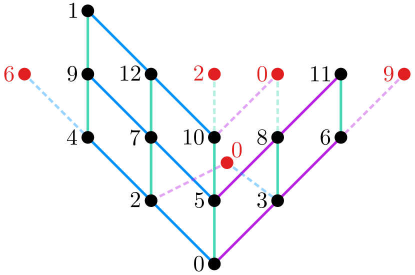

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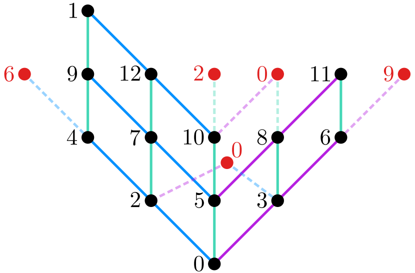

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Need: decrementing any coordinate lands in $\mathrm{Ap}(S)$

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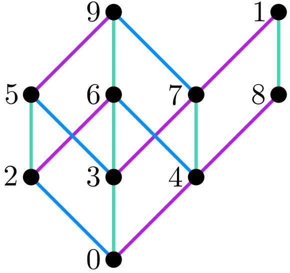

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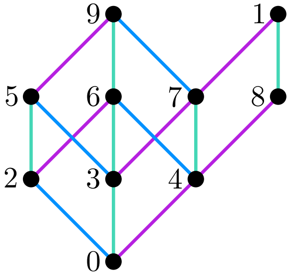

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"inner" trade at a_6:
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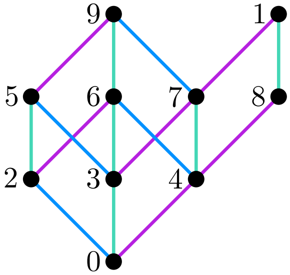

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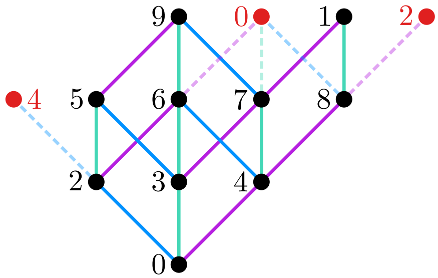

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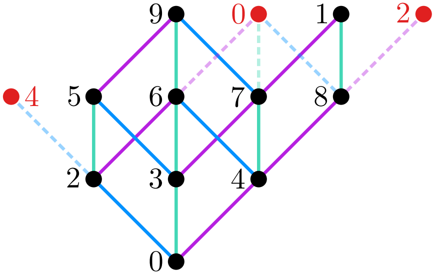

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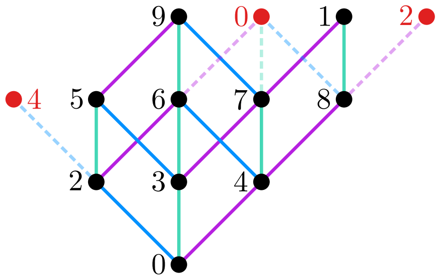

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Moral: use **sets** of factorizations, avoids overcounting minimal trades

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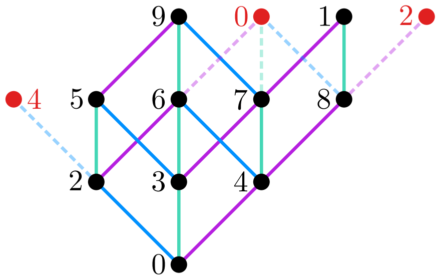

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Moral: use **sets** of factorizations, avoids overcounting minimal trades

0: $\{(0, 0, 2, 1), (0, 1, 0, 2)\}$
2: $\{(0, 0, 0, 3)\}$, 4: $\{(0, 2, 0, 0)\}$

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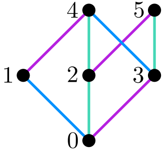

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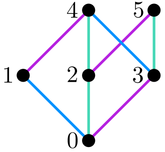

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"inner" trade at a_4:
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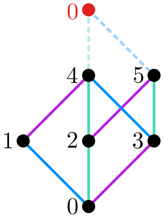

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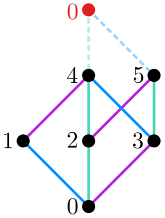

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No trades in $\mathsf{Z}(25)$:
$\{(0, 0, 2, 1), (0, 1, 0, 2), (3, 1, 0, 0)\}$

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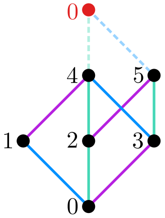

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The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \mathrm{supp}(B)$.

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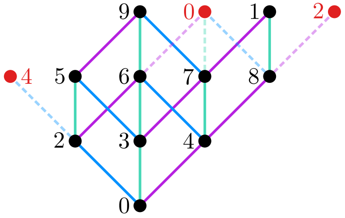

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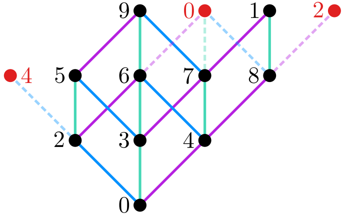

$B = \{(0,0,2,1), (0,1,0,2)\}$
$B - e_2 = \{(0,0,0,2)\} = \mathsf{Z}(a_8)$
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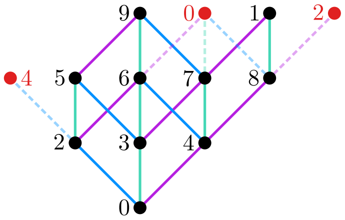

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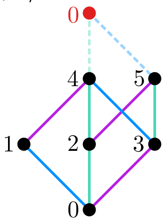

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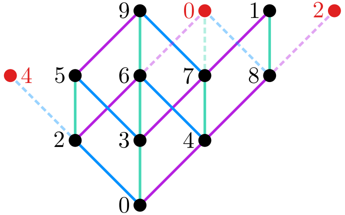

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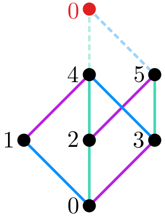

$B = \{(0,0,2,1)\}$?

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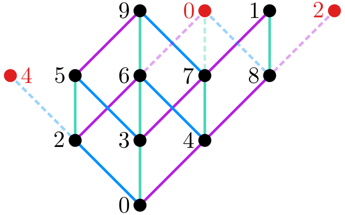


$B = \{(0,0,2,1), (0,1,0,2)\}$

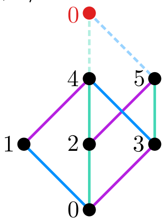
$B - e_2 = \{(0,0,0,2)\} = \mathsf{Z}(a_8)$

$B - e_3 = \{(0,0,1,1)\} = \mathsf{Z}(a_7)$

$B - e_4 = \{(0,0,2,0), (0,1,0,1)\}$

$= \mathsf{Z}(a_6)$

$S = \langle 6, 7, 8, 9 \rangle$



$B = \{(0,0,2,1)\}$?

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The main theorem

Definition

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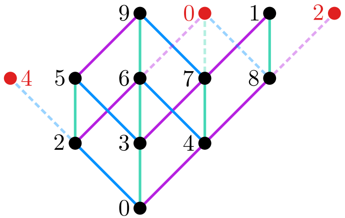

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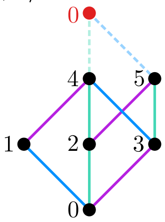

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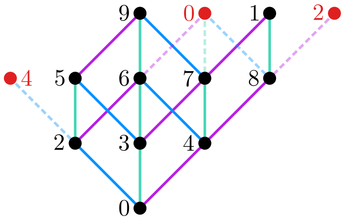

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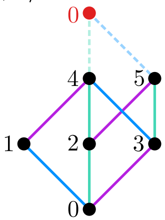

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$B = \{(0,0,2,1),(0,1,0,2)\}$?
$B - e_3 = \{(0,0,1,1)\} \not\subseteq \mathsf{Z}(a_i)$

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Theorem (Gomes–O.–Torres Davila)

If S has Kunz poset P, each minimal trade of S not ocurring in $\mathrm{Ap}(S)$ *contains a factorization from a distinct outer Betti element of P.*

In particular, if S, S' have identical Kunz poset, then S and S' have the same number of minimal trades.

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Bonus: new tool for classifying minimal presentations

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$\beta_1(I_S) = \#$ minimal trades $\le \binom{m}{2}$

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Bonus: new tool for classifying minimal presentations

$\beta_1(I_S) = \#$ minimal trades $\le \binom{m}{2}$

For $m = 4$, $\quad \beta_1(I_S) \in \{1, 2, 3, 6\}$

For $m = 5$, $\quad \beta_1(I_S) \in \{1, 2, 3, 5, 6, 10\}$

For $m = 6$, $\quad \beta_1(I_S) \in \{1, 2, 3, 4, 5, 6, 9, 10, 15\}$

Achievable minimal presentation cardinalities

Question

Given the multiplicity $m = \mathrm{m}(S)$ and $\#$ minimal generators $e = \mathrm{e}(S)$ of a numerical semigroup S, what can $\beta_1(I_S) = \#$ minimal trades be?

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m																								
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$\mathsf{e}(S) = 3$ $\mathsf{e}(S) = 4$ $\mathsf{e}(S) = 5$ $\mathsf{e}(S) = 6$ $\mathsf{e}(S) = 7$

m																								
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Given the multiplicity $m = \mathsf{m}(S)$ and # minimal generators $e = \mathsf{e}(S)$ of a numerical semigroup S, what can $\beta_1(I_S) =$ # minimal trades be?

$\mathsf{e}(S) = 3$, $\mathsf{e}(S) = 4$, $\mathsf{e}(S) = 5$, $\mathsf{e}(S) = 6$, $\mathsf{e}(S) = 7$

m																								
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Well known: $\beta_1(S) \le \binom{m}{2}$, with equality if and only if $e = m$

if $e = 3$, then $\beta_1(S) = 2, 3$

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Prior work: a family has $\beta_1(S) = \binom{e}{2}$ for $3 \le e \le m$ (Rosales)

if $r = m - e \le 2$, then $\beta_1(S) \in [\binom{e}{2} - r, \binom{e}{2}]$ (GS-R)

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Using Kunz posets: a family hits each $\beta_1(S) \in [\binom{e}{2} - r, \binom{e}{2}]$

for $r = m - e \leq e - 2$

a family hits $\beta_1(S) = \binom{e}{2} + 1$ for each $m \geq e + 3$

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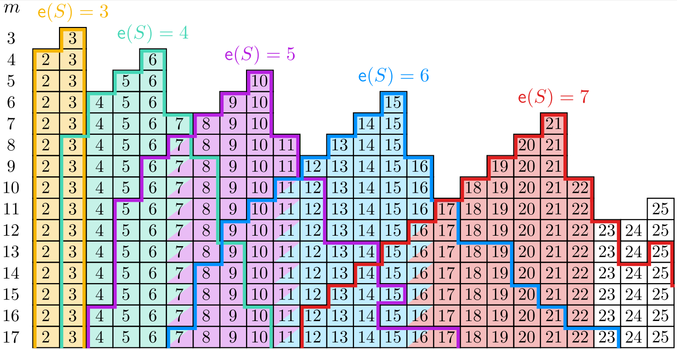

Bounds from Kunz posets: $\beta_1(S) \ge \binom{e}{2} - r$, where $r = m - e$

if $m - e = 3$, then $\beta_1(S) \in [\binom{e}{2} - 3, \binom{e}{2} + 1]$

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Given the multiplicity $m = \mathsf{m}(S)$ and # minimal generators $e = \mathsf{e}(S)$ of a numerical semigroup S, what can $\beta_1(I_S) =$ # minimal trades be?

$\mathsf{e}(S) = 4$

m																								
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One more family: for $e = 4$, achieves each $\beta_1(S)$ with $(\beta_1(S) - 2)^2 \leq 4m$
conjectured to achieve every possible $\beta_1(S)$ for $e = 4$

References

T. Gomes, C. O'Neill, E. Torres Davila (2021)
Numerical semigroups, polyhedra, and posets III: minimal presentations and face dimension
under review. (arXiv:2009.05921)

C. Elmacioglu, K. Hilmer, H. Koufmann, C. O'Neill, M. Okandan (2022)
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GAP Numerical Semigroups Package
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Thanks!