

Numerical semigroups, minimal presentations, and posets

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September 12, 2023

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$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \dots \\ \dots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

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Multiplicity: $m(S) =$ smallest nonzero element

Apéry sets

Fix a numerical semigroup S with $m(S) = m$.

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For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \dots\} \cap S = \{20, 26, 32, \dots\}$

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- The elements of $\text{Ap}(S)$ are distinct modulo m
- $|\text{Ap}(S)| = m$

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Theorem

If $A = \{0, a_1, \dots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \pmod{m}$, then there exists a numerical semigroup S with $\text{Ap}(S) = A$ if and only if

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Big idea: the inequalities “ $a_i + a_j \geq a_{i+j}$ ” to define a **cone** C_m .

Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

$$a_i + a_j \geq a_{i+j} \quad \text{whenever} \quad i + j \neq 0.$$

$$\begin{aligned} \{S \subseteq \mathbb{Z}_{\geq 0} : m(S) = m\} &\longrightarrow C_m \\ \text{Ap}(S) = \{0, a_1, \dots, a_{m-1}\} &\longmapsto (a_1, \dots, a_{m-1}) \end{aligned}$$

Kunz cone

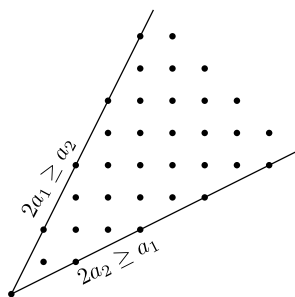
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Example: C_3



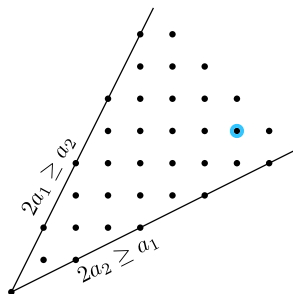
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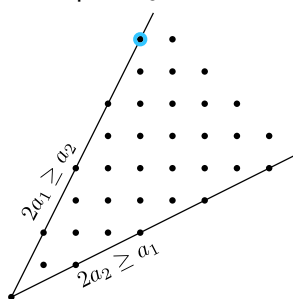
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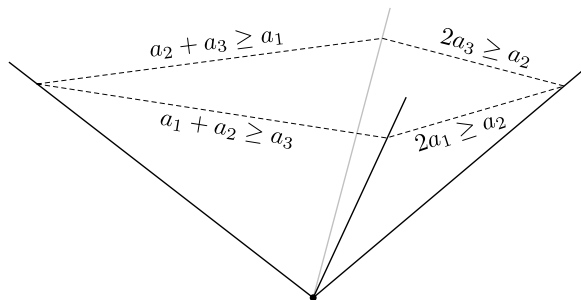
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Example: C_4



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When are numerical semigroups in (the relative interior of) the same face?

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Big picture: “moduli space” approach for studying XYZ 's

- Define a space with XYZ 's as points
Small changes to an $XYZ \rightsquigarrow$ small movements in space
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Basic example: $GL_n(\mathbb{R}) \hookrightarrow \mathbb{R}^{n^2}$

Faces of the Kunz cone

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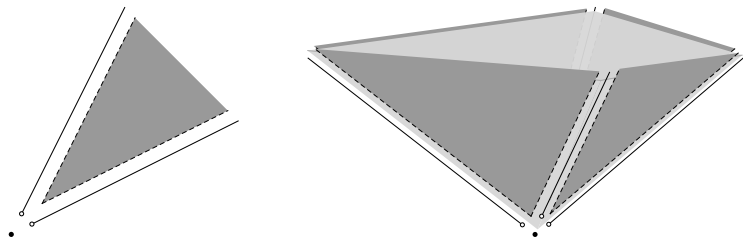
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More interesting example: C_m



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What about the other faces?

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Example: $S = \langle 4, 10, 11, 13 \rangle$

$$\text{Ap}(S) = \{0, 13, 10, 11\}$$

$$a_1 = 13, \quad a_2 = 10, \quad a_3 = 11$$

$$2a_1 > a_2 \quad a_1 + a_2 > a_3$$

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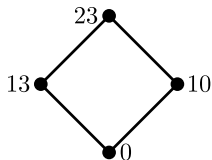
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Definition

The *Apéry poset* of S : define $a \preceq a'$ whenever $a' - a \in S$.

$$\text{Ap}(S) = \{0, 13, 10, 23\}$$



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$$S' = \langle 6, 26, 27 \rangle$$
$$\text{Ap}(S') = \{0, 79, 26, 27, 52, 53\}$$

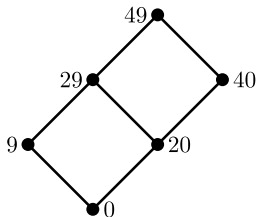
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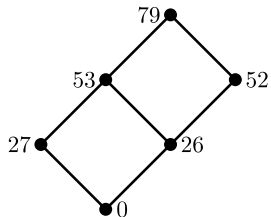
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$$\text{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$



$$S' = \langle 6, 26, 27 \rangle$$

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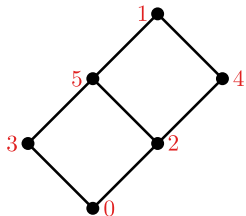
Faces of the Kunz polyhedron

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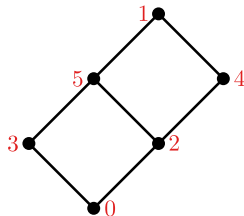
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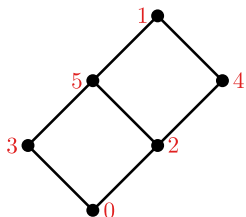
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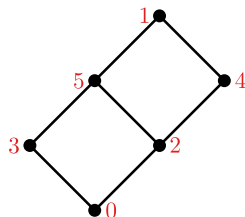
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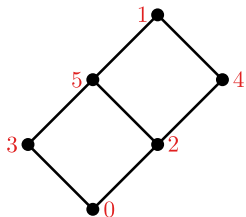
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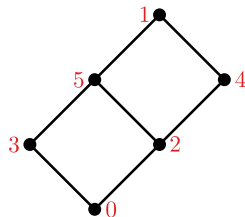
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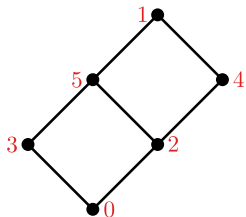
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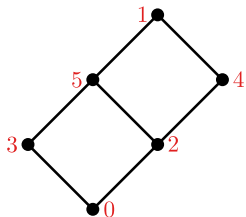
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Defining facet equations:

$$2a_2 = a_4$$

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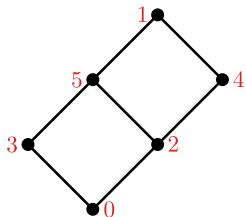
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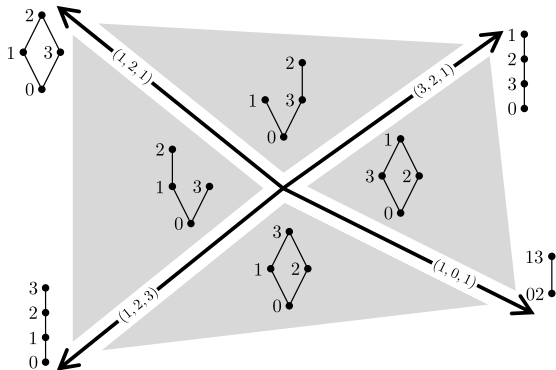
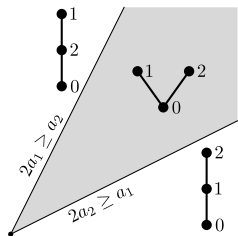
$$\begin{array}{ll} 2a_2 = a_4 & 2 \preceq 4 \\ a_2 + a_3 = a_5 & 2 \preceq 5 \\ & 3 \preceq 5 \\ a_2 + a_5 = a_1 & 2 \preceq 1 \\ & 5 \preceq 1 \\ a_3 + a_4 = a_1 & 3 \preceq 1 \\ & 4 \preceq 1 \end{array}$$

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C_3 and C_4



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Spoiler

If S, S' have identical Kunz posets, then S and S' have the same number of minimal trades.

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \dots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$Z(n) = \left\{ \mathbf{a} \in \mathbb{Z}_{\geq 0}^k : n = a_1 n_1 + \dots + a_k n_k \right\}$$

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$S = \langle 6, 9, 20 \rangle$:

$$Z(60) = \{(10, 0, 0), (7, 2, 0), (4, 4, 0), (1, 6, 0), (0, 0, 3)\}$$

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$$x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S \Rightarrow x^{\mathbf{b}} - x^{\mathbf{a}} \in I_S$$

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Minimal presentations

Fix a numerical semigroup $S = \langle n_1, \dots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

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Definition

The *kernel* $\ker \pi$ is the relation \sim on $\mathbb{Z}_{\geq 0}^k$ with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b}) \quad x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S = \ker \varphi$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$

$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

that is closed under *translation*

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

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$$x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S \Rightarrow x^{\mathbf{c}}(x^{\mathbf{a}} - x^{\mathbf{b}}) \in I_S$$

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Example

$$S = \langle 6, 9, 20 \rangle:$$

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$Z(18)$:

(3, 0, 0)



(0, 2, 0)

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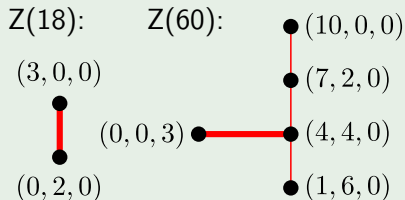
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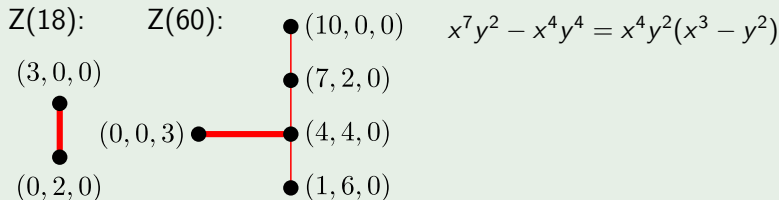
minimal presentation of S

\Leftrightarrow

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$Z(18):$ $(3, 0, 0)$ $(0, 2, 0)$	$Z(60):$ $(10, 0, 0)$ $(7, 2, 0)$ $(0, 0, 3)$ $(4, 4, 0)$ $(1, 6, 0)$	$x^7 y^2 - x^4 y^4 = x^4 y^2 (x^3 - y^2)$ $x^7 y^2 - z^3 = (x^7 y^2 - x^4 y^4) + (x^4 y^4 - z^3)$
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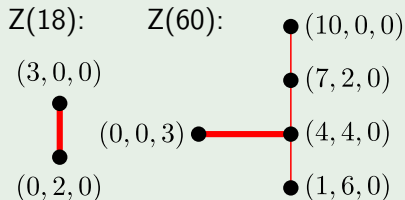
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$$\begin{aligned} x^7 y^2 - x^4 y^4 &= x^4 y^2 (x^3 - y^2) \\ x^7 y^2 - z^3 &= (x^7 y^2 - x^4 y^4) \\ &\quad + (x^4 y^4 - z^3) \end{aligned}$$

Generating set for $I_S \iff Z(n)$ connected for all $n \in S$

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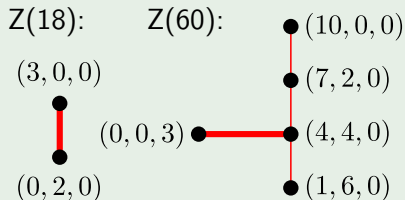
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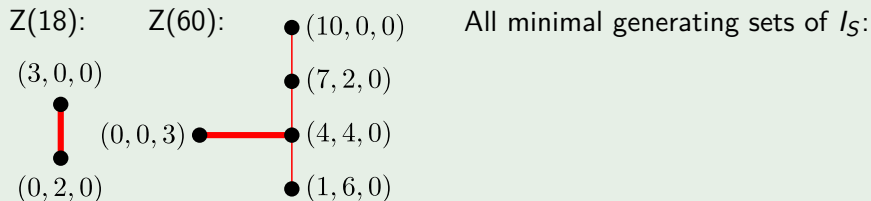
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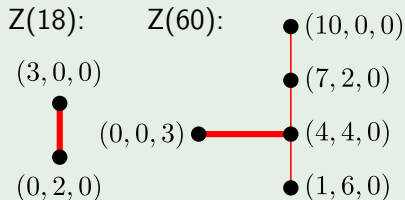
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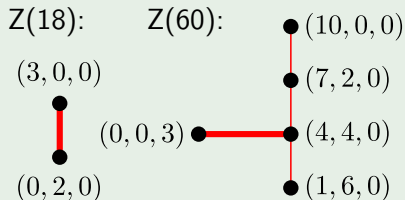
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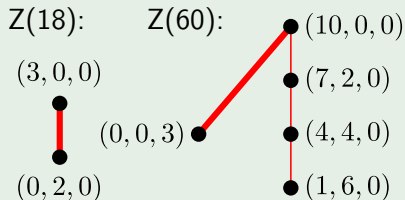
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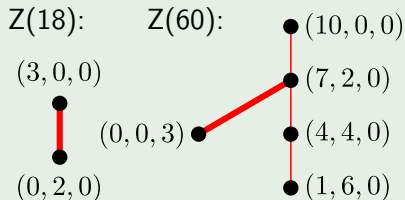
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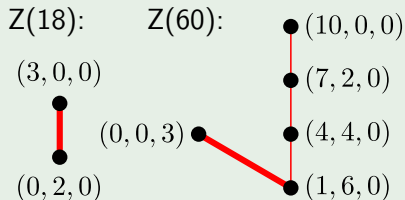
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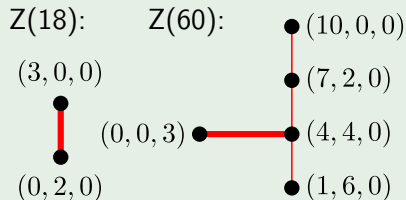
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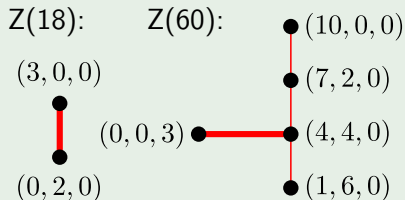
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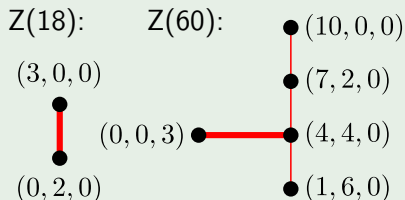
Monomial map:

$$\begin{aligned} \varphi : \mathbb{k}[x_1, \dots, x_k] &\longrightarrow \mathbb{k}[w] \\ x_i &\longmapsto w^{n_i} \end{aligned}$$

minimal presentation of S \iff *minimal generating set of I_S*

Example

$$S = \langle 6, 9, 20 \rangle: \quad I_S = \langle x^3 - y^2, x^4 y^4 - z^3 \rangle \subseteq \mathbb{k}[x, y, z]$$



All minimal generating sets of \sim :

$$(3, 0, 0) \sim (0, 2, 0), (10, 0, 0) \sim (0, 0, 3)$$

$$(3, 0, 0) \sim (0, 2, 0), (7, 2, 0) \sim (0, 0, 3)$$

$$(3, 0, 0) \sim (0, 2, 0), (4, 4, 0) \sim (0, 0, 3)$$

$$(3, 0, 0) \sim (0, 2, 0), (1, 6, 0) \sim (0, 0, 3)$$

Minimal presentations and Betti elements

$$S = \langle n_1, \dots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0} \quad \pi : \mathbb{Z}_{\geq 0}^k \longrightarrow S$$

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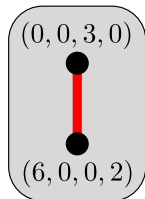
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Minimal presentations and Betti elements

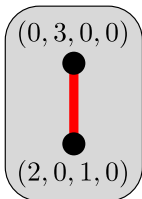
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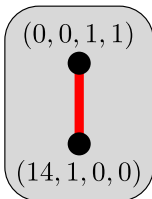
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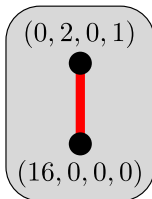
Z(132)



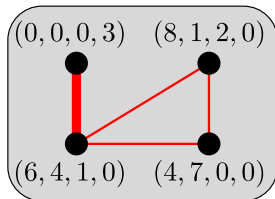
Z(318)



Z(226)



Z(208)



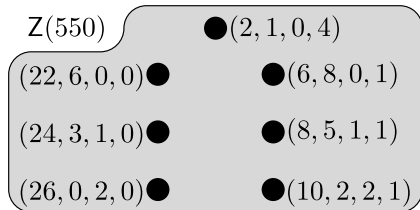
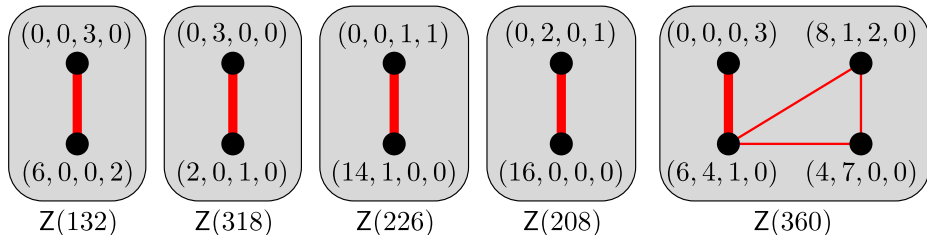
Z(360)

Minimal presentations and Betti elements

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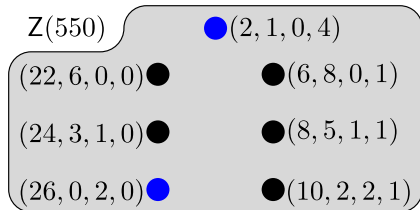
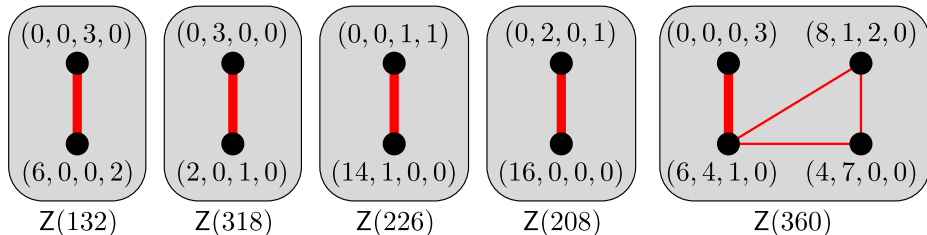


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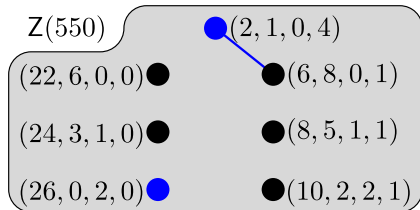
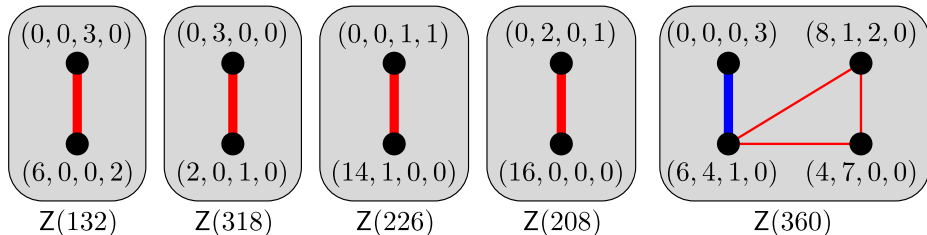


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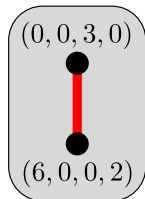


Minimal presentations and Betti elements

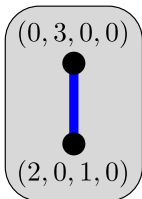
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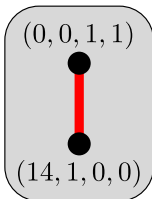
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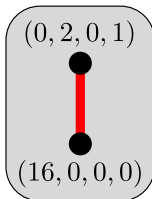
Z(132)



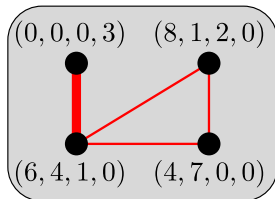
Z(318)



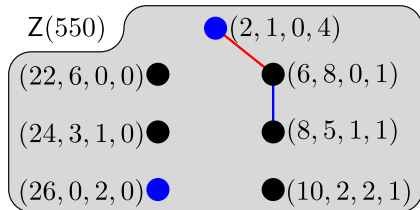
Z(226)



Z(208)



Z(360)



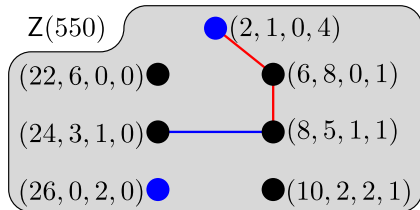
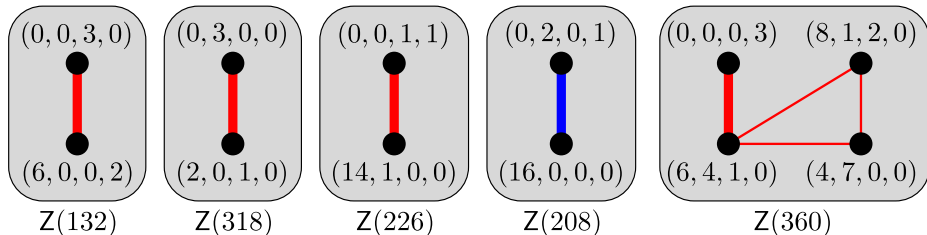
Z(550)

Minimal presentations and Betti elements

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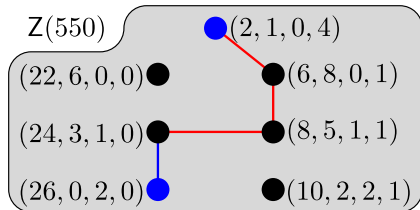
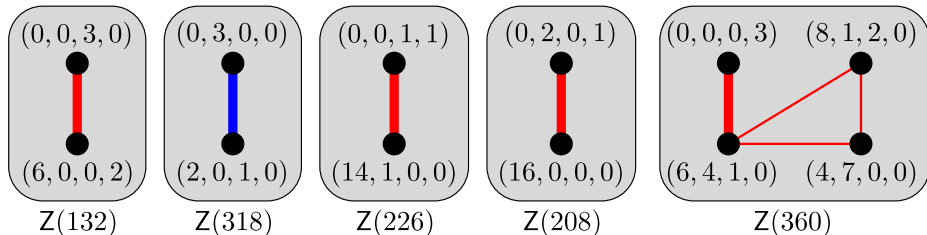


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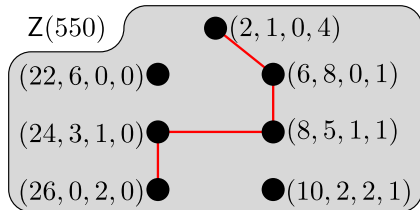
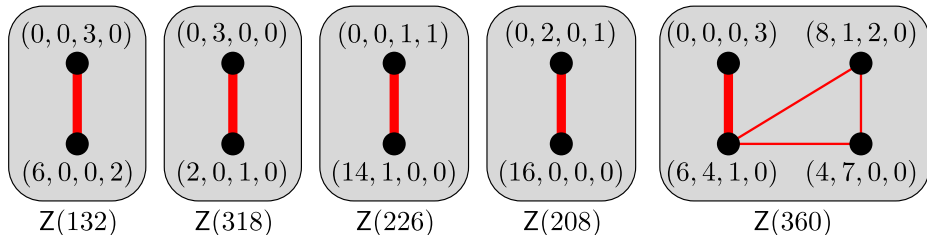


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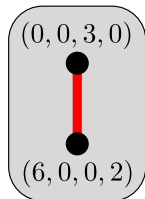


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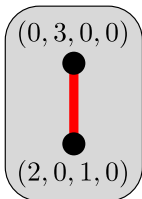
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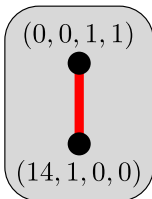
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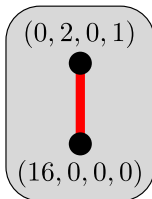
$\mathbb{Z}(132)$



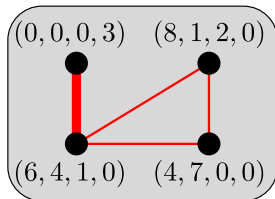
$\mathbb{Z}(318)$



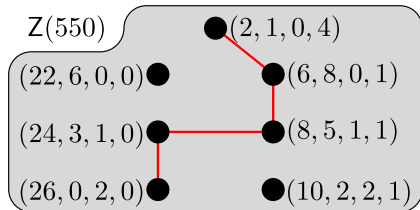
$\mathbb{Z}(226)$



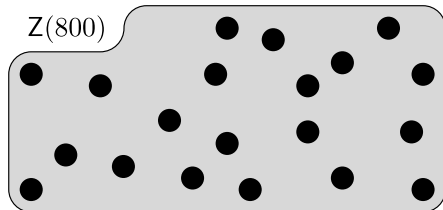
$\mathbb{Z}(208)$



$\mathbb{Z}(360)$



$\mathbb{Z}(550)$



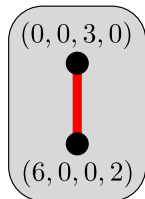
$\mathbb{Z}(800)$

Minimal presentations and Betti elements

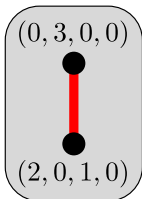
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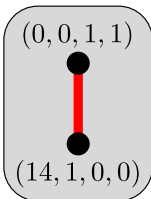
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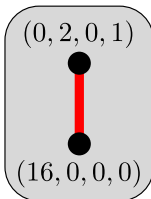
Z(132)



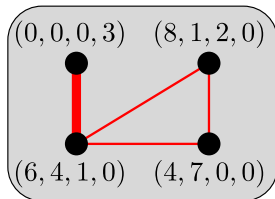
Z(318)



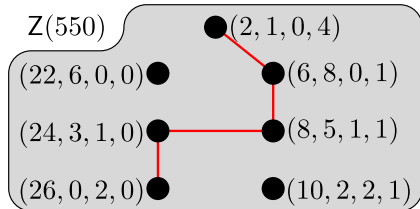
Z(226)



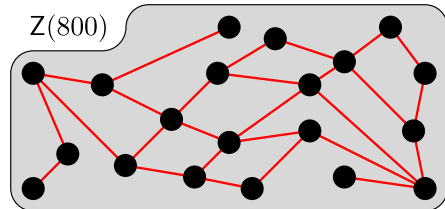
Z(208)



Z(360)



Z(550)



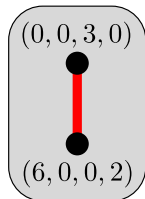
Z(800)

Minimal presentations and Betti elements

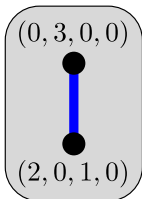
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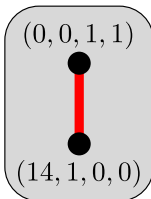
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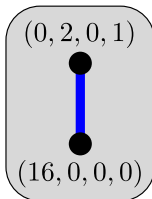
Z(132)



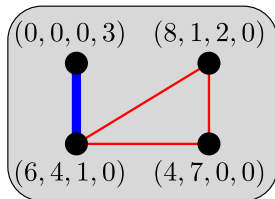
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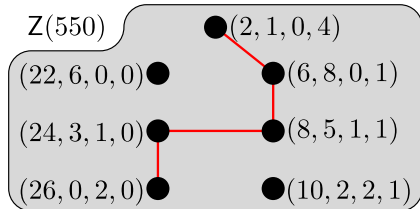
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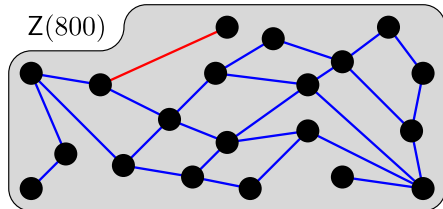
Z(208)



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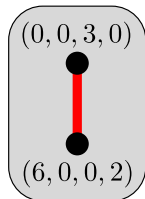
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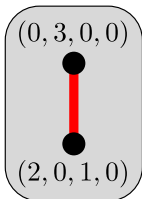
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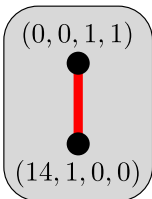
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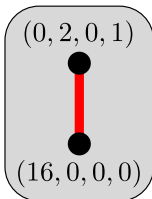
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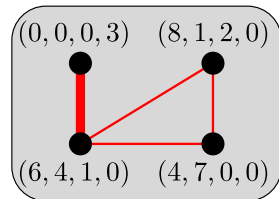
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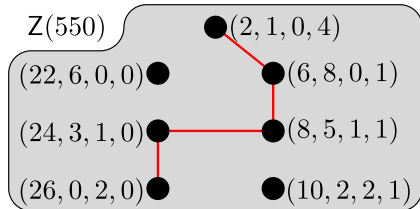
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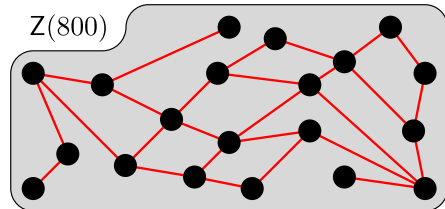
Z(208)



Z(360)



Z(550)



Z(800)

Minimal trades and Kunz posets

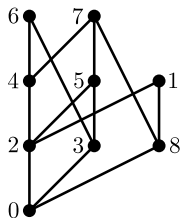
Question

How can one recover minimal trade structure from the Kunz poset?

Minimal trades and Kunz posets

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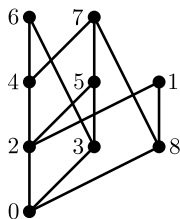


Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

$$\text{Ap}(S) = \{0, a_1, a_2, \dots, a_8\}$$



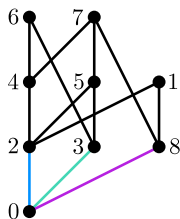
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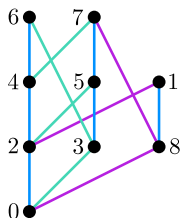
$$S = \langle 9, a_2, a_3, a_8 \rangle$$



Minimal trades and Kunz posets

Question

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$$\text{Ap}(S) = \{0, a_1, a_2, \dots, a_8\}$$

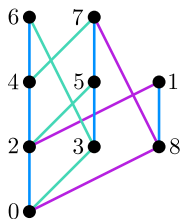
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Cover relations: add a generator

Minimal trades and Kunz posets

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How can one recover minimal trade structure from the Kunz poset?



$$\text{Ap}(S) = \{0, a_1, a_2, \dots, a_8\}$$

$$S = \langle 9, a_2, a_3, a_8 \rangle$$

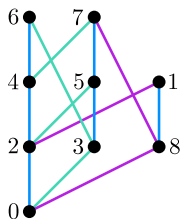
Cover relations: add a generator

$$Z(a_6) = \{(0, 3, 0, 0), (0, 0, 2, 0)\}$$

Minimal trades and Kunz posets

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2 “inner” minimal trades:

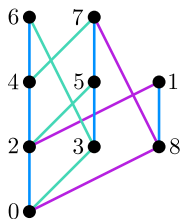
$$(0, 3, 0, 0) \sim (0, 0, 2, 0) \text{ (at } a_6)$$

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Moral: can recover

- factorizations of $a \in \text{Ap}(S)$
- (minimal) trades at $a \in \text{Ap}(S)$

Minimal trades and Kunz posets

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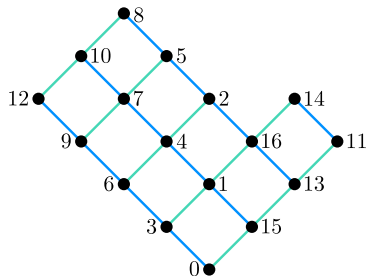
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Minimal trades and Kunz posets

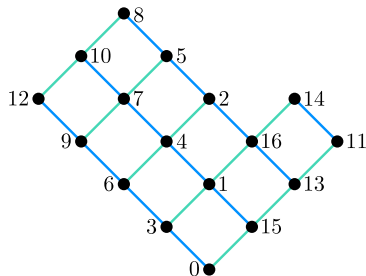
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3 minimal trades, none in $\text{Ap}(S)$



Minimal trades and Kunz posets

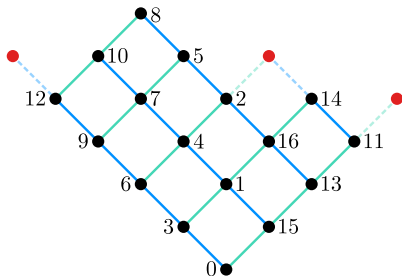
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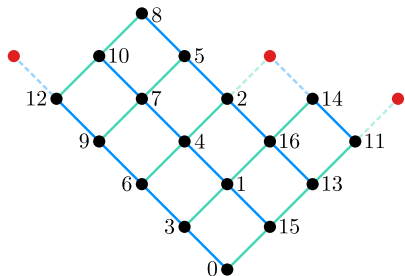


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$$a_{12} + a_3: (0, 5, 0) \sim (12, 10, 8)$$

$$a_{11} + a_{15}: (0, 0, 4) \sim (11, 13, 15)$$

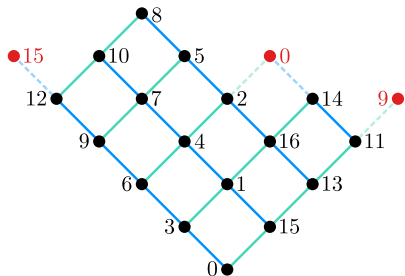
$$a_2 + a_{15}: (0, 2, 3) \sim (1, 16, 14)$$

Minimal trades and Kunz posets

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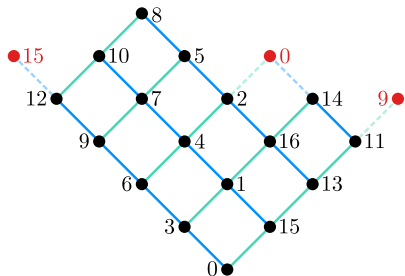
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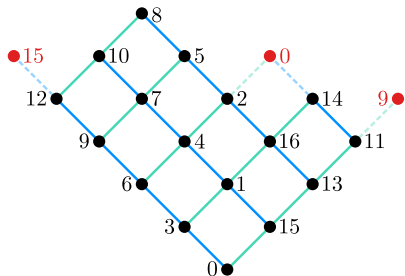
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Possible method to locate the “outer” trades:

- factorizations of $a \in \text{Ap}(S)$ form a monomial staircase
- one “outer” minimal trade for each monomial generator

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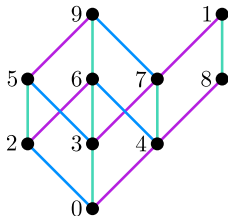
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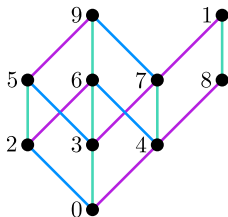


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“inner” trade at a_6 :

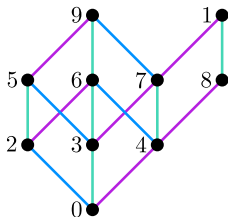
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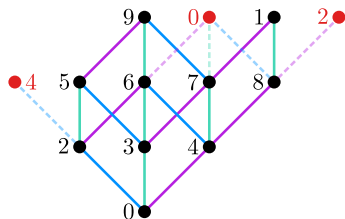
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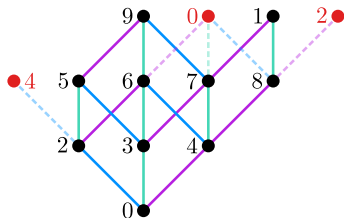
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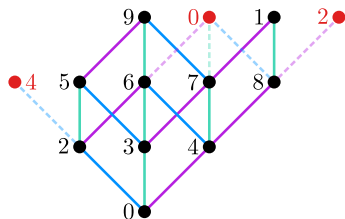
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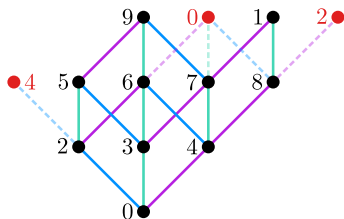
Moral: use **sets** of factorizations,
avoids overcounting minimal trades

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Moral: use **sets** of factorizations,
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$$0: \{(0, 0, 2, 1), (0, 1, 0, 2)\}$$

$$2: \{(0, 0, 0, 3)\}, \quad 4: \{(0, 2, 0, 0)\}$$

Minimal trades and Kunz posets

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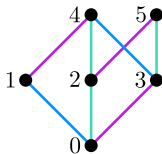
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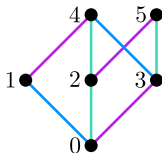
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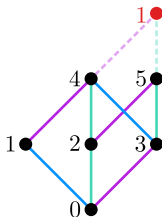


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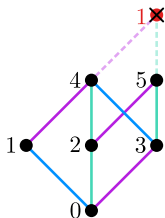
$$(0, 0, 2, 1) \in Z(25)$$

Minimal trades and Kunz posets

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$$S = \langle 6, 7, 8, 9 \rangle$$

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No trades in $Z(25)$:

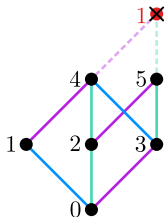
$$\{(0, 0, 2, 1), (0, 1, 0, 2), (3, 1, 0, 0)\}$$

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A technical definition

Definition

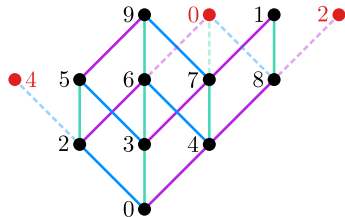
An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = Z(a_i)$ for each $i \in \text{supp}(B)$.

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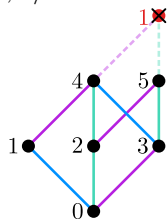
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$$S = \langle 10, a_2, a_3, a_4 \rangle$$



$$\begin{aligned} B &= \{(0, 0, 2, 1), (0, 1, 0, 2)\} \\ B - e_2 &= \{(0, 0, 0, 2)\} = Z(a_8) \\ B - e_3 &= \{(0, 0, 1, 1)\} = Z(a_7) \\ B - e_4 &= \{(0, 0, 2, 0), (0, 1, 0, 1)\} \\ &= Z(a_6) \end{aligned}$$

$$S = \langle 6, 7, 8, 9 \rangle$$



$$\begin{aligned} B &= \{(0, 0, 2, 1)\} \\ B - e_4 &= \{(0, 0, 2, 0)\} \subsetneq Z(a_4) \\ B &= \{(0, 0, 2, 1), (0, 1, 0, 2)\} \\ B - e_3 &= \{(0, 0, 1, 1)\} \not\subseteq Z(a_i) \end{aligned}$$

The main theorem

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Recovering minimal presentation from the Kunz poset P of S :

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- trades occurring at $a \in \text{Ap}(S)$ recovered from factorizations of $\bar{a} \in P$

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Theorem (Gomes–O.–Torres Davila)

If S has Kunz poset P , each minimal trade of S not occurring in $\text{Ap}(S)$ contains a factorization from a distinct outer Betti element of P .

In particular, if S, S' have identical Kunz poset, then S and S' have the same number of minimal trades.

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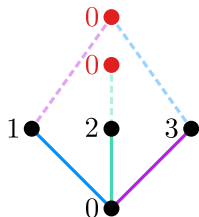
Another subtlety: distinct outer Betti elements can **coincide** for some S

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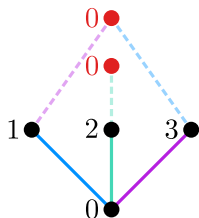
$$B_2 = \{(0, 1, 0, 1)\}$$

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$$S = \langle 4, 9, 14, 11 \rangle$$

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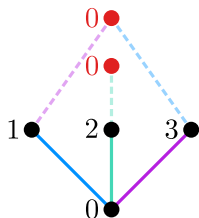
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$$S = \langle 4, 9, 14, 11 \rangle$$

$$20: (0, 1, 0, 1), (5, 0, 0, 0)$$

$$28: (0, 0, 2, 0), (2, 1, 0, 0), (5, 0, 0, 0)$$

$$B_1 = \{(0, 0, 2, 0)\}$$

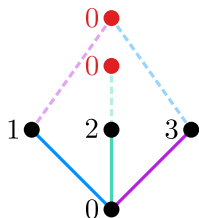
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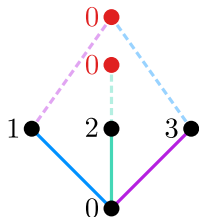
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An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = Z(a_i)$ for each $i \in \text{supp}(B)$.

Theorem (Gomes–O.–Torres Davila)

If S has Kunz poset P , each minimal trade of S not occurring in $\text{Ap}(S)$ contains a factorization from a distinct outer Betti element of P .

In particular, if S, S' have identical Kunz poset, then S and S' have the same number of minimal trades.

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For $m = 6$: $\#$ minimal trades $\in \{1, 2, 3, 4, 5, 6, 9, 10, 15\}$

Application: classifying minimal trades

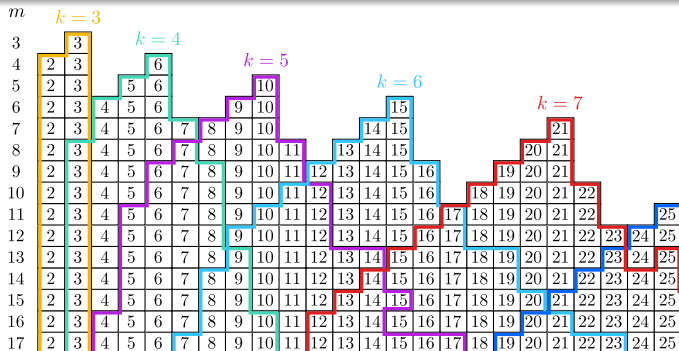
Question

Given the multiplicity $m = m(S)$ and $\#$ minimal generators k of a numerical semigroup S , what can $\beta_1(I_S) = \#$ minimal trades be?

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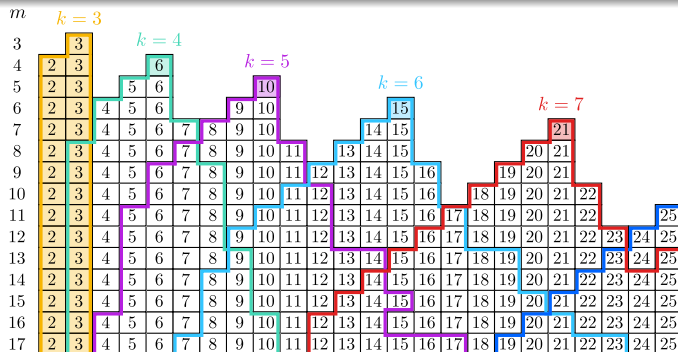
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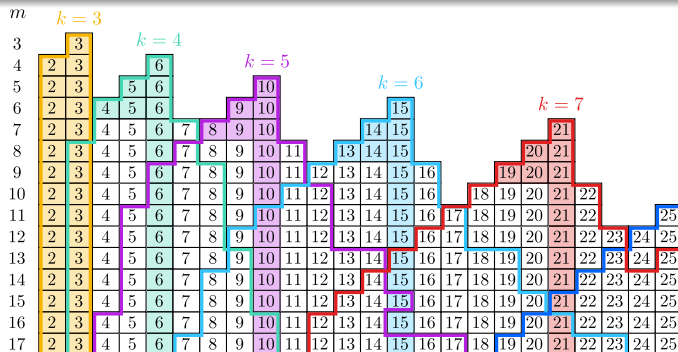


Well known: $\beta_1(S) \leq \binom{m}{2}$, with equality if and only if $k = m$
 if $k = 3$, then $\beta_1(S) = 2, 3$

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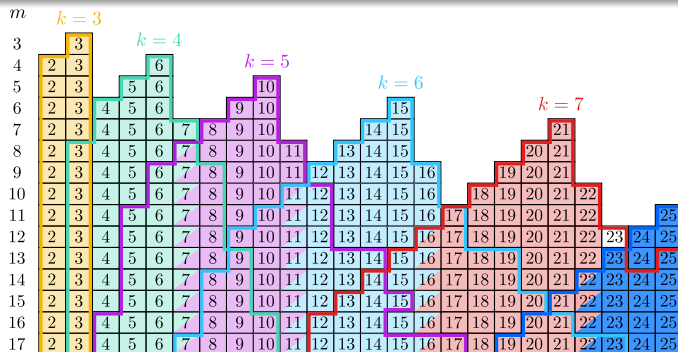
Prior work: a family has $\beta_1(S) = \binom{k}{2}$ for $3 \leq k \leq m$ (Rosales)

if $r = m - k \leq 2$, then $\beta_1(S) \in [\binom{k}{2} - r, \binom{k}{2}]$ (GS-R)

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Using Kunz posets: a family hits each $\beta_1(S) \in \left[\binom{k}{2} - r, \binom{k}{2} \right]$

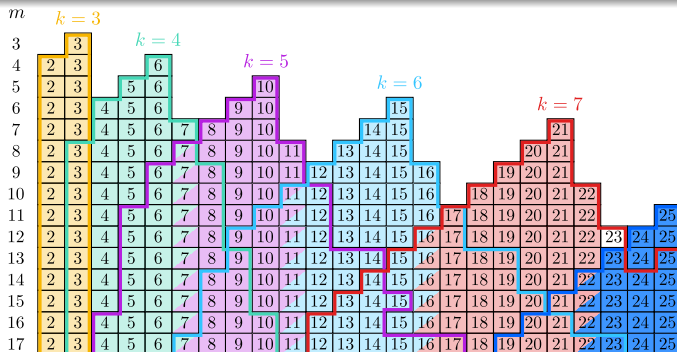
for $r = m - k \leq k - 2$

a family hits $\beta_1(S) = \binom{k}{2} + 1$ for each $m \geq k + 3$

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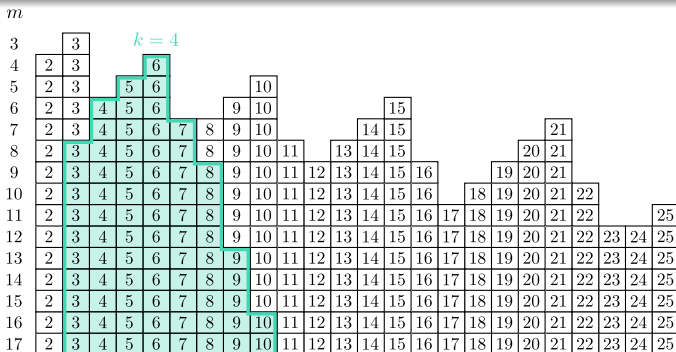


Bounds from Kunz posets: $\beta_1(S) \geq \binom{k}{2} - r$, where $r = m - k$
 if $m - k = 3$, then $\beta_1(S) \in [\binom{k}{2} - 3, \binom{k}{2} + 1]$

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One more family: for $k = 4$, achieves each $\beta_1(S)$ with $(\beta_1(S) - 2)^2 \leq 4m$ conjectured to achieve every possible $\beta_1(S)$ for $k = 4$

References

-  W. Bruns, P. García-Sánchez, C. O'Neill, D. Wilburne (2020)
Wilf's conjecture in fixed multiplicity
International Journal of Algebra and Computation **30** (2020), no. 4, 861–882.
(arXiv:1903.04342)
-  N. Kaplan, C. O'Neill, (2021)
Numerical semigroups, polyhedra, and posets I: the group cone
Combinatorial Theory **1** (2021), #19. (arXiv:1912.03741)
-  T. Gomes, C. O'Neill, E. Torres Davila (2022)
Numerical semigroups, polyhedra, and posets III: minimal presentations and face dimension.
Electronic Journal of Combinatorics **30** (2023), no. 2, #P2.5. (arXiv:2009.05921)
-  C. Elmacioglu, K. Hilmer, H. Koufmann, C. O'Neill, M. Okandan (2022)
On the cardinality of minimal presentations of numerical semigroups
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References

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Thanks!