Numerical semigroups, minimal presentations, and posets

Christopher O'Neill

San Diego State University

cdoneill@sdsu.edu

Joint with (i) Winfred Bruns, Pedro García-Sánchez, and Dane Wilbourne;
(ii) Nathan Kaplan; (iii) *Tara Gomes and *Eduardo Torres Davila;
(iv) *Ceyhun Elmacioglu, Kieran Hilmer, *Hannah Koufmann, and *Melin Okandan

* = undergraduate student

Slides available: `https://cdoneill.sdsu.edu/`

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$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

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Multiplicity: $\mathsf{m}(S) =$ smallest nonzero element

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Theorem

If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup S with $\mathrm{Ap}(S) = A$ if and only if

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Big idea: the inequalities "$a_i + a_j \geq a_{i+j}$" to define a **cone** C_m.

Kunz cone

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The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

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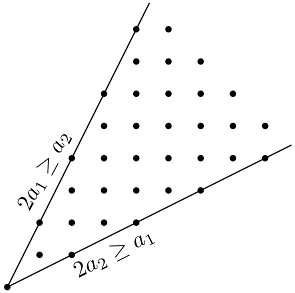
Example: C_3



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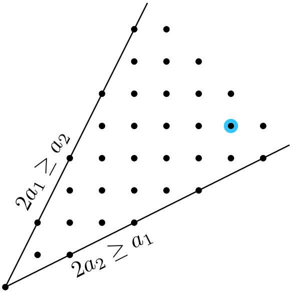

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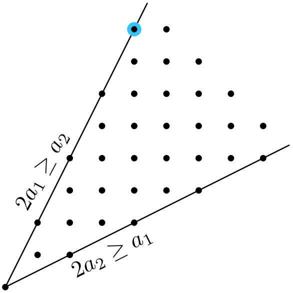

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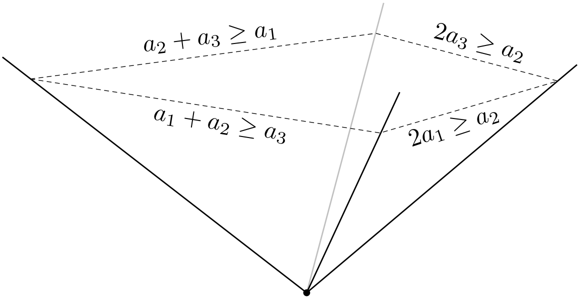

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When are numerical semigroups in (the relative interior of) the same face?

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Big picture: "moduli space" approach for studying XYZ's

- Define a space with XYZ's as points
  Small changes to an XYZ $\rightsquigarrow$ small movements in space
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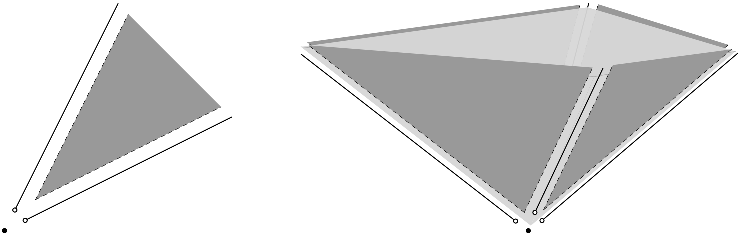
More interesting example: C_m



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What about the other faces?

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Faces of the Kunz cone

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The *Apéry poset* of S: define $a \preceq a'$ whenever $a' - a \in S$.

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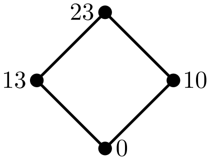

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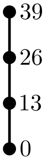

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$$S = \langle 6, 9, 20 \rangle \qquad S' = \langle 6, 26, 27 \rangle$$

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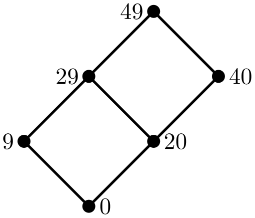

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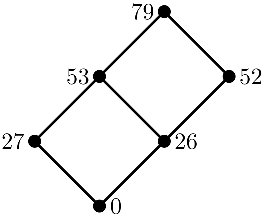

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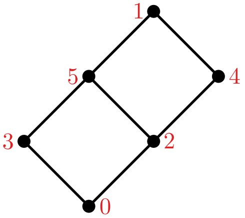

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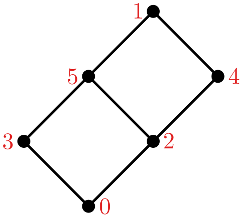

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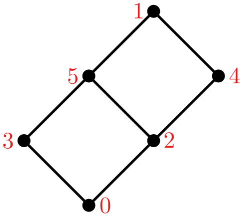

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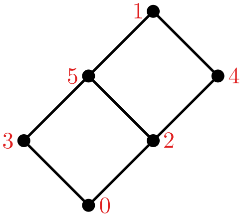

The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

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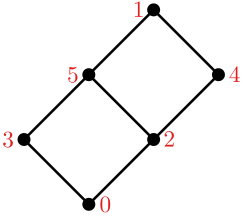

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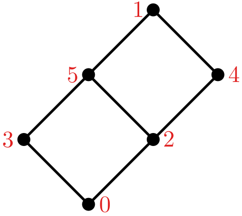

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Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

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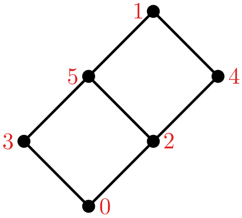

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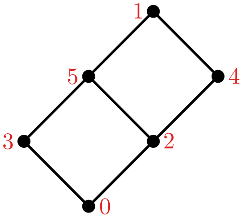

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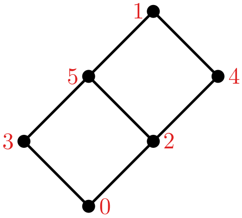

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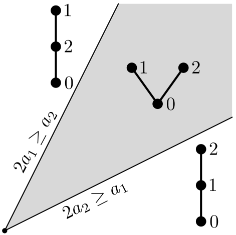

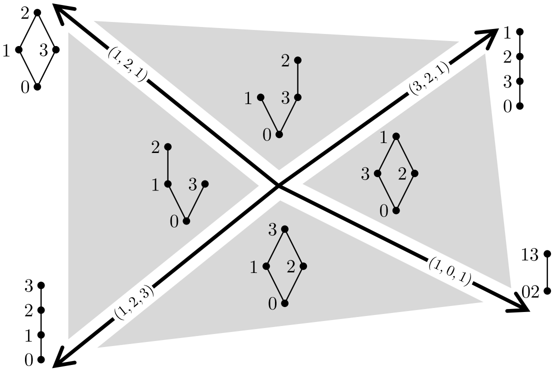

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Spoiler

If S, S' have identical Kunz posets, then S and S' have the same number of minimal trades.

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$\mathsf{Z}(n) = \left\{ \mathbf{a} \in \mathbb{Z}_{\geq 0}^k : n = a_1 n_1 + \cdots + a_k n_k \right\}$$

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Example

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$$\mathsf{Z}(60) = \{(10,0,0), (7,2,0), (4,4,0), (1,6,0), (0,0,3)\}$$

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$\ker \pi$ is a *congruence*: an equivalence relation

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$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

that is closed under *translation*

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

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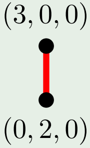

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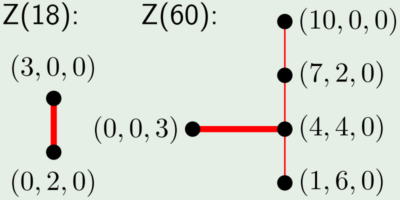

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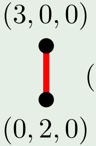

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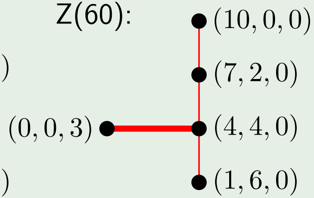

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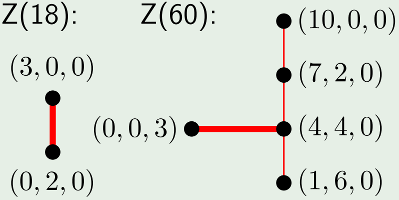

$$\begin{aligned} x^7y^2 - x^4y^4 &= x^4y^2(x^3 - y^2) \\ x^7y^2 - z^3 &= (x^7y^2 - x^4y^4) \\ &\quad + (x^4y^4 - z^3) \end{aligned}$$

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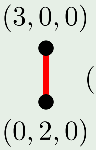

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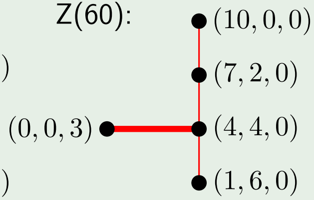

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Generating set for I_S $\Leftrightarrow$ $Z(n)$ connected for all $n \in S$

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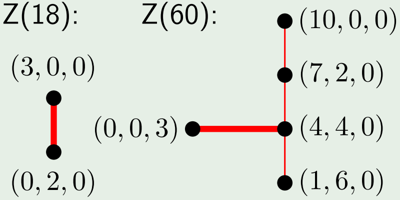

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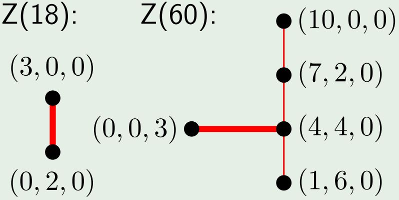

All minimal generating sets of I_S:

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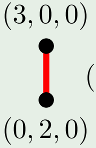

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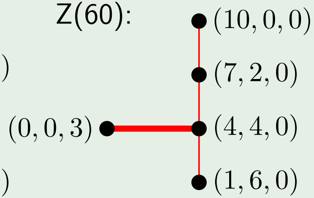

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$$\begin{aligned} I_S &= \langle x^3 - y^2,\, x^{10} - z^3 \rangle \\ &= \langle x^3 - y^2,\, x^7 y^2 - z^3 \rangle \\ &= \langle x^3 - y^2,\, x^4 y^4 - z^3 \rangle \\ &= \langle x^3 - y^2,\, x^6 y - z^3 \rangle \end{aligned}$$

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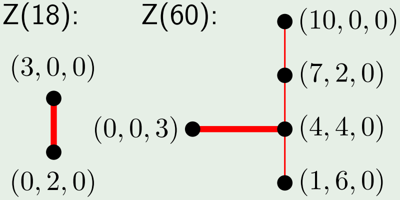

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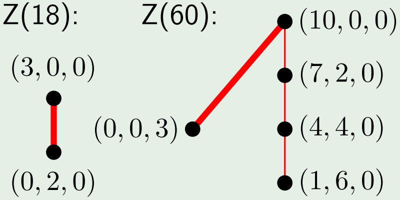

All minimal generating sets of I_S:

$$\begin{aligned} I_S &= \langle x^3 - y^2,\ x^{10} - z^3 \rangle \\ &= \langle x^3 - y^2,\ x^7y^2 - z^3 \rangle \\ &= \langle x^3 - y^2,\ x^4y^4 - z^3 \rangle \\ &= \langle x^3 - y^2,\ x^6y - z^3 \rangle \end{aligned}$$

Minimal presentations

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \rightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

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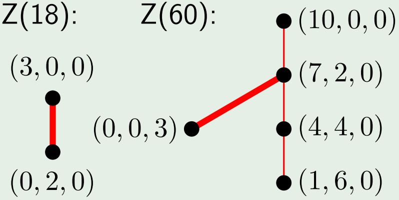

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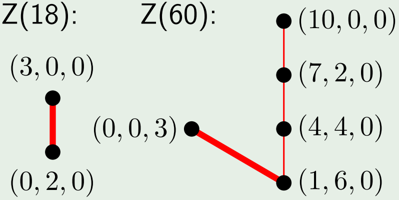

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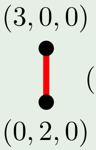

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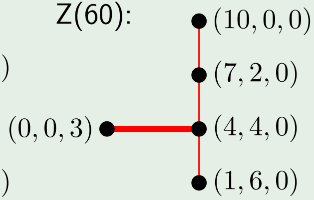

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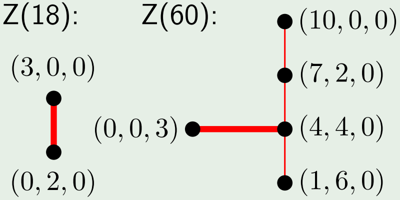

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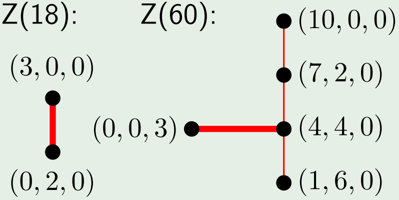

All minimal generating sets of $\sim$:

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Minimal presentations and Betti elements

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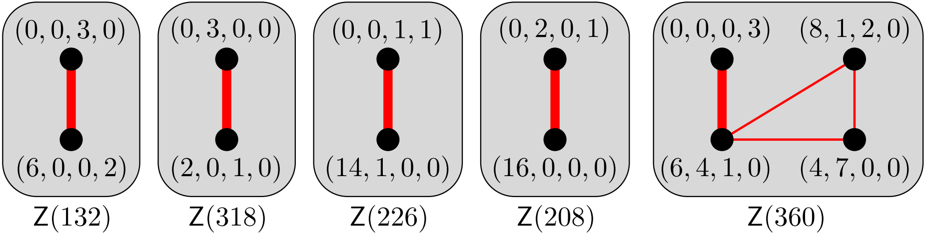

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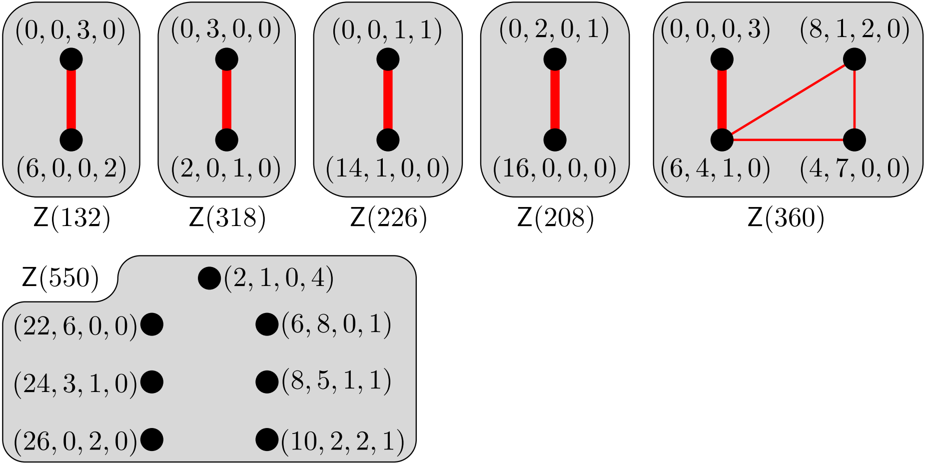

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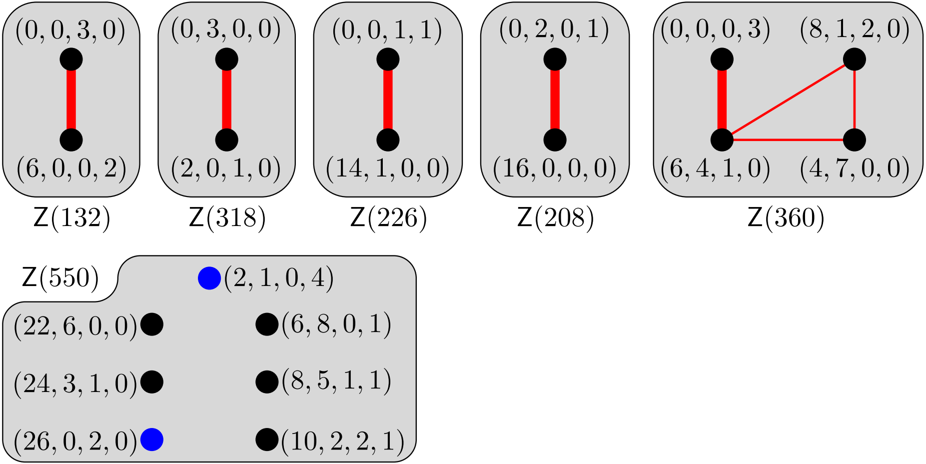

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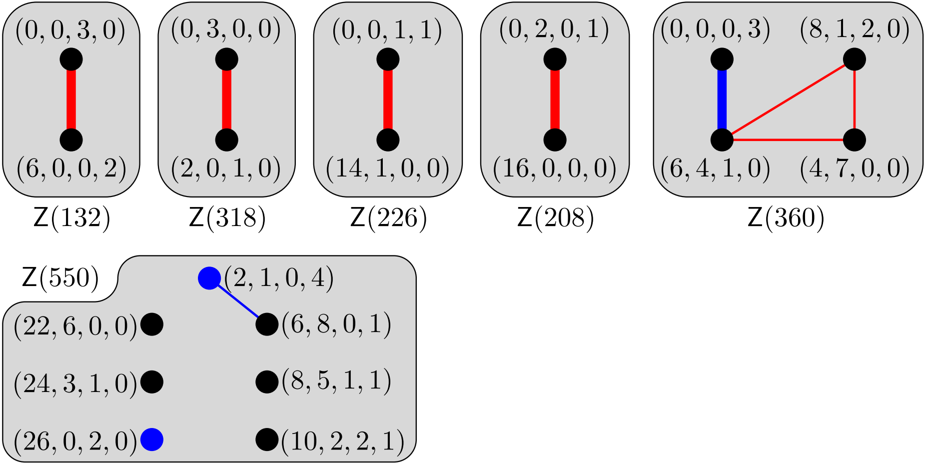

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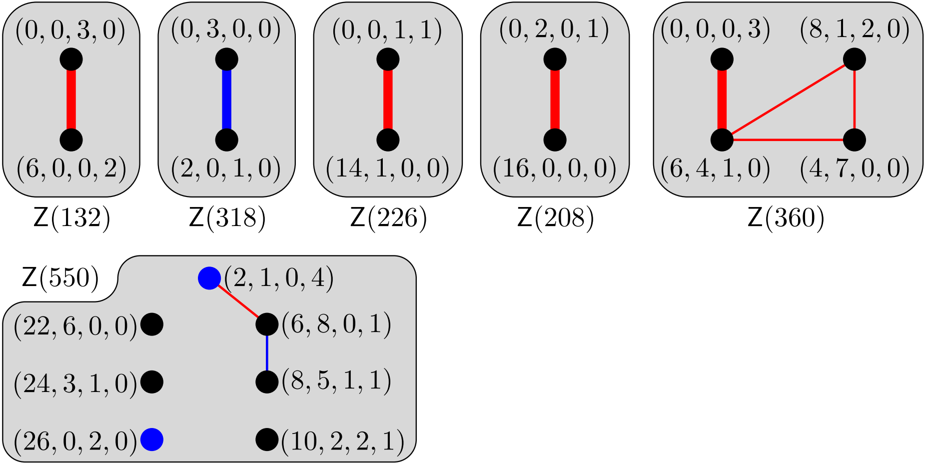

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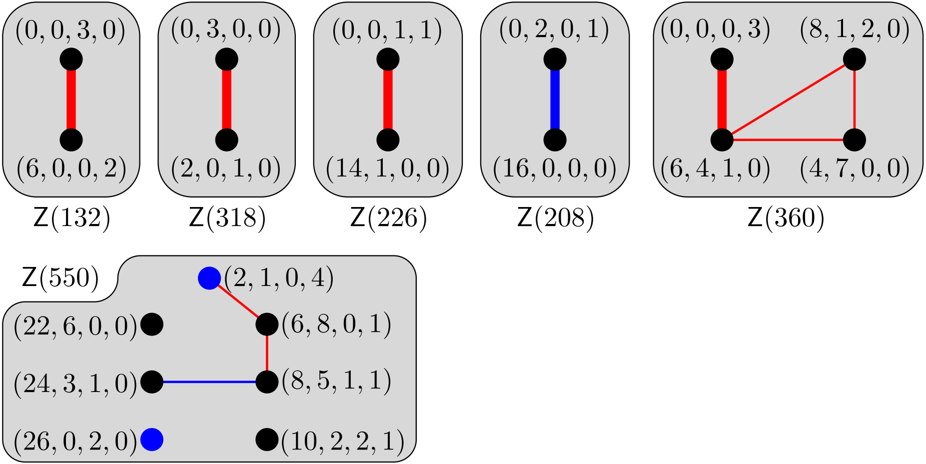

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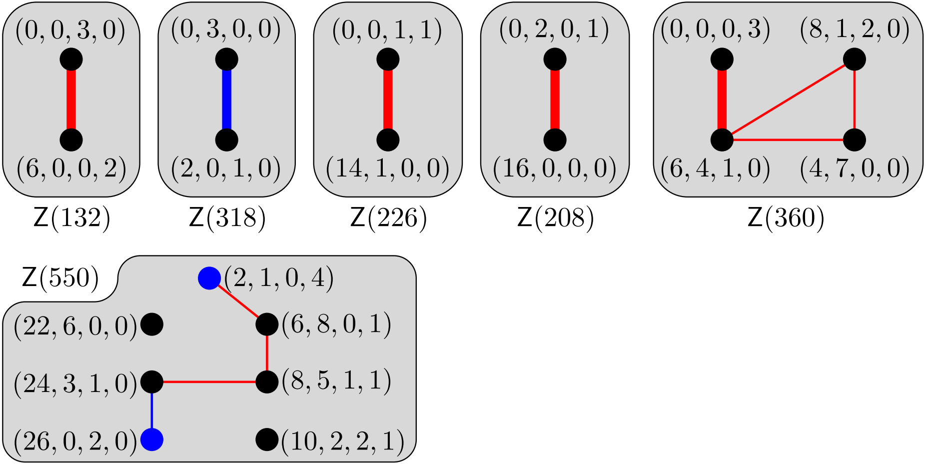

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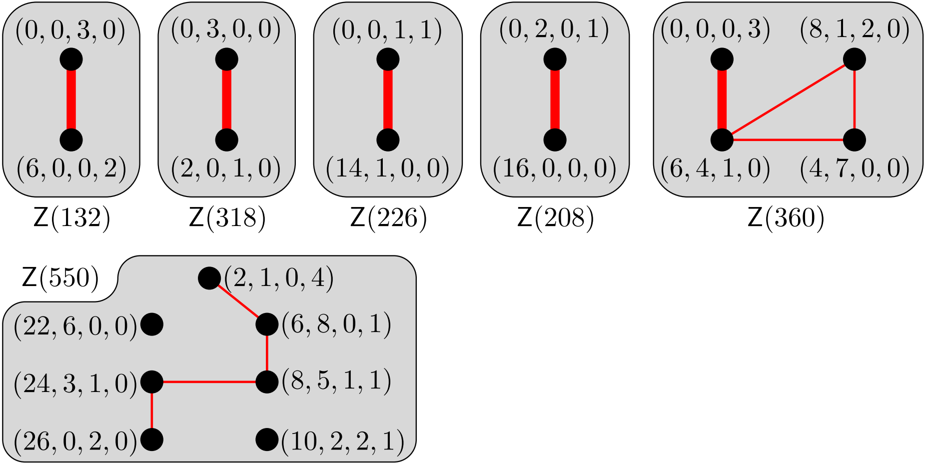

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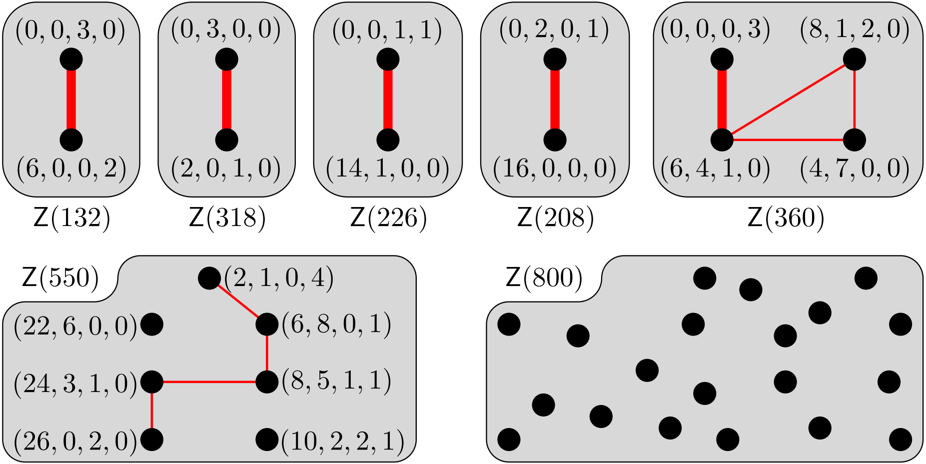

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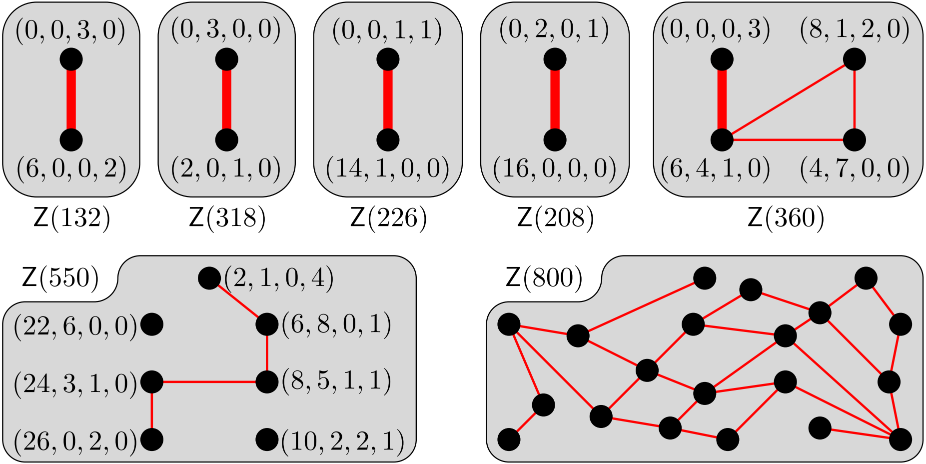

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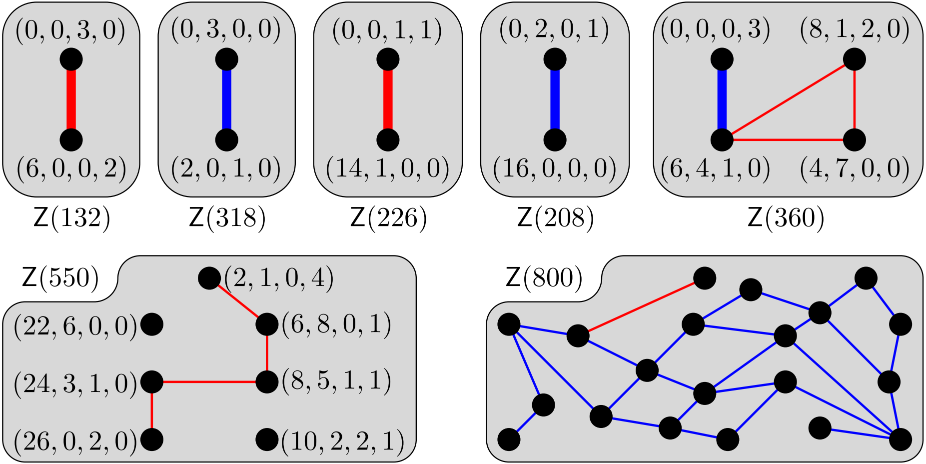

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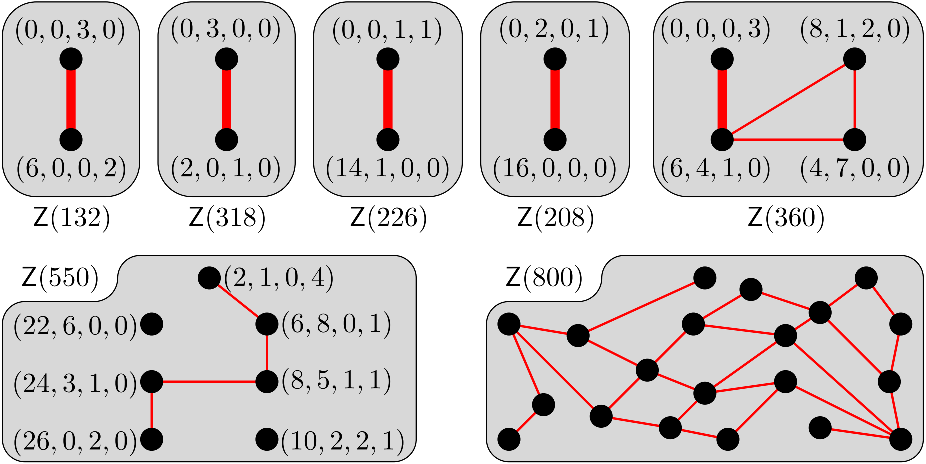

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Question

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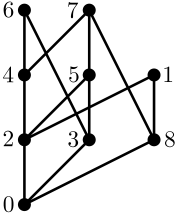

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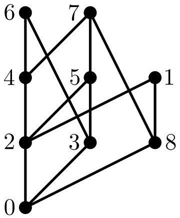

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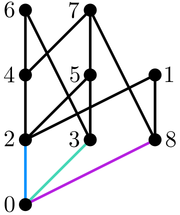

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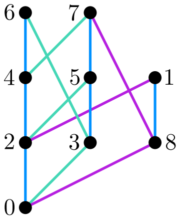

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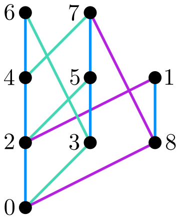

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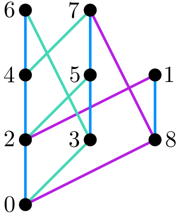

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2 "inner" minimal trades:
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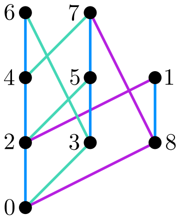

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Moral: can recover

- factorizations of $a \in \mathrm{Ap}(S)$
- (minimal) trades at $a \in \mathrm{Ap}(S)$

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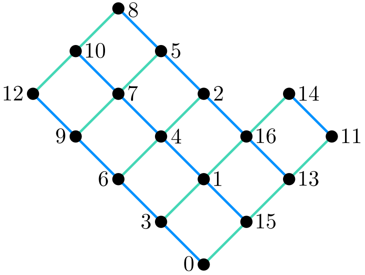

$$S = \langle 17, a_3, a_{15} \rangle$$

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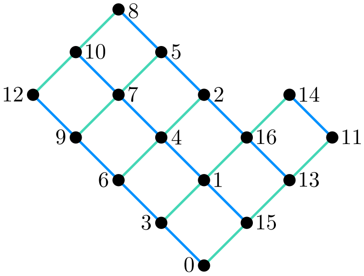

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3 minimal trades, none in $\mathrm{Ap}(S)$

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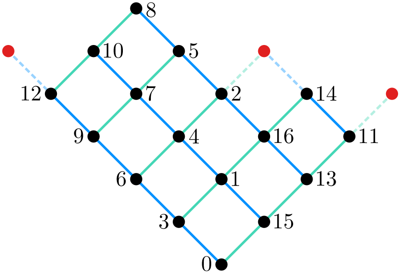

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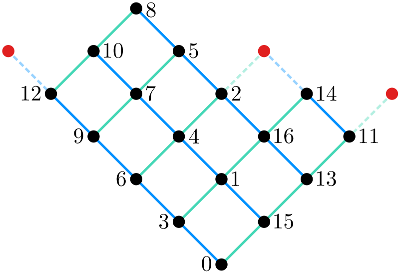

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$a_{12} + a_3$: $(0, 5, 0) \sim (\ ,\ ,\ )$

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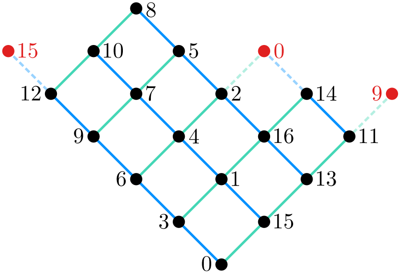

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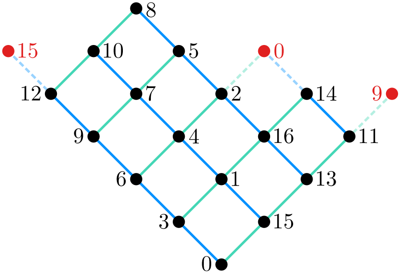

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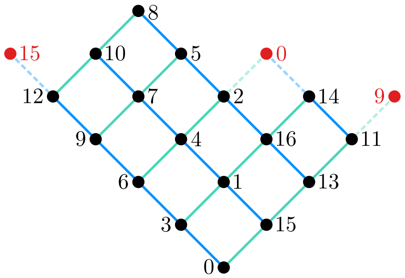

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Possible method to locate the "outer" trades:

- factorizations of $a \in \mathrm{Ap}(S)$ form a monomial staircase
- one "outer" minimal trade for each monomial generator

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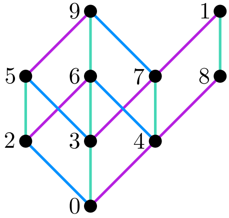

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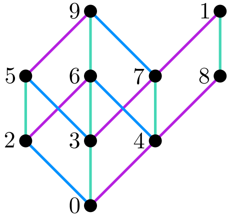

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"inner" trade at a_6:
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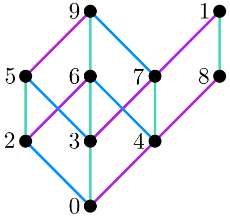

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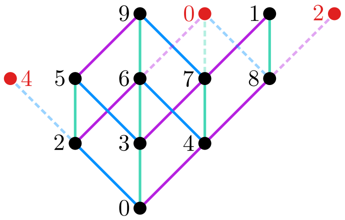

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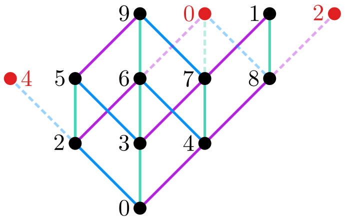

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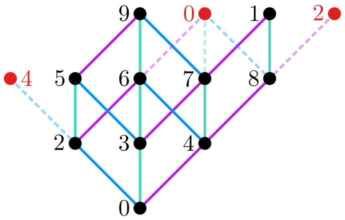

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Moral: use **sets** of factorizations, avoids overcounting minimal trades

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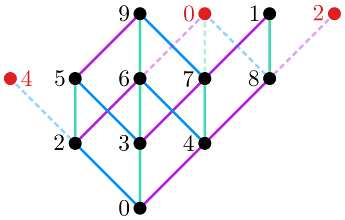

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Moral: use **sets** of factorizations, avoids overcounting minimal trades

0: $\{(0,0,2,1),(0,1,0,2)\}$
2: $\{(0,0,0,3)\}$, 4: $\{(0,2,0,0)\}$

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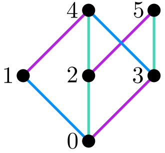

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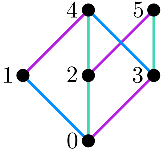

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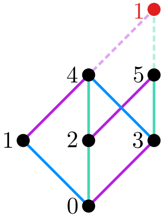

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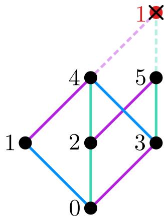

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No trades in $\mathsf{Z}(25)$:
$\{(0, 0, 2, 1), (0, 1, 0, 2), (3, 1, 0, 0)\}$

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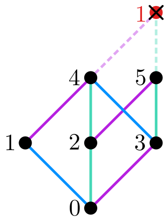

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A technical definition

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An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \operatorname{supp}(B)$.

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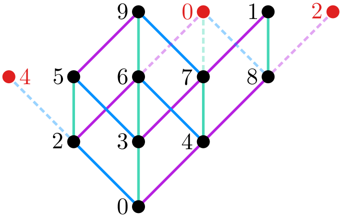

$B = \{(0,0,2,1), (0,1,0,2)\}$
$B - e_2 = \{(0,0,0,2)\} = \mathsf{Z}(a_8)$
$B - e_3 = \{(0,0,1,1)\} = \mathsf{Z}(a_7)$
$B - e_4 = \{(0,0,2,0), (0,1,0,1)\}$
$= \mathsf{Z}(a_6)$

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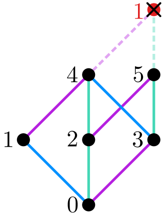

$B = \{(0,0,2,1)\}$?
$B - e_4 = \{(0,0,2,0)\} \subsetneq \mathsf{Z}(a_4)$

$B = \{(0,0,2,1), (0,1,0,2)\}$?
$B - e_3 = \{(0,0,1,1)\} \not\subseteq \mathsf{Z}(a_i)$

The main theorem

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Recovering minimal presentation from the Kunz poset P of S:

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- trades occuring at $a \in \mathrm{Ap}(S)$ recovered from factorizations of $\overline{a} \in P$

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Theorem (Gomes–O.–Torres Davila)

If S has Kunz poset P, each minimal trade of S not ocurring in $\mathrm{Ap}(S)$ contains a factorization from a distinct outer Betti element of P.

In particular, if S, S' have identical Kunz poset, then S and S' have the same number of minimal trades.

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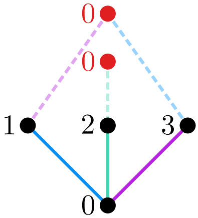

$B_1 = \{(0, 0, 2, 0)\}$

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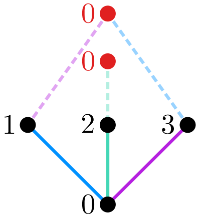

$S = \langle 4, 9, 14, 11 \rangle$

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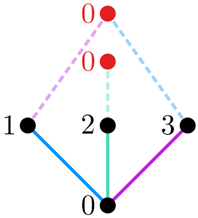

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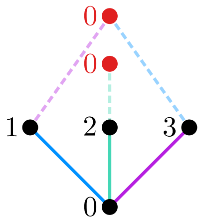

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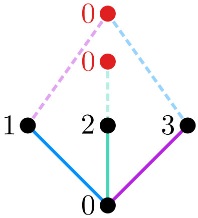

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The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \mathrm{supp}(B)$.

Theorem (Gomes–O.–Torres Davila)

If S has Kunz poset P, each minimal trade of S not ocurring in $\mathrm{Ap}(S)$ contains a factorization from a distinct outer Betti element of P.

In particular, if S, S' have identical Kunz poset, then S and S' have the same number of minimal trades.

For $m = 6$: $\#$ minimal trades $\in \{1, 2, 3, 4, 5, 6, 9, 10, 15\}$

Application: classifying minimal trades

Question

Given the multiplicity $m = \mathrm{m}(S)$ and # minimal generators $e = \mathrm{e}(S)$ of a numerical semigroup S, what can $\beta_1(I_S) =$ # minimal trades be?

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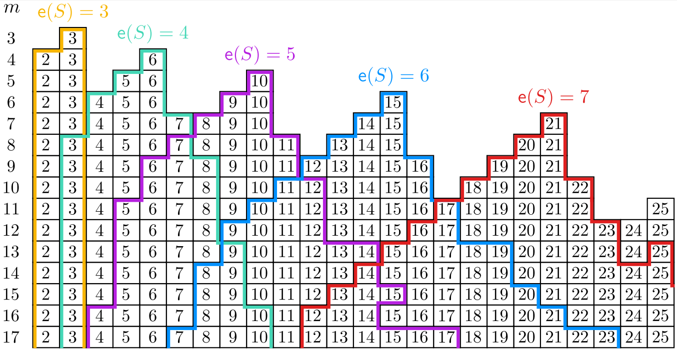

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Given the multiplicity $m = \mathsf{m}(S)$ and # minimal generators $e = \mathsf{e}(S)$ of a numerical semigroup S, what can $\beta_1(I_S)$ = # minimal trades be?

$\mathsf{e}(S) = 3$ $\mathsf{e}(S) = 4$ $\mathsf{e}(S) = 5$ $\mathsf{e}(S) = 6$ $\mathsf{e}(S) = 7$

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Well known: $\beta_1(S) \le \binom{m}{2}$, with equality if and only if $e = m$
if $e = 3$, then $\beta_1(S) = 2, 3$

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$\mathsf{e}(S) = 3$, $\mathsf{e}(S) = 4$, $\mathsf{e}(S) = 5$, $\mathsf{e}(S) = 6$, $\mathsf{e}(S) = 7$

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Prior work: a family has $\beta_1(S) = \binom{e}{2}$ for $3 \le e \le m$ (Rosales)

if $r = m - e \le 2$, then $\beta_1(S) \in [\binom{e}{2} - r, \binom{e}{2}]$ (GS-R)

Application: classifying minimal trades

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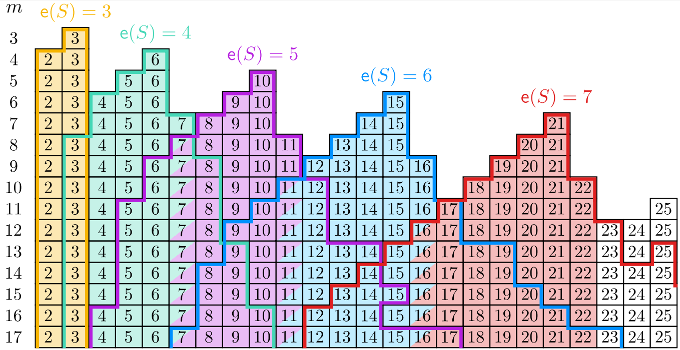

Using Kunz posets: a family hits each $\beta_1(S) \in [\binom{e}{2} - r, \binom{e}{2}]$

for $r = m - e \leq e - 2$

a family hits $\beta_1(S) = \binom{e}{2} + 1$ for each $m \geq e + 3$

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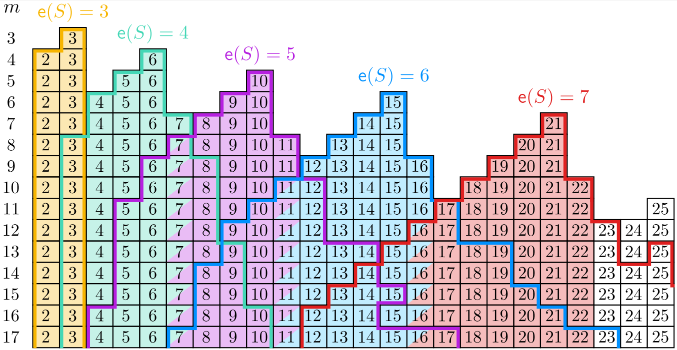

Bounds from Kunz posets: $\beta_1(S) \geq \binom{e}{2} - r$, where $r = m - e$

if $m - e = 3$, then $\beta_1(S) \in [\binom{e}{2} - 3, \binom{e}{2} + 1]$

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Given the multiplicity $m = \mathsf{m}(S)$ and # minimal generators $e = \mathsf{e}(S)$ of a numerical semigroup S, what can $\beta_1(I_S) =$ # minimal trades be?

$\mathsf{e}(S) = 4$

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One more family: for $e = 4$, achieves each $\beta_1(S)$ with $(\beta_1(S) - 2)^2 \le 4m$

conjectured to achieve every possible $\beta_1(S)$ for $e = 4$

References

N. Kaplan, C. O'Neill, (2021)
Numerical semigroups, polyhedra, and posets I: the group cone
Combinatorial Theory **1** (2021), #19. (arXiv:1912.03741)

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Thanks!