Numerical semigroups, minimal presentations, and posets

Christopher O'Neill

San Diego State University

cdoneill@sdsu.edu

Joint with (i) Winfred Bruns, Pedro García-Sánchez, and Dane Wilbourne;
(ii) Nathan Kaplan; (iii) *Tara Gomes and *Eduardo Torres Davila;
(iv) *Ceyhun Elmacioglu, Kieran Hilmer, *Hannah Koufmann, and *Melin Okandan

* = undergraduate student

Slides available: `https://cdoneill.sdsu.edu/`

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$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

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Every numerical semigroup has a unique minimal generating set.

Embedding dimension: $\mathrm{e}(S) = \#$ minimal generators
Multiplicity: $\mathrm{m}(S) =$ smallest nonzero element

Apéry sets

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Theorem

If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup S with $\mathrm{Ap}(S) = A$ if and only if

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Big idea: the inequalities "$a_i + a_j \ge a_{i+j}$" to define a **cone** C_m.

Kunz cone

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The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

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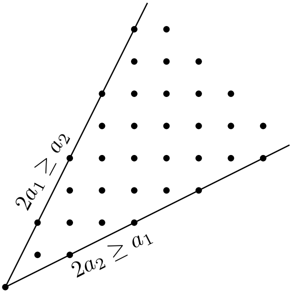
Example: C_3



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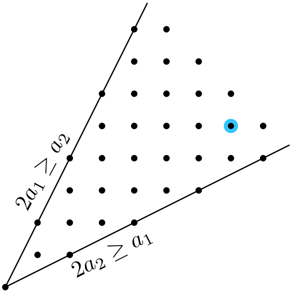

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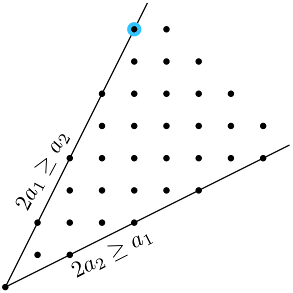

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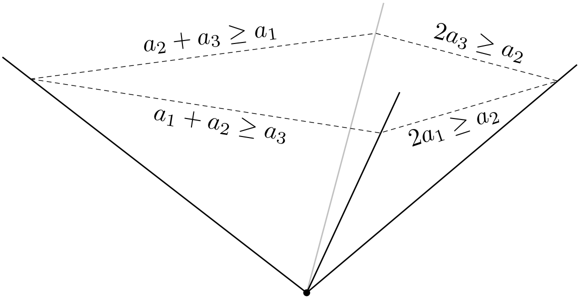

Faces of the Kunz cone

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When are numerical semigroups in (the relative interior of) the same face?

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Big picture: "moduli space" approach for studying XYZ's

- Define a space with XYZ's as points
  Small changes to an XYZ $\rightsquigarrow$ small movements in space
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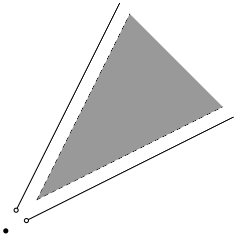
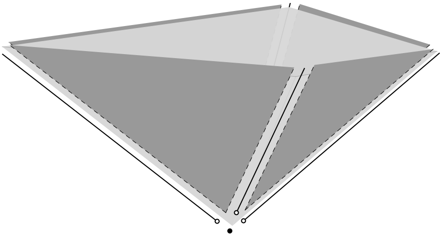
More interesting example: C_m





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What about the other faces?

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Faces of the Kunz cone

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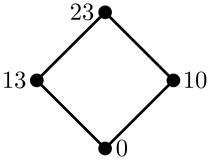

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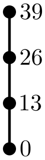

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$$S = \langle 6, 9, 20 \rangle \qquad S' = \langle 6, 26, 27 \rangle$$

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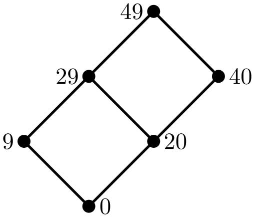

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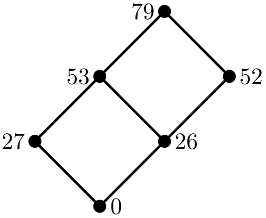

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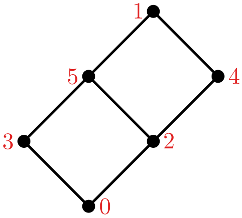

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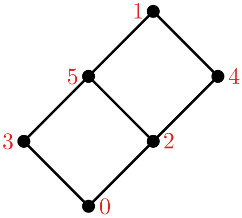

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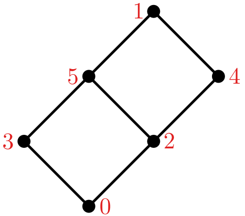

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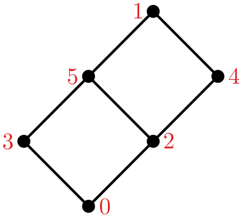

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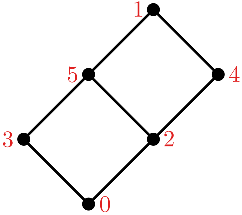

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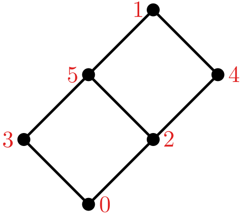

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Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

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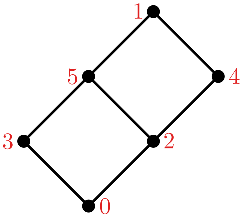

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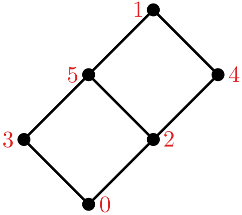

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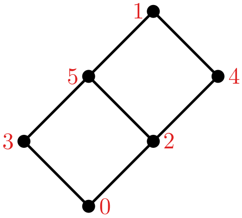

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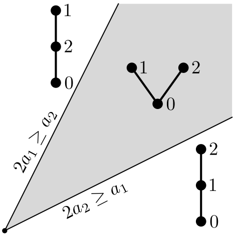
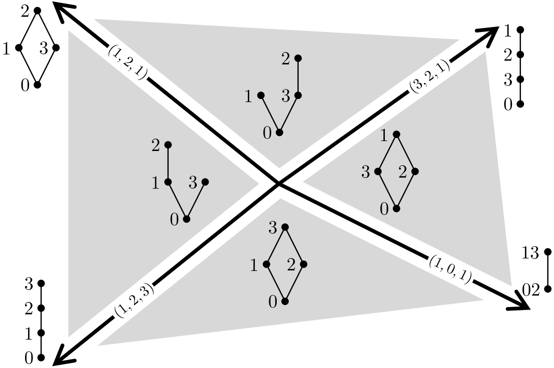
C_3 and C_4



1
2
0
1
2
0
$2a_1 \geq a_2$
$2a_2 \geq a_1$
2
1
0



(1, 2, 1)
(3, 2, 1)
(1, 2, 3)
(1, 0, 1)
2
1
3
0
1
2
3
0
13
02
3
2
1
0

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Spoiler

If S, S' have identical Kunz posets, then S and S' have the same number of minimal trades.

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$\mathsf{Z}(n) = \left\{ \mathbf{a} \in \mathbb{Z}_{\geq 0}^k : n = a_1 n_1 + \cdots + a_k n_k \right\}$$

is the *set of factorizations* of $n \in S$.

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Example

$S = \langle 6, 9, 20 \rangle$:

$$\mathsf{Z}(60) = \{(10,0,0), (7,2,0), (4,4,0), (1,6,0), (0,0,3)\}$$

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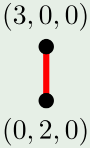

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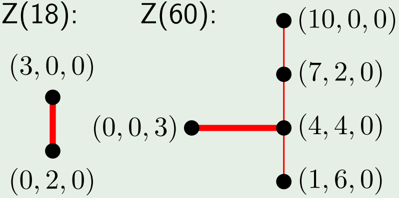

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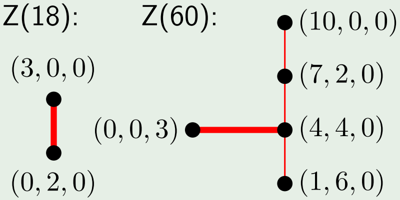

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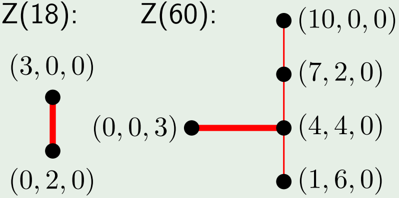

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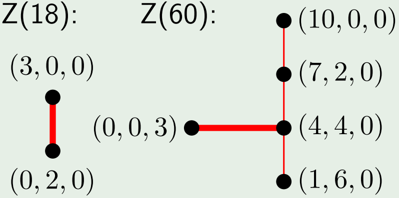

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Generating set for I_S $\Leftrightarrow$ Z(n) connected for all $n \in S$

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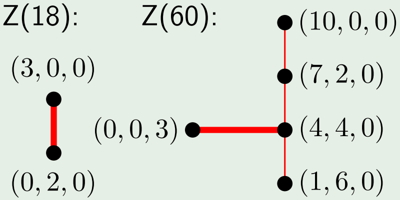

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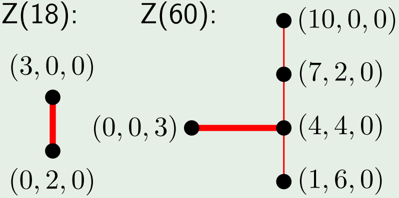

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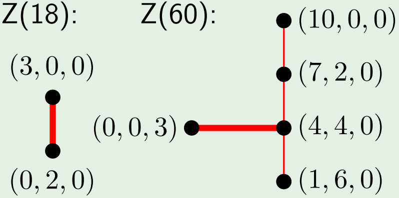

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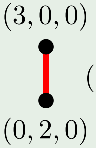

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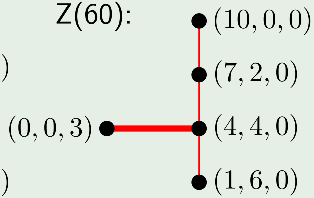

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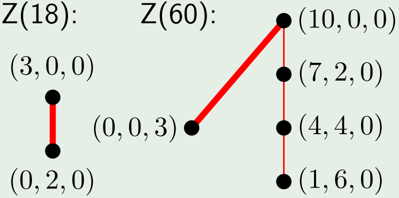

All minimal generating sets of I_S:

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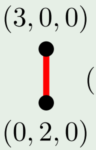

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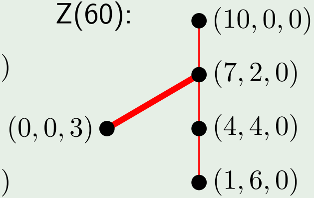

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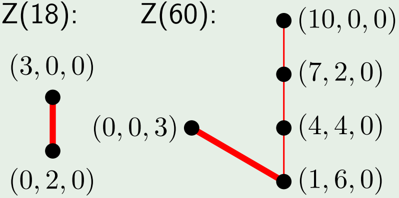

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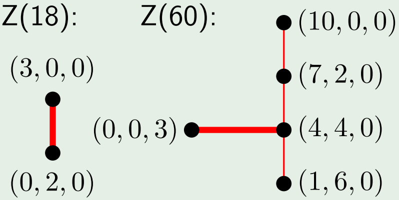

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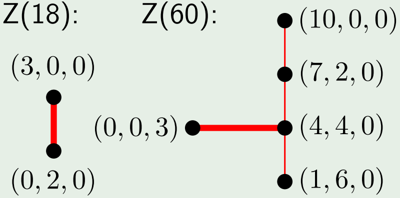

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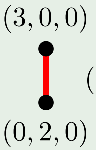

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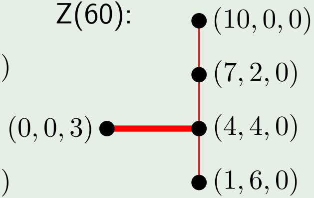

All minimal generating sets of $\sim$:

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Minimal presentations and Betti elements

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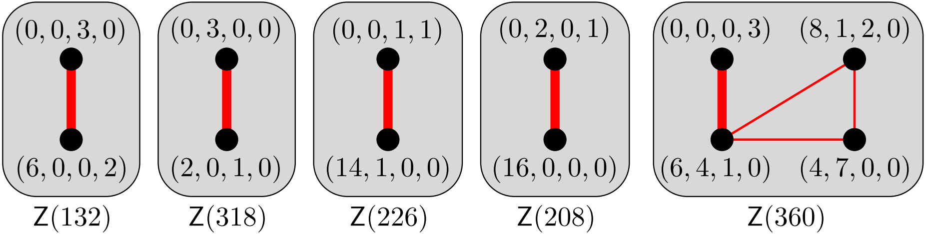

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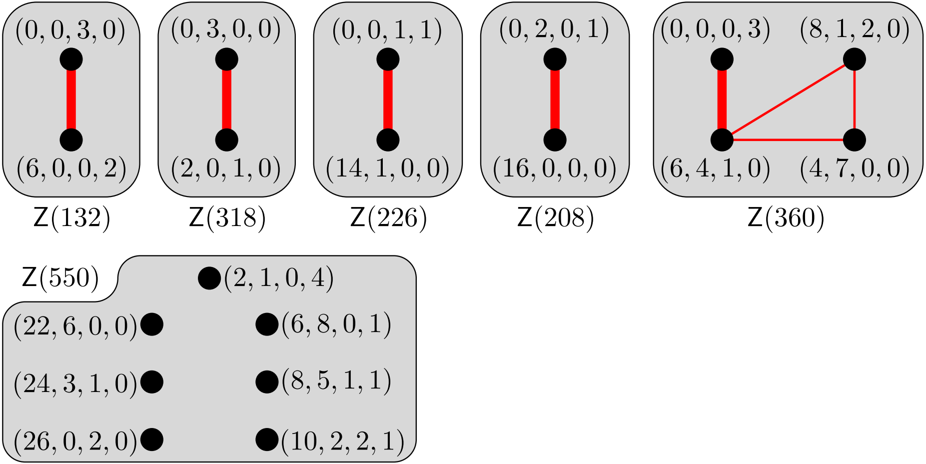

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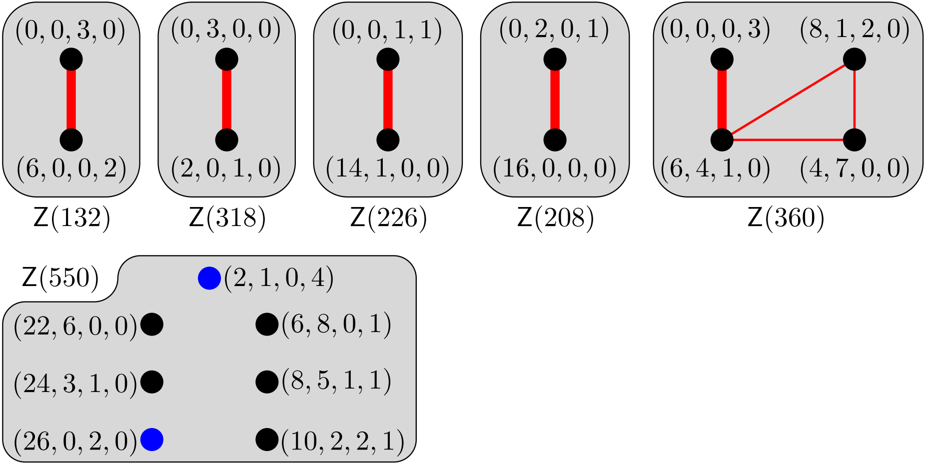

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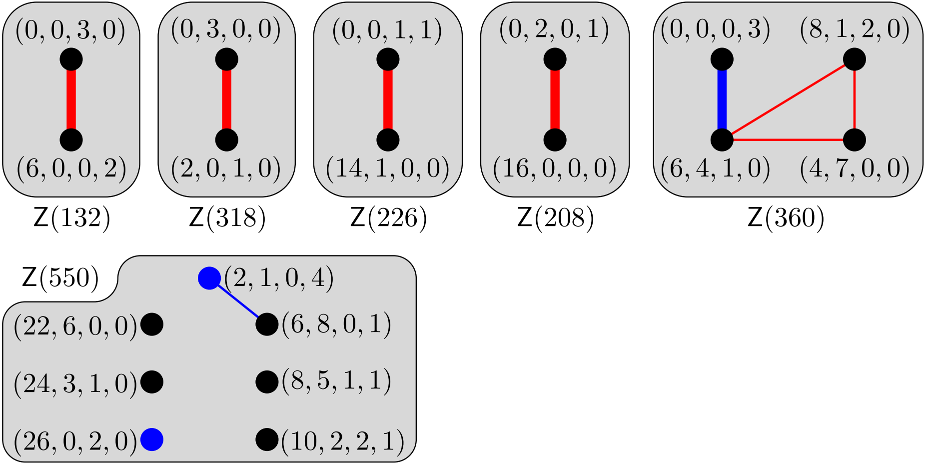

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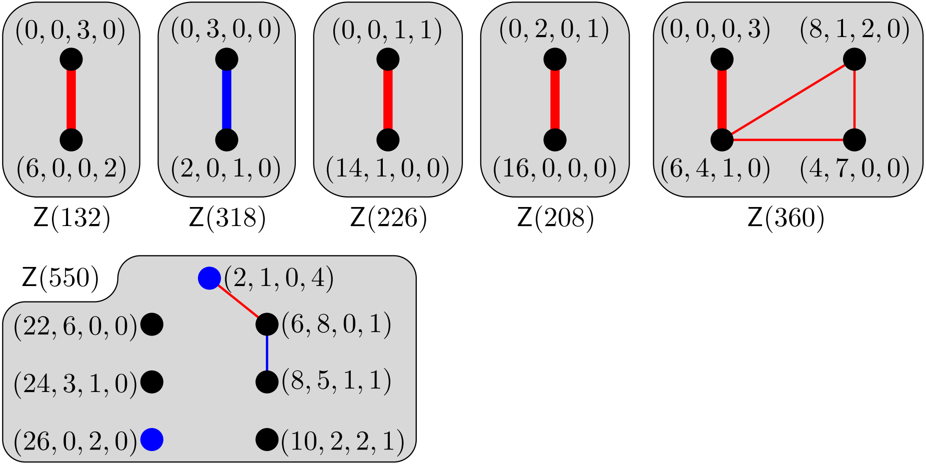

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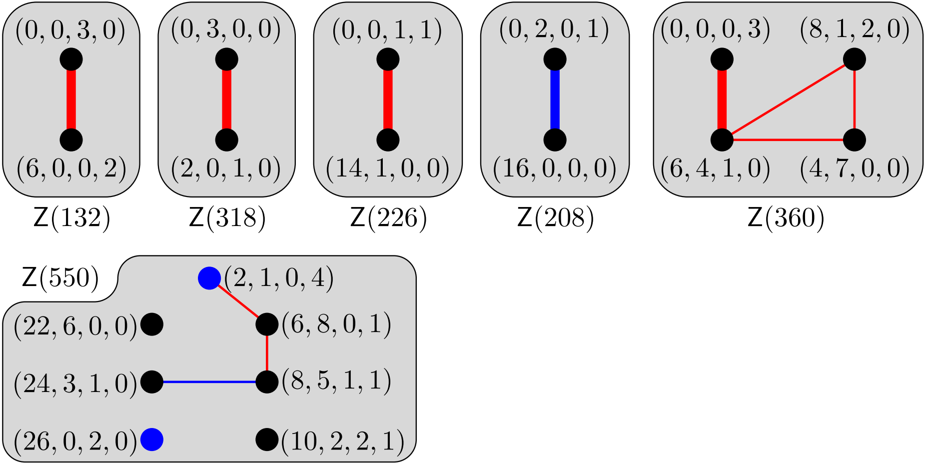

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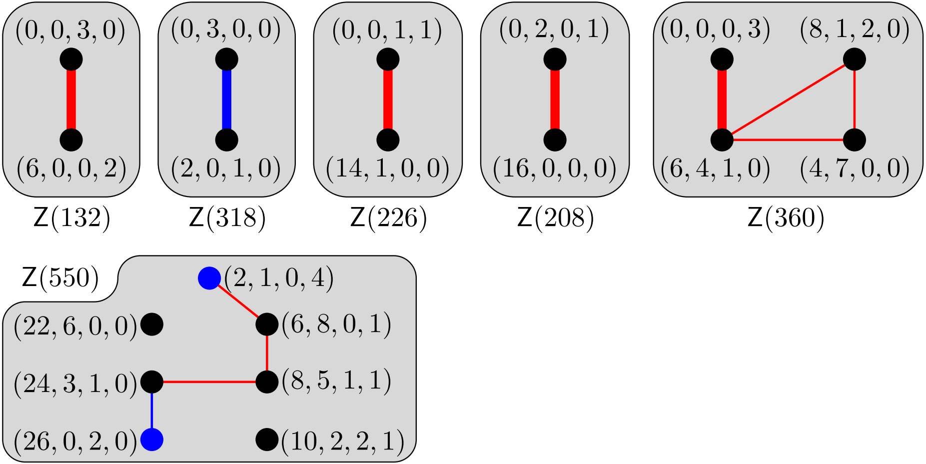

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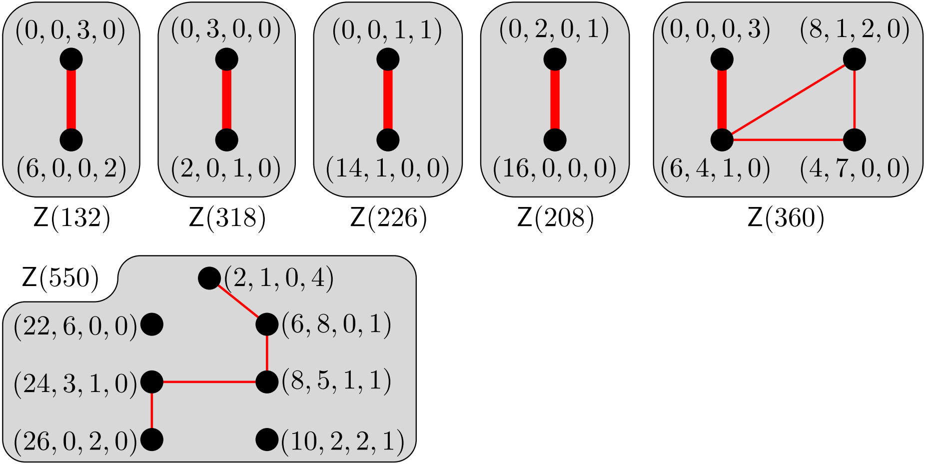

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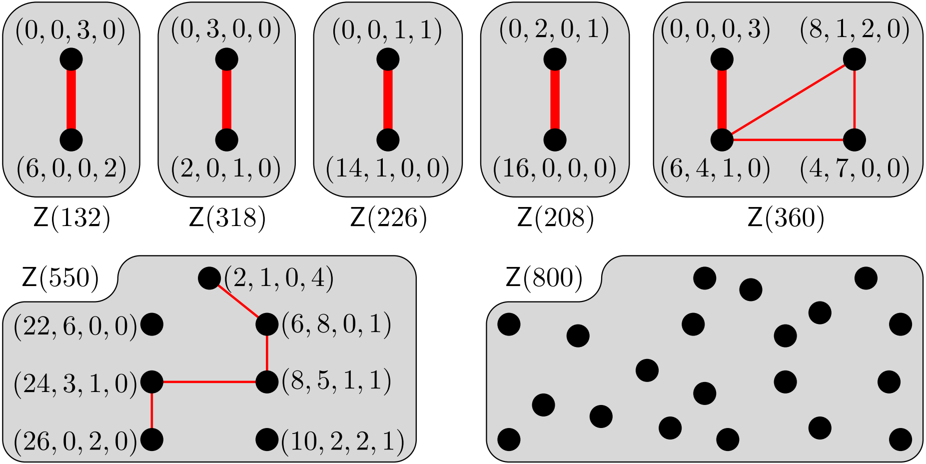

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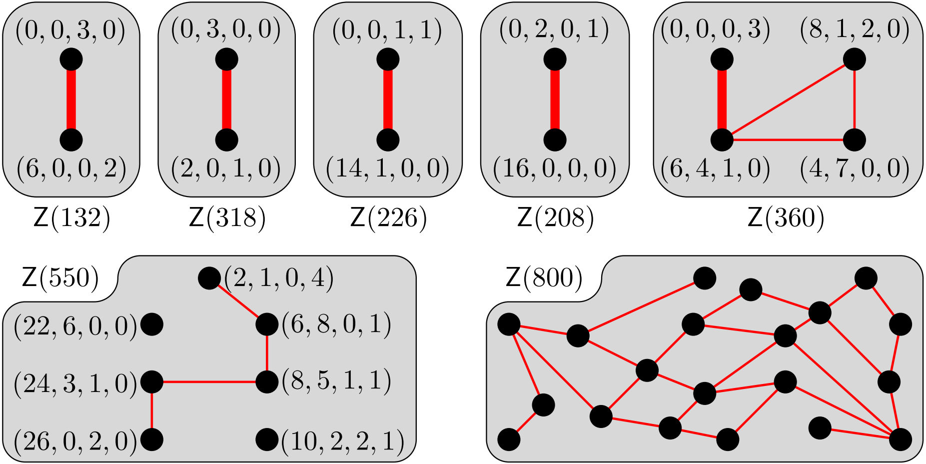

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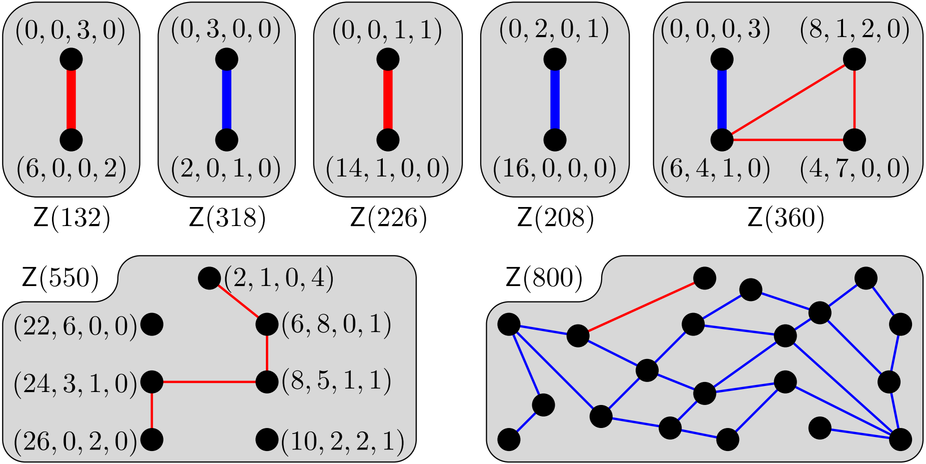

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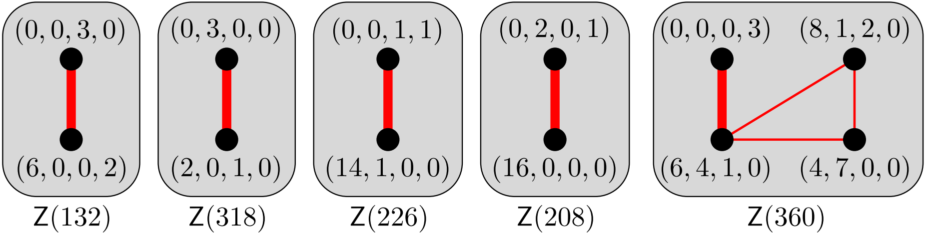

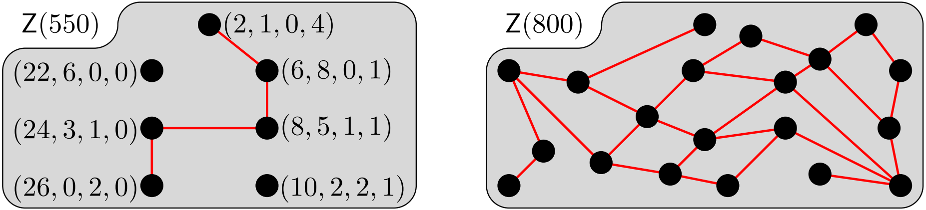

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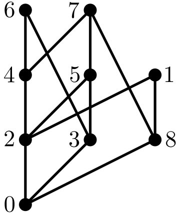

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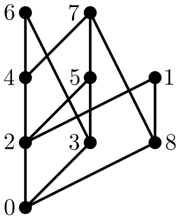

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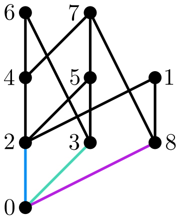

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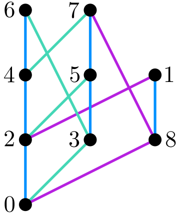

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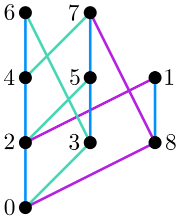

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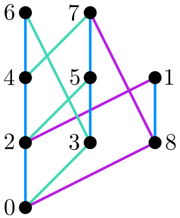

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2 "inner" minimal trades:
$(0,3,0,0) \sim (0,0,2,0)$ (at a_6)
$(0,2,1,0) \sim (0,0,0,2)$ (at a_7)

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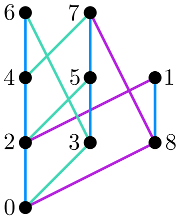

$\mathrm{Ap}(S) = \{0, a_1, a_2, \ldots, a_8\}$

$S = \langle 9, a_2, a_3, a_8 \rangle$

Cover relations: add a generator

$\mathsf{Z}(a_6) = \{(0,3,0,0), (0,0,2,0)\}$

2 "inner" minimal trades:
$(0,3,0,0) \sim (0,0,2,0)$ (at a_6)
$(0,2,1,0) \sim (0,0,0,2)$ (at a_7)

Moral: can recover

- factorizations of $a \in \mathrm{Ap}(S)$
- (minimal) trades at $a \in \mathrm{Ap}(S)$

Minimal trades and Kunz posets

Question

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Minimal trades and Kunz posets

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Key fact: each trade occurs at $a_i + n_j$ for some $a_i \in \mathrm{Ap}(S)$, generator n_j

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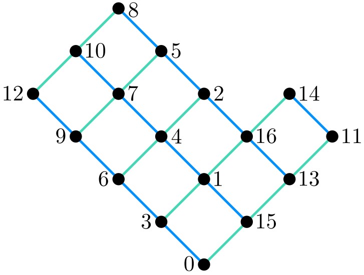

$$S = \langle 17, a_3, a_{15} \rangle$$

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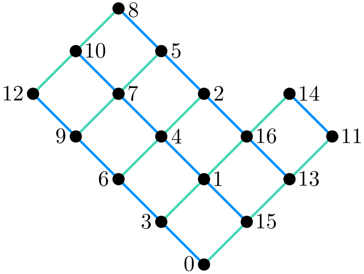

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3 minimal trades, none in $\mathrm{Ap}(S)$

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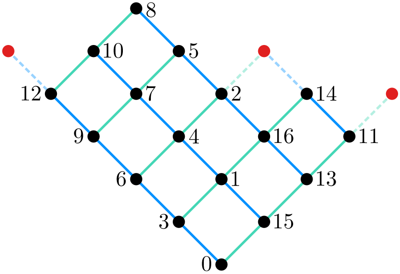

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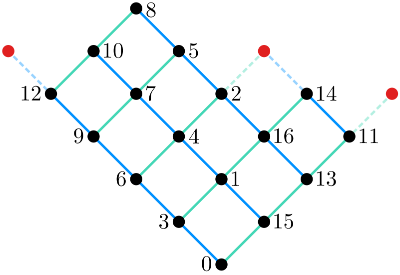

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3 minimal trades, none in $\mathrm{Ap}(S)$

$a_{12} + a_3$: $(0, 5, 0) \sim (\ \ ,\ \ ,\ \ )$

$a_{11} + a_{15}$: $(0, 0, 4) \sim (\ \ ,\ \ ,\ \ )$

$a_2 + a_{15}$: $(0, 2, 3) \sim (\ \ ,\ \ ,\ \ )$

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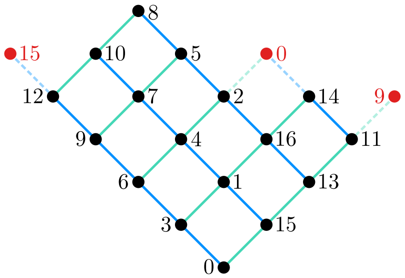

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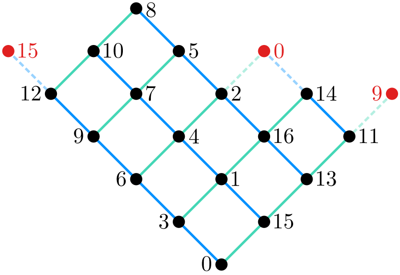

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$a_{12} + a_3$: $(0, 5, 0) \sim (*, 0, 1)$

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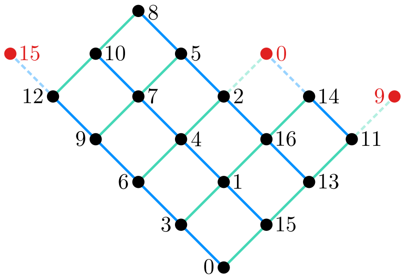

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Possible method to locate the "outer" trades:

- factorizations of $a \in \mathrm{Ap}(S)$ form a monomial staircase
- one "outer" minimal trade for each monomial generator

Minimal trades and Kunz posets

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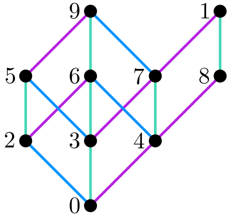

$$S = \langle 10, a_2, a_3, a_4 \rangle$$

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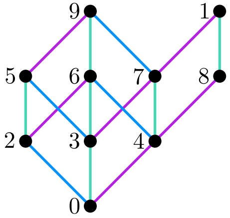

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"inner" trade at a_6:
$(0, 0, 2, 0) \sim (0, 1, 0, 1)$

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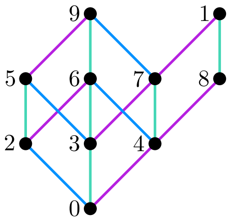

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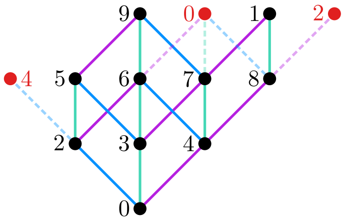

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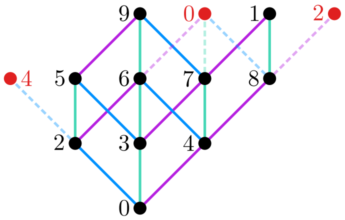

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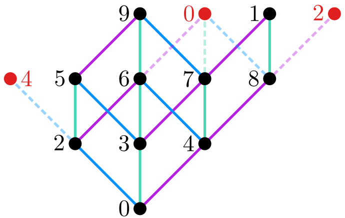

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Moral: use **sets** of factorizations, avoids overcounting minimal trades

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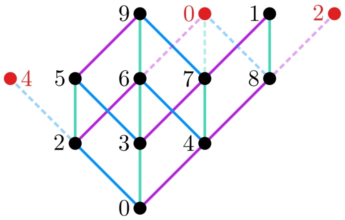

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$(0, 0, 2, 1)$, $(0, 1, 0, 2)$,
$(0, 0, 0, 3)$, $(0, 2, 0, 0)$

Moral: use **sets** of factorizations, avoids overcounting minimal trades

0: $\{(0, 0, 2, 1), (0, 1, 0, 2)\}$
2: $\{(0, 0, 0, 3)\}$, 4: $\{(0, 2, 0, 0)\}$

Minimal trades and Kunz posets

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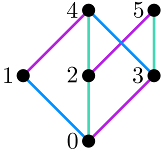
Key fact: each trade occurs at $a_i + n_j$ for some $a_i \in \mathrm{Ap}(S)$, generator n_j

$$S = \langle 6, 7, 8, 9 \rangle$$



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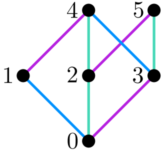

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"inner" trade at a_4:
$(0, 0, 2, 0) \sim (0, 1, 0, 1)$

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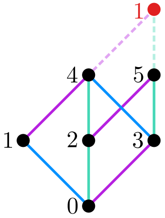

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"inner" trade at a_4:
$(0, 0, 2, 0) \sim (0, 1, 0, 1)$

candidate for "outer" trade:
$(0, 0, 2, 1) \in \mathsf{Z}(25)$

Minimal trades and Kunz posets

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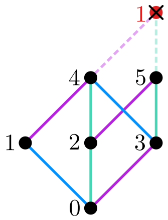

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"inner" trade at a_4:
$(0, 0, 2, 0) \sim (0, 1, 0, 1)$

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$(0, 0, 2, 1) \in \mathsf{Z}(25)$

No trades in $\mathsf{Z}(25)$:
$\{(0, 0, 2, 1), (0, 1, 0, 2), (3, 1, 0, 0)\}$

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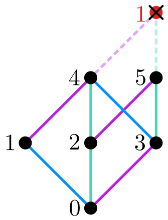

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A technical definition

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \mathrm{supp}(B)$.

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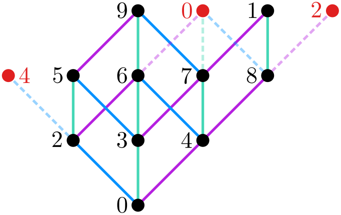


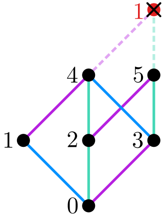
$B = \{(0,0,2,1), (0,1,0,2)\}$

$B - e_2 = \{(0,0,0,2)\} = \mathsf{Z}(a_8)$

$B - e_3 = \{(0,0,1,1)\} = \mathsf{Z}(a_7)$

$B - e_4 = \{(0,0,2,0), (0,1,0,1)\}$
$= \mathsf{Z}(a_6)$

$S = \langle 6, 7, 8, 9 \rangle$



$B = \{(0,0,2,1)\}$?

$B - e_4 = \{(0,0,2,0)\} \subsetneq \mathsf{Z}(a_4)$

$B = \{(0,0,2,1), (0,1,0,2)\}$?

$B - e_3 = \{(0,0,1,1)\} \not\subseteq \mathsf{Z}(a_i)$

The main theorem

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Recovering minimal presentation from the Kunz poset P of S:

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Recovering minimal presentation from the Kunz poset P of S:

- trades occuring at $a \in \mathrm{Ap}(S)$ recovered from factorizations of $\overline{a} \in P$

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Recovering minimal presentation from the Kunz poset P of S:

- trades occuring at $a \in \mathrm{Ap}(S)$ recovered from factorizations of $\overline{a} \in P$
- each trade occuring outside $\mathrm{Ap}(S)$ corresponds to an outer Betti element of P

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Recovering minimal presentation from the Kunz poset P of S:

- trades occuring at $a \in \mathrm{Ap}(S)$ recovered from factorizations of $\overline{a} \in P$
- each trade occuring outside $\mathrm{Ap}(S)$ corresponds to an outer Betti element of P

Theorem (Gomes–O.–Torres Davila)

If S has Kunz poset P, each minimal trade of S not ocurring in $\mathrm{Ap}(S)$ contains a factorization from a distinct outer Betti element of P.

In particular, if S, S' have identical Kunz poset, then S and S' have the same number of minimal trades.

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Another subtlety: distinct outer Betti elements can **coincide** for some S

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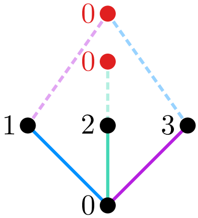

$B_1 = \{(0, 0, 2, 0)\}$

$B_2 = \{(0, 1, 0, 1)\}$

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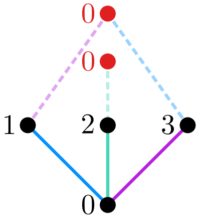

$S = \langle 4, 9, 14, 11 \rangle$

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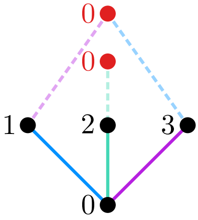

$S = \langle 4, 9, 14, 11 \rangle$

20: $(0,1,0,1), (5,0,0,0)$
28: $(0,0,2,0), (2,1,0,0), (5,0,0,0)$

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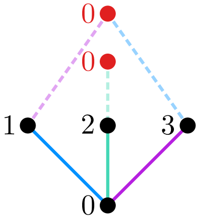

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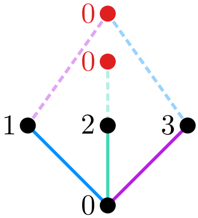

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$S = \langle 4, 9, 14, 11 \rangle$

20: $(0,1,0,1), (5,0,0,0)$
28: $(0,0,2,0), (2,1,0,0), (5,0,0,0)$

$S = \langle 4, 9, 10, 11 \rangle$

20: $(0,0,2,0), (0,1,0,1), (5,0,0,0)$

The main theorem

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An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \operatorname{supp}(B)$.

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Theorem (Gomes–O.–Torres Davila)

If S has Kunz poset P, each minimal trade of S not ocurring in $\mathrm{Ap}(S)$ contains a factorization from a distinct outer Betti element of P.

In particular, if S, S' have identical Kunz poset, then S and S' have the same number of minimal trades.

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If S has Kunz poset P, each minimal trade of S not ocurring in $\mathrm{Ap}(S)$ *contains a factorization from a distinct outer Betti element of P.*

In particular, if S, S' have identical Kunz poset, then S and S' have the same number of minimal trades.

For $m = 6$: $\quad\#$ minimal trades $\in \{1, 2, 3, 4, 5, 6, 9, 10, 15\}$

Application: classifying minimal trades

Question

Given the multiplicity $m = \mathrm{m}(S)$ and # minimal generators $e = \mathrm{e}(S)$ of a numerical semigroup S, what can $\beta_1(I_S) = \#$ minimal trades be?

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5	2	3		5	6				10															
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15	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
16	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
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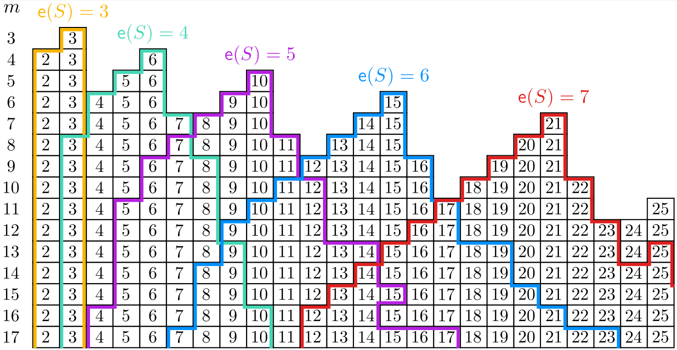

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$\mathsf{e}(S) = 3$ $\mathsf{e}(S) = 4$ $\mathsf{e}(S) = 5$ $\mathsf{e}(S) = 6$ $\mathsf{e}(S) = 7$

m																								
3		3																						
4	2	3			6																			
5	2	3		5	6				10															
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Well known: $\beta_1(S) \le \binom{m}{2}$, with equality if and only if $e = m$
if $e = 3$, then $\beta_1(S) = 2, 3$

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$\mathsf{e}(S) = 3$, $\mathsf{e}(S) = 4$, $\mathsf{e}(S) = 5$, $\mathsf{e}(S) = 6$, $\mathsf{e}(S) = 7$

m																								
3		3																						
4	2	3			6																			
5	2	3		5	6				10															
6	2	3	4	5	6			9	10					15										
7	2	3	4	5	6	7	8	9	10				14	15						21				
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14	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
15	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
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17	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

Prior work: a family has $\beta_1(S) = \binom{e}{2}$ for $3 \le e \le m$ (Rosales)

if $r = m - e \le 2$, then $\beta_1(S) \in [\binom{e}{2} - r, \binom{e}{2}]$ (GS-R)

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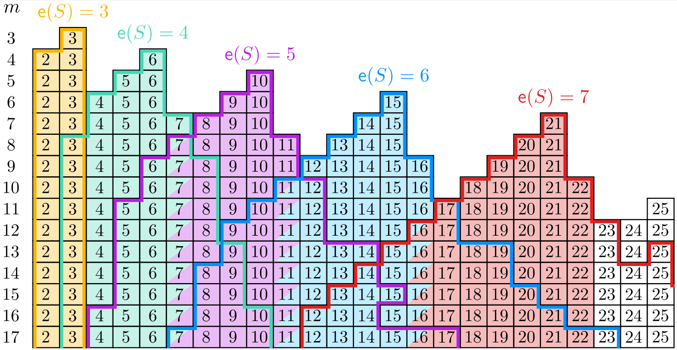

Using Kunz posets: a family hits each $\beta_1(S) \in [\binom{e}{2} - r, \binom{e}{2}]$

for $r = m - e \le e - 2$

a family hits $\beta_1(S) = \binom{e}{2} + 1$ for each $m \ge e + 3$

Application: classifying minimal trades

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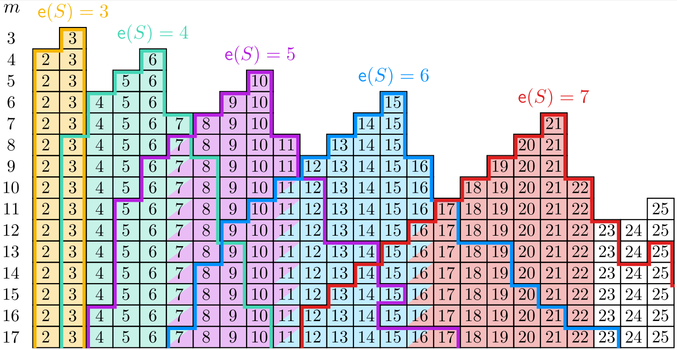

Bounds from Kunz posets: $\beta_1(S) \ge \binom{e}{2} - r$, where $r = m - e$

if $m - e = 3$, then $\beta_1(S) \in [\binom{e}{2} - 3, \binom{e}{2} + 1]$

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Given the multiplicity $m = \mathsf{m}(S)$ and # minimal generators $e = \mathsf{e}(S)$ of a numerical semigroup S, what can $\beta_1(I_S) =$ # minimal trades be?

$\mathsf{e}(S) = 4$

m																								
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One more family: for $e = 4$, achieves each $\beta_1(S)$ with $(\beta_1(S) - 2)^2 \leq 4m$

conjectured to achieve every possible $\beta_1(S)$ for $e = 4$

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Numerical semigroups, polyhedra, and posets I: the group cone
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Thanks!