Numerical semigroups, minimal presentations, and posets

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Slides available: `https://cdoneill.sdsu.edu/`

Novemer 5, 2023

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$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{c} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

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Multiplicity: $\mathrm{m}(S) =$ smallest nonzero element

Minimal presentations

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$\mathsf{Z}(n) = \left\{ \mathbf{a} \in \mathbb{Z}_{\geq 0}^k : n = a_1 n_1 + \cdots + a_k n_k \right\}$$

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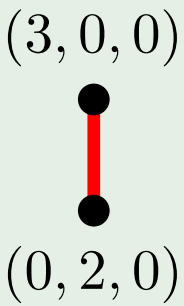

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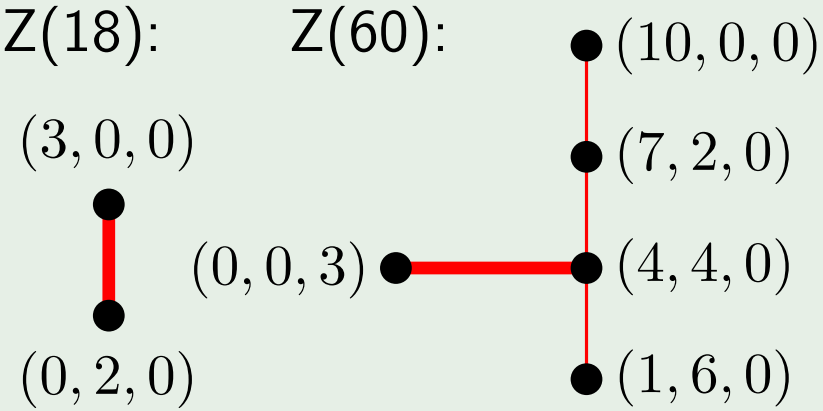

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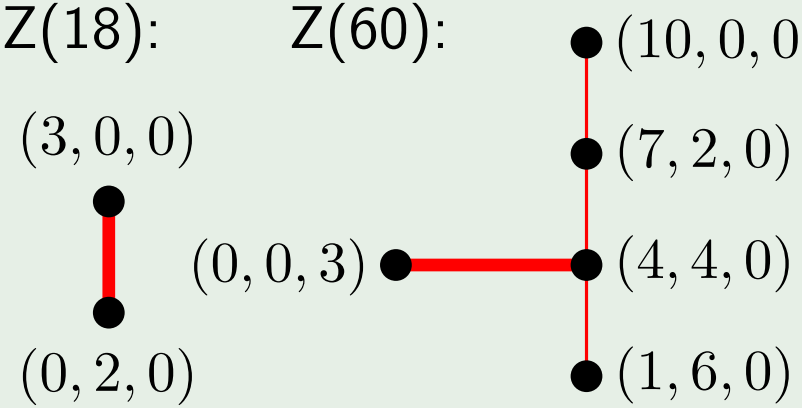

$x^7 y^2 - x^4 y^4 = x^4 y^2 (x^3 - y^2) \in I_S$

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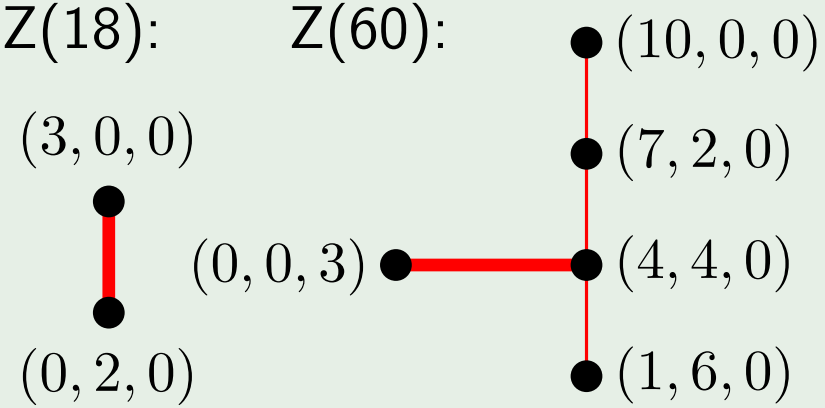

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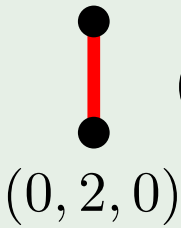

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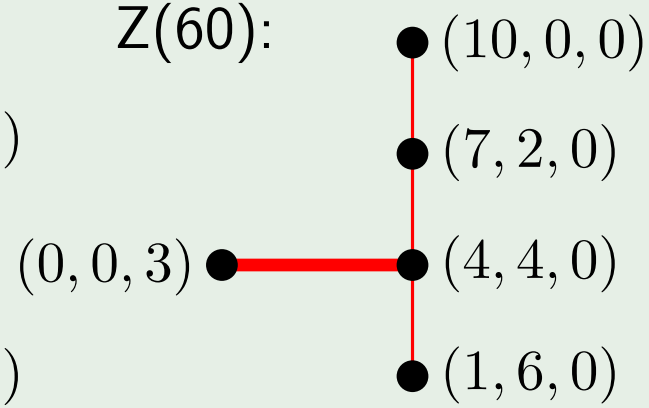

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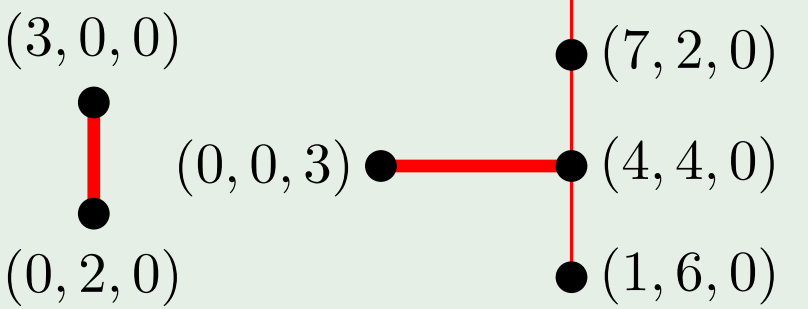

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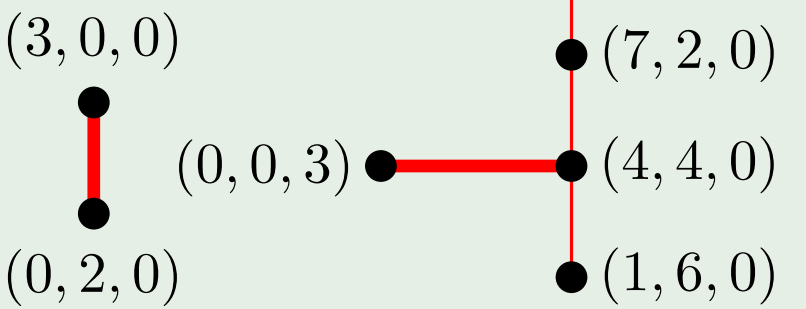

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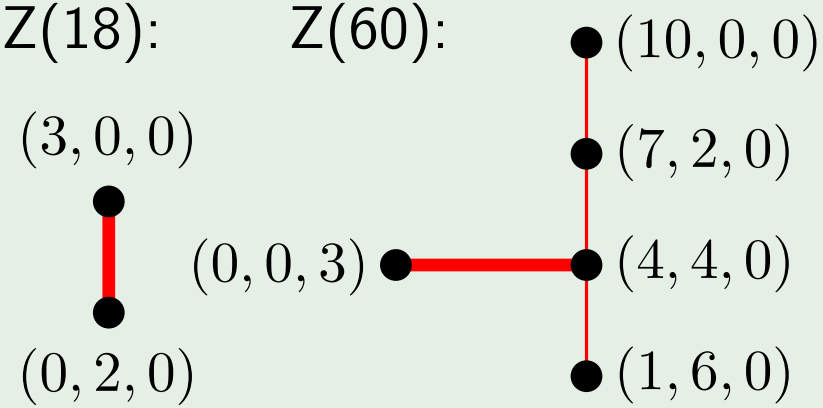

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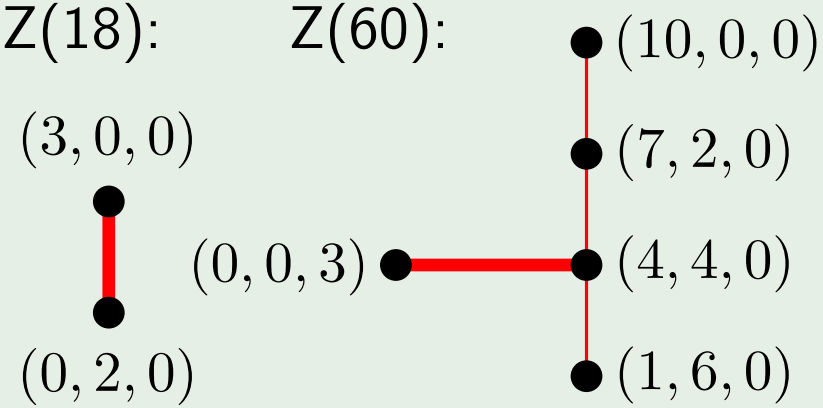

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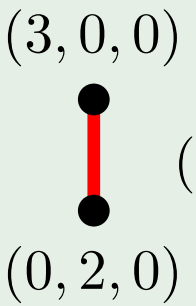

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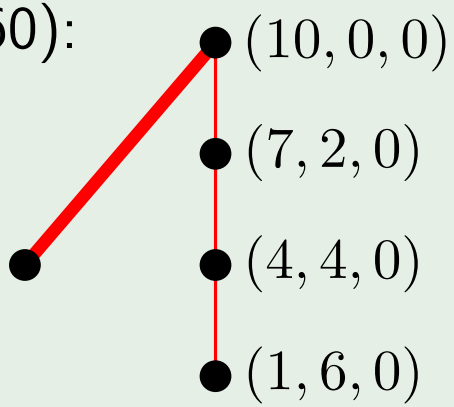

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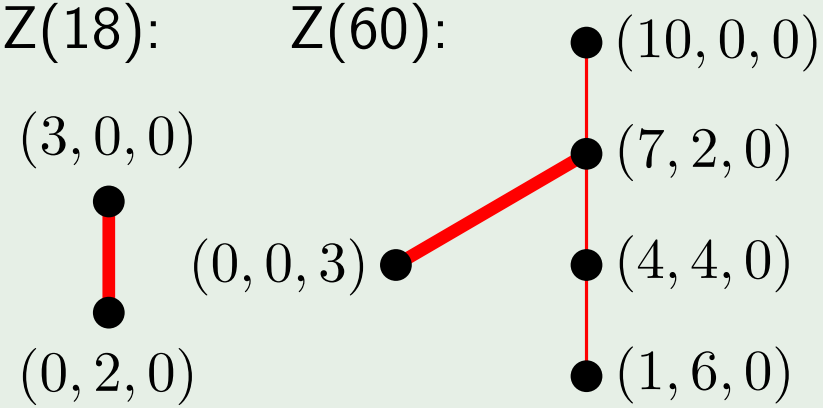

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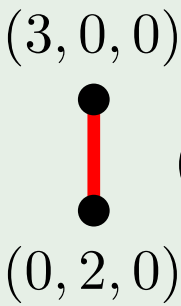

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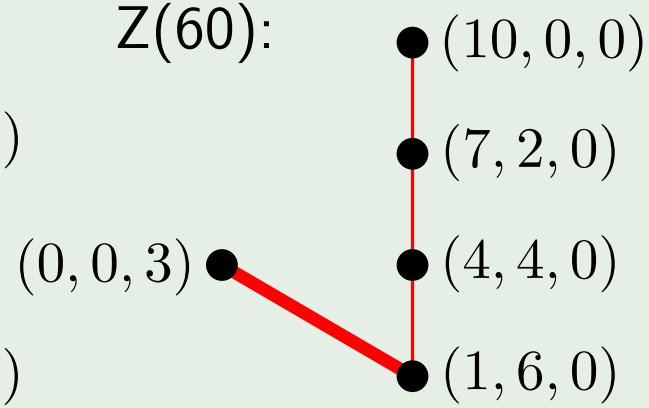

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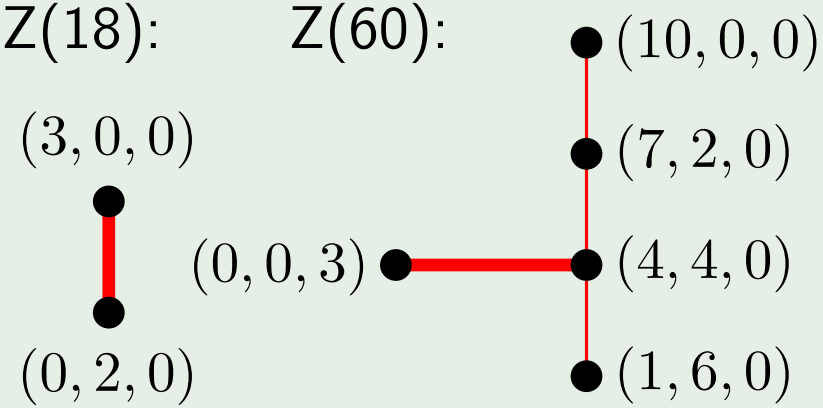

All minimal generating sets of I_S:

$$\begin{aligned} I_S &= \langle x^3 - y^2,\ x^{10} - z^3 \rangle \\ &= \langle x^3 - y^2,\ x^7y^2 - z^3 \rangle \\ &= \langle x^3 - y^2,\ x^4y^4 - z^3 \rangle \\ &= \langle x^3 - y^2,\ x^6y - z^3 \rangle \end{aligned}$$

Minimal presentations

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{Z}_{\geq 0}^k & \longrightarrow & \langle n_1, \ldots, n_k \rangle \\ \mathbf{a} & \longmapsto & a_1 n_1 + \cdots + a_k n_k \end{array}$$

Monomial map:

$$\begin{array}{rcl} \varphi : \mathbb{k}[x_1, \ldots, x_k] & \longrightarrow & \mathbb{k}[w] \\ x_i & \longmapsto & w^{n_i} \end{array}$$

minimal presentation of S $\quad \leftrightsquigarrow \quad$ minimal generating set of I_S

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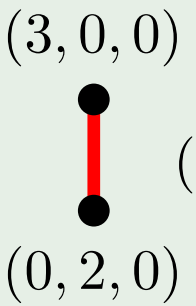

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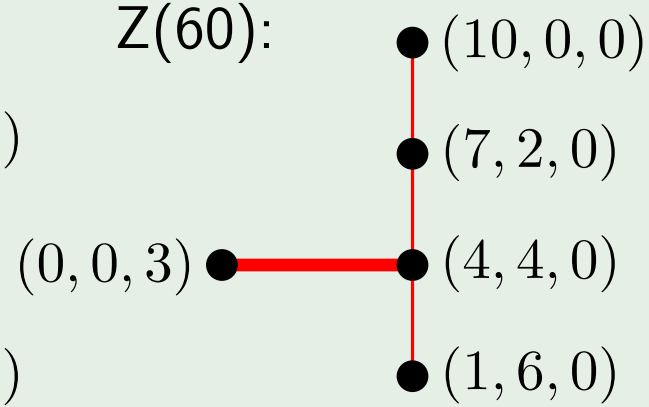

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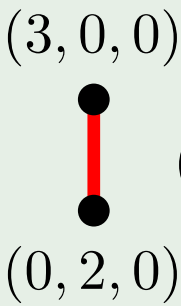

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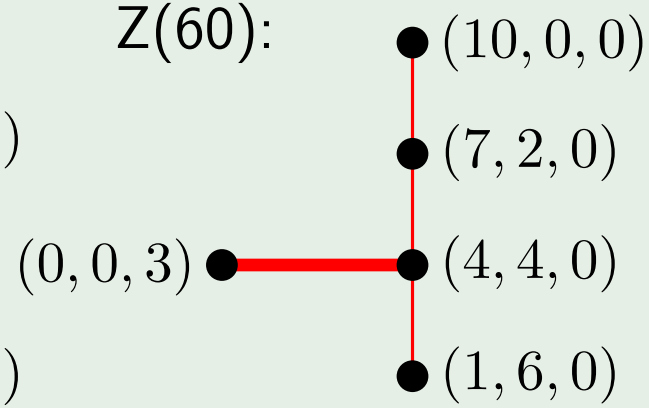

All minimal generating sets of $\sim$:

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Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

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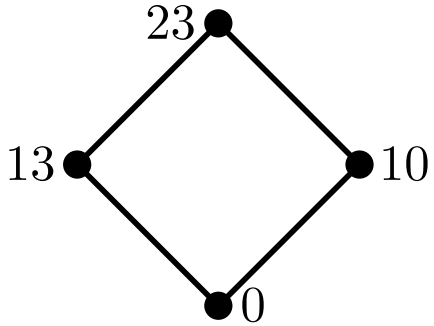
- $|\mathrm{Ap}(S)| = m$

Apéry posets and Kunz posets

Definition

The *Apéry poset* of S: define $a \preceq a'$ whenever $a' - a \in S$.

$$S = \langle 4, 10, 13 \rangle$$
$$\mathrm{Ap}(S) = \{0, 13, 10, 23\}$$



$$S = \langle 4, 13 \rangle$$
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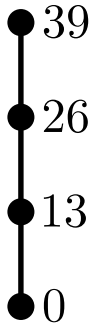

Faces of the Kunz polyhedron

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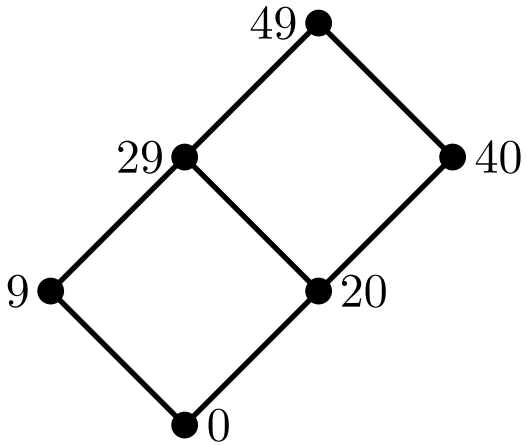

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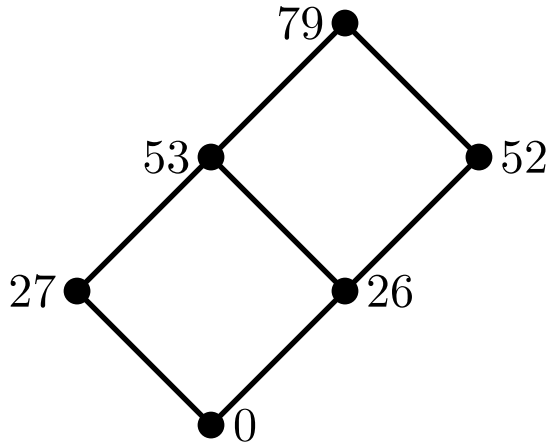

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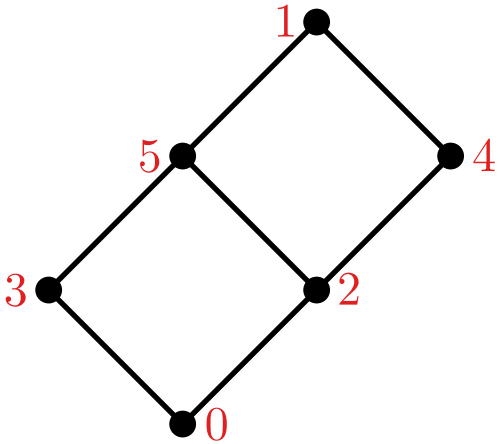

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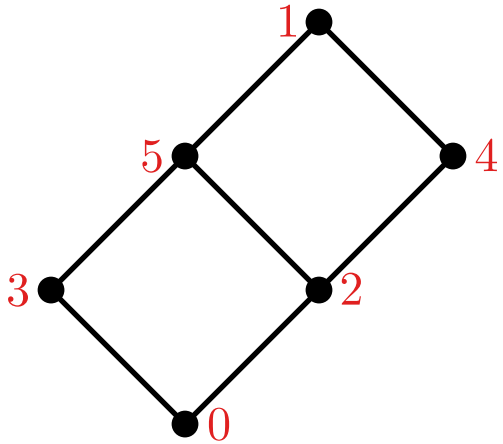

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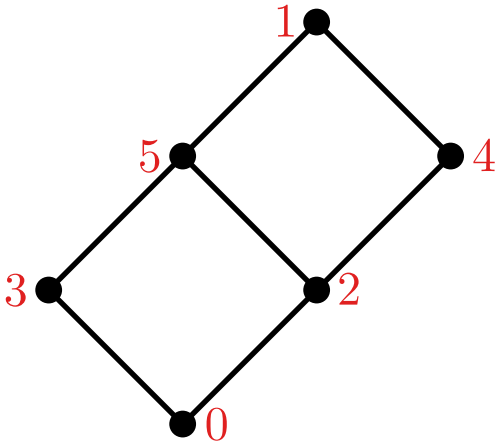

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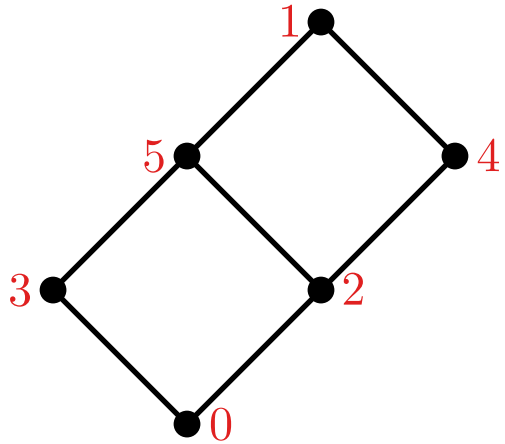

The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

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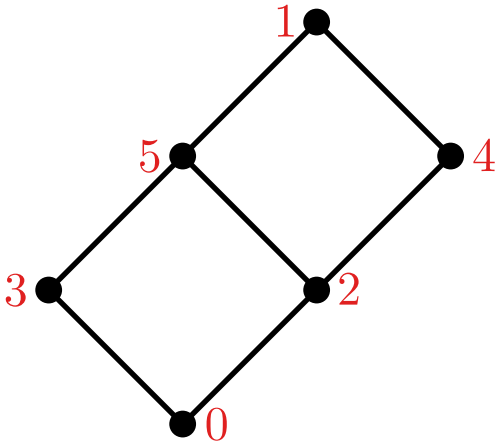

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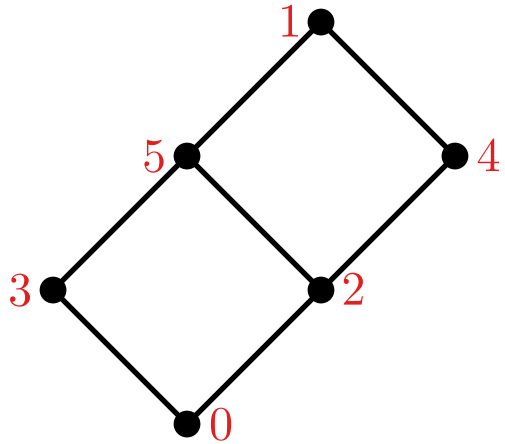

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Theorem (Gomes–O.–Torres Davila)

If S, S' have identical Kunz poset, then S and S' have the same number of minimal trades.

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

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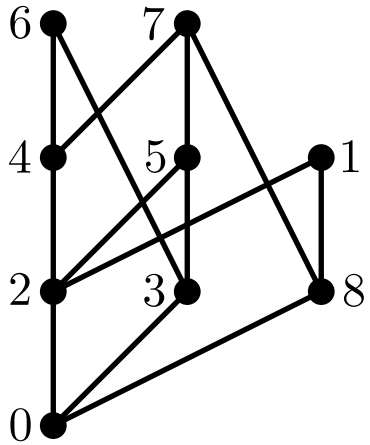

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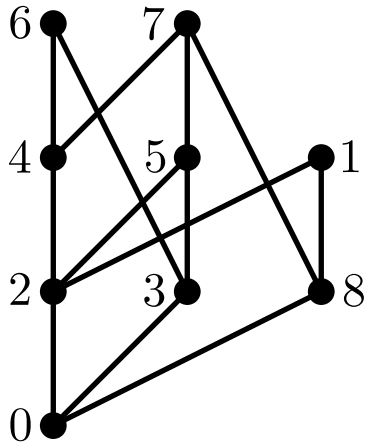
$$\mathrm{Ap}(S) = \{0, a_1, a_2, \ldots, a_8\}$$



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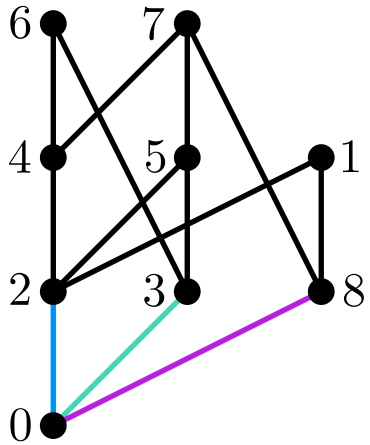

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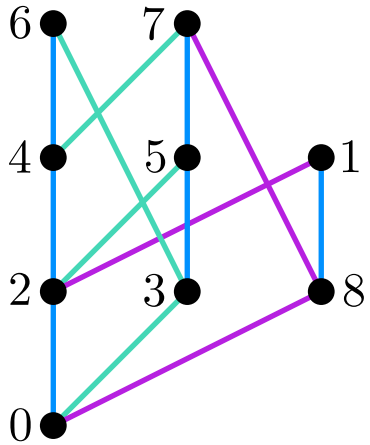

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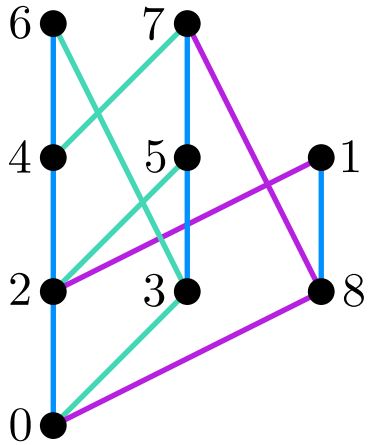

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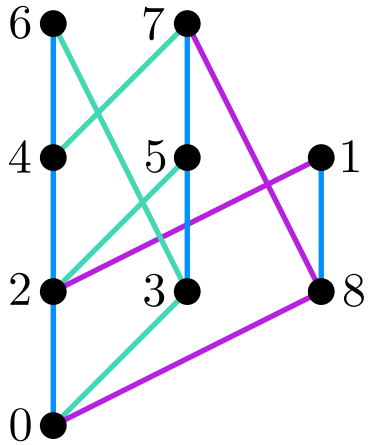

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2 "inner" minimal trades:
$(0,3,0,0) \sim (0,0,2,0)$ (at a_6)
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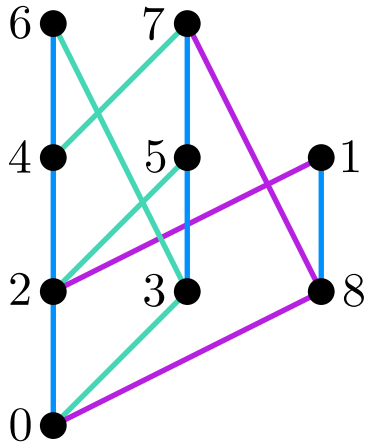

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Moral: can recover

- $\mathrm{Z}(a)$ for $a \in \mathrm{Ap}(S)$
- (minimal) trades at $a \in \mathrm{Ap}(S)$

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Key fact: each trade occurs at $a_i + n_j$ for some $a_i \in \mathrm{Ap}(S)$, generator n_j

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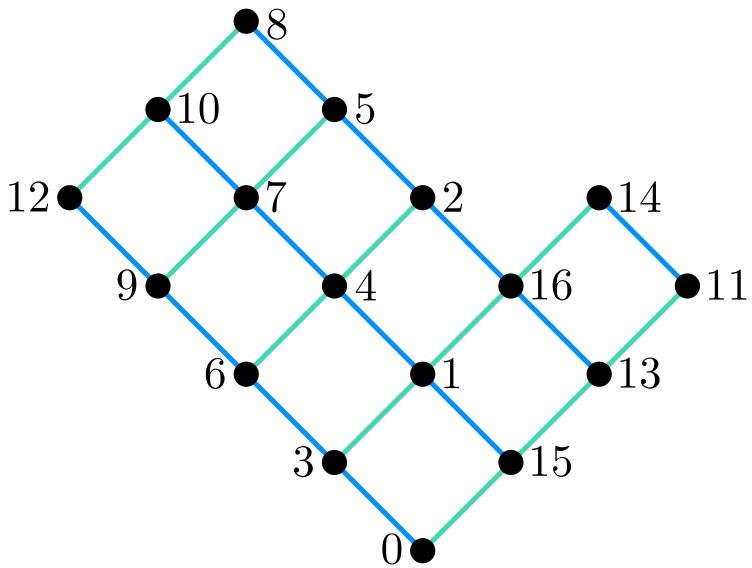

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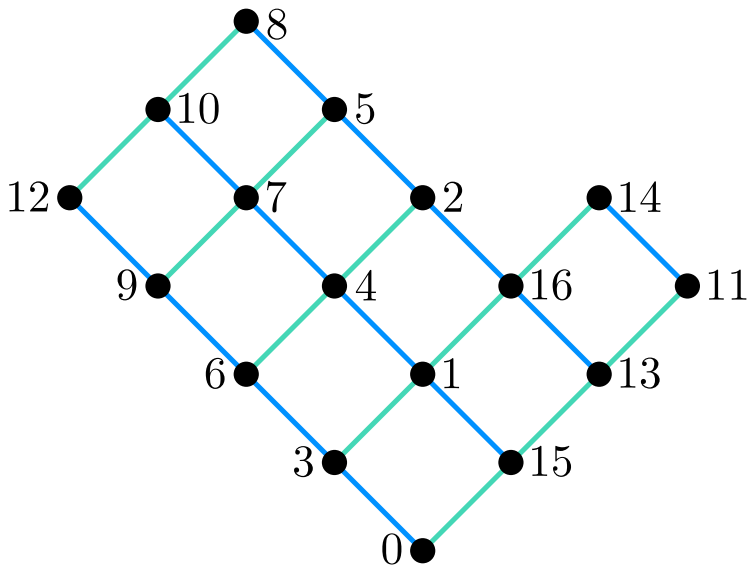

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3 minimal trades, none in $\mathrm{Ap}(S)$

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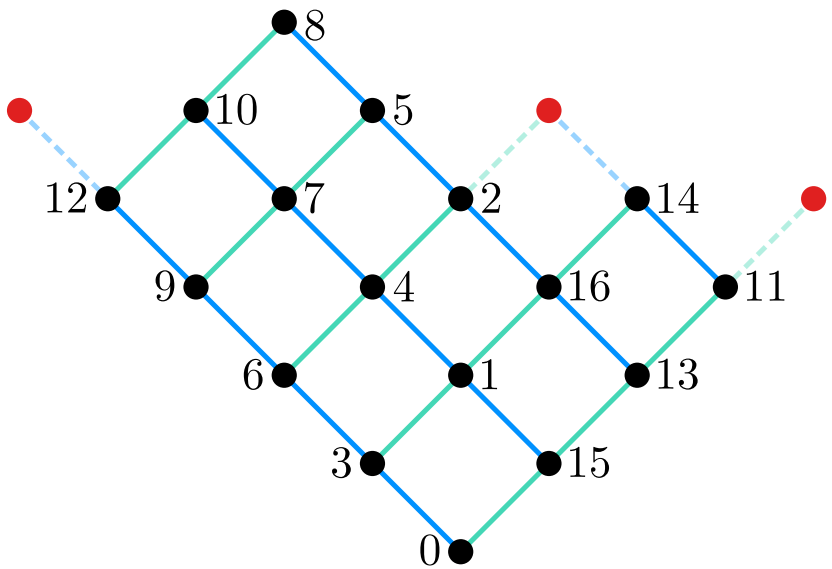

$S = \langle 17, a_3, a_{15} \rangle$

3 minimal trades, none in $\mathrm{Ap}(S)$

Minimal trades and Kunz posets

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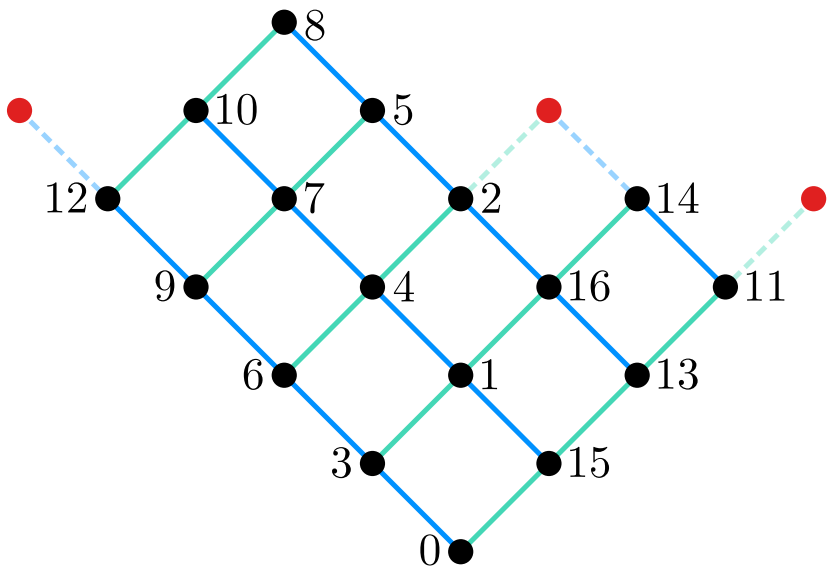

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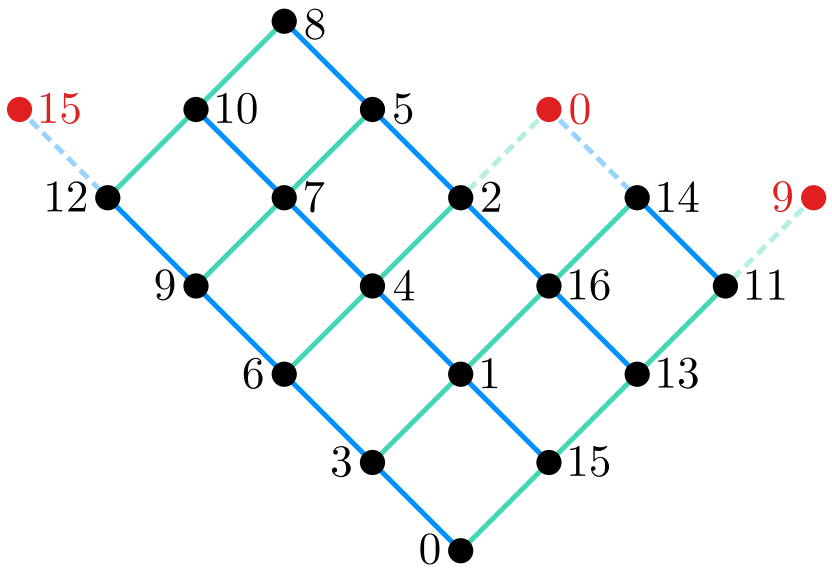

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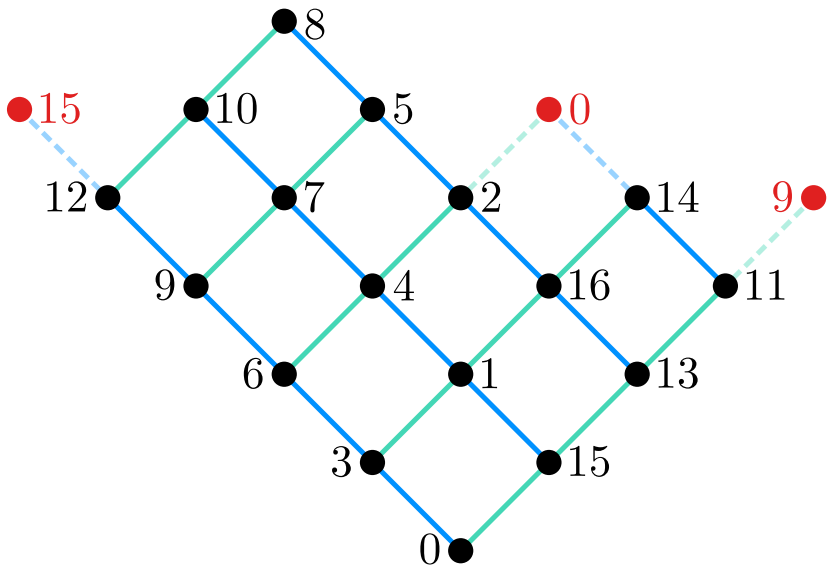

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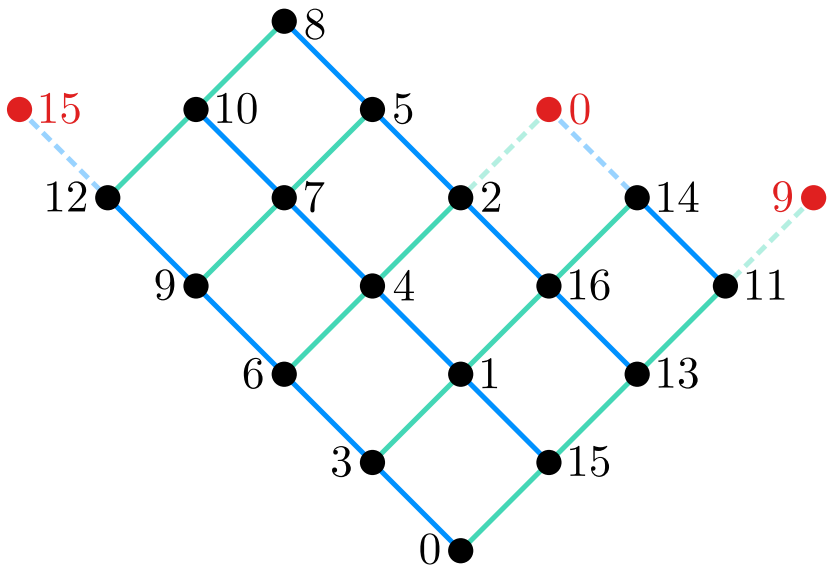

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If S has an *Apéry set of unique expression*:

- factorizations of $a \in \mathrm{Ap}(S)$ form monomial staircase
- one "outer" minimal trade for each monomial generator

Minimal trades and Kunz posets

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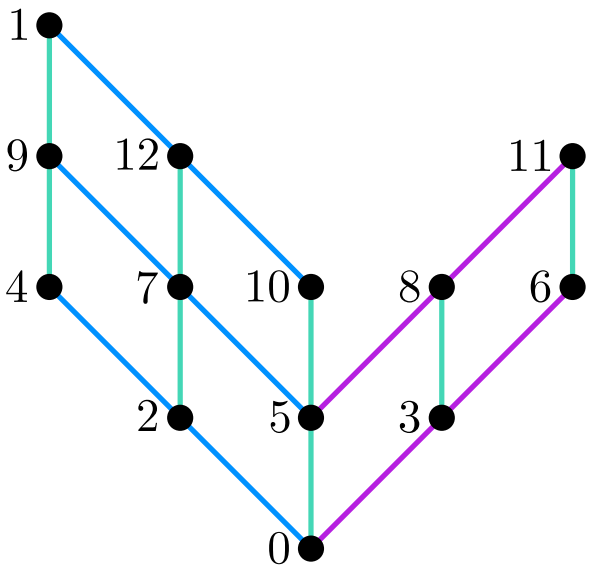

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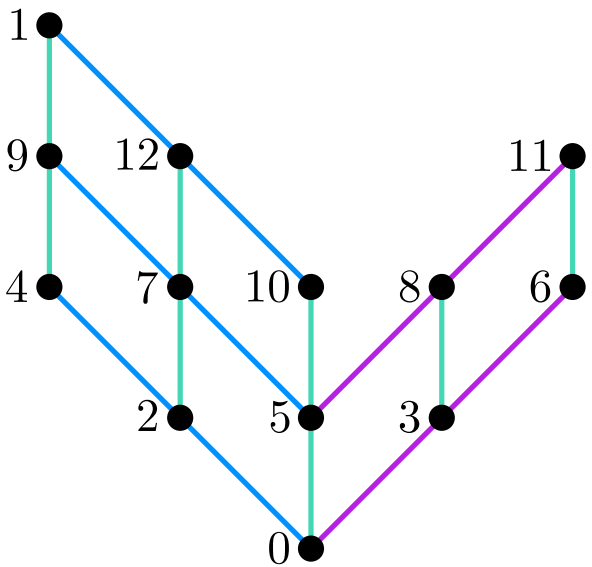

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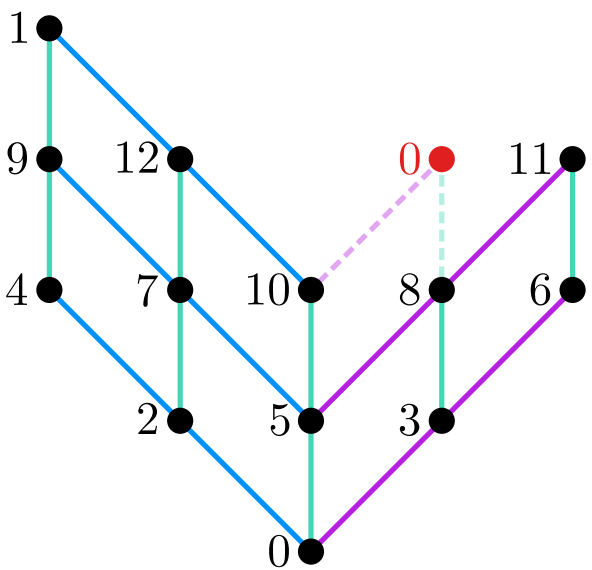

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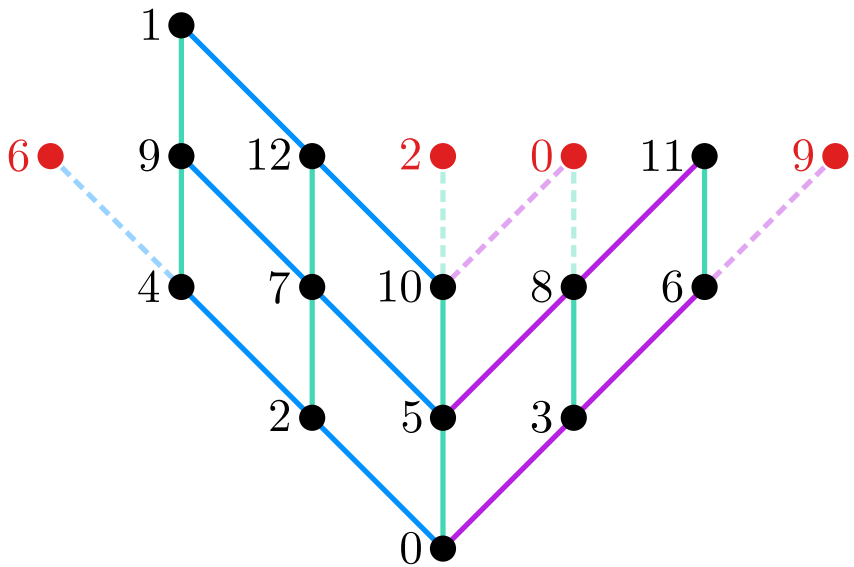

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9: $(0, 0, 0, 3)$

Minimal trades and Kunz posets

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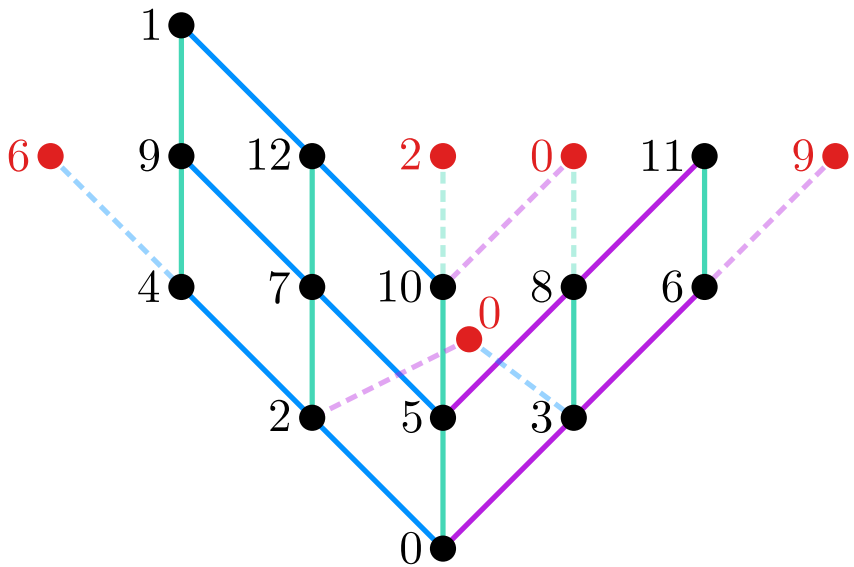

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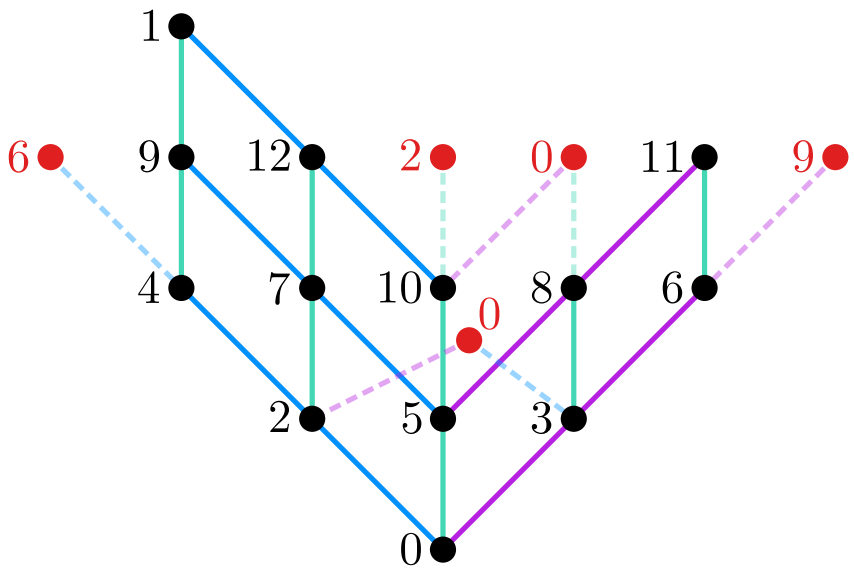

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Need: decrementing any coordinate lands in $\mathrm{Ap}(S)$

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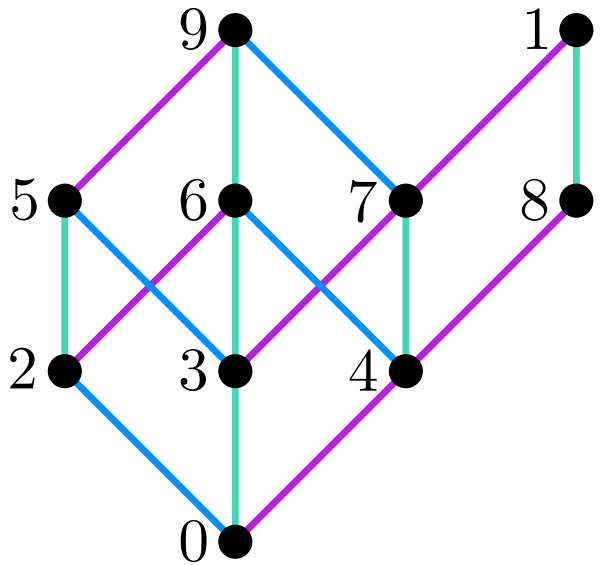

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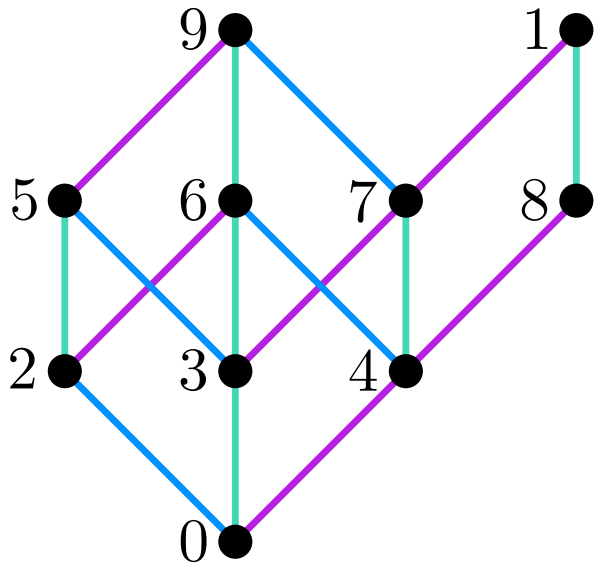

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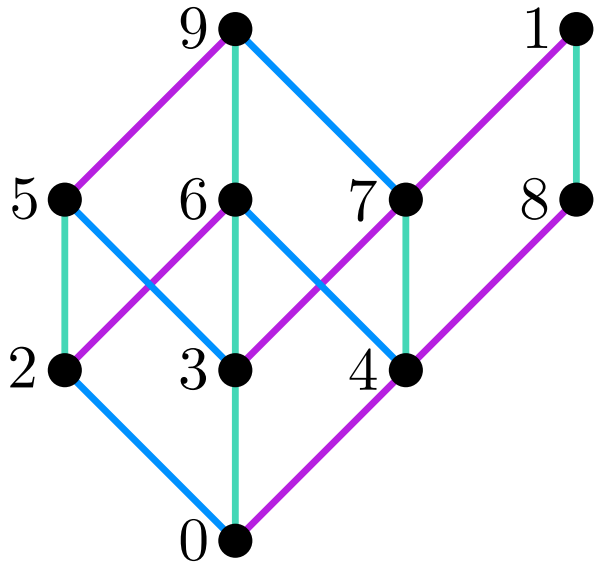

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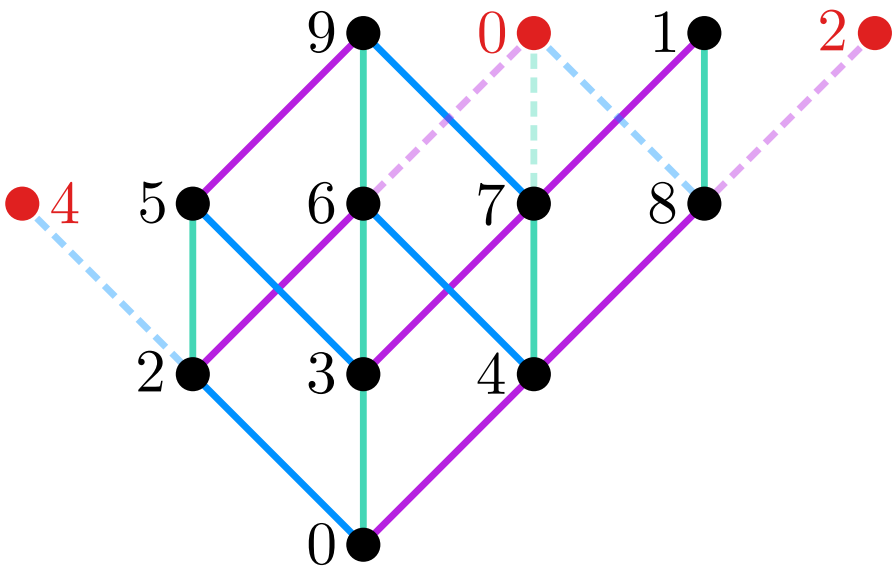

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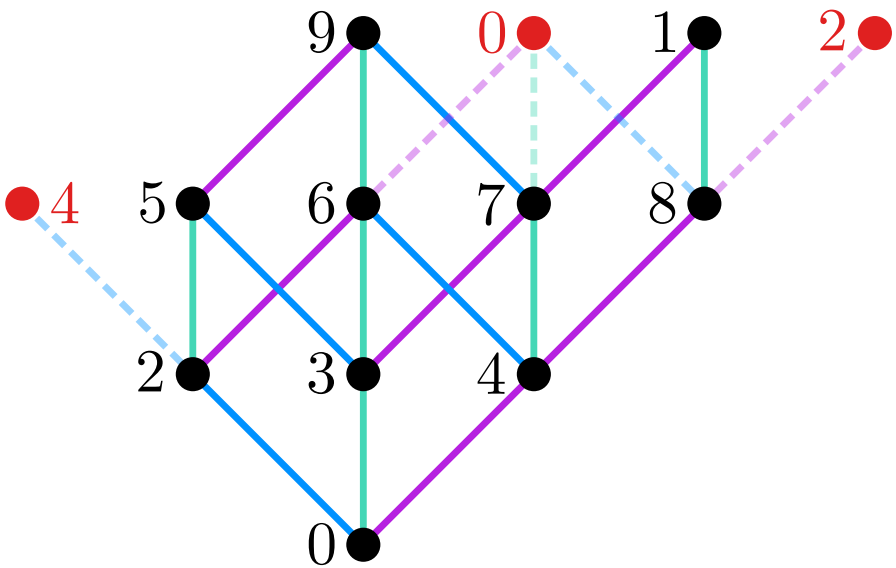

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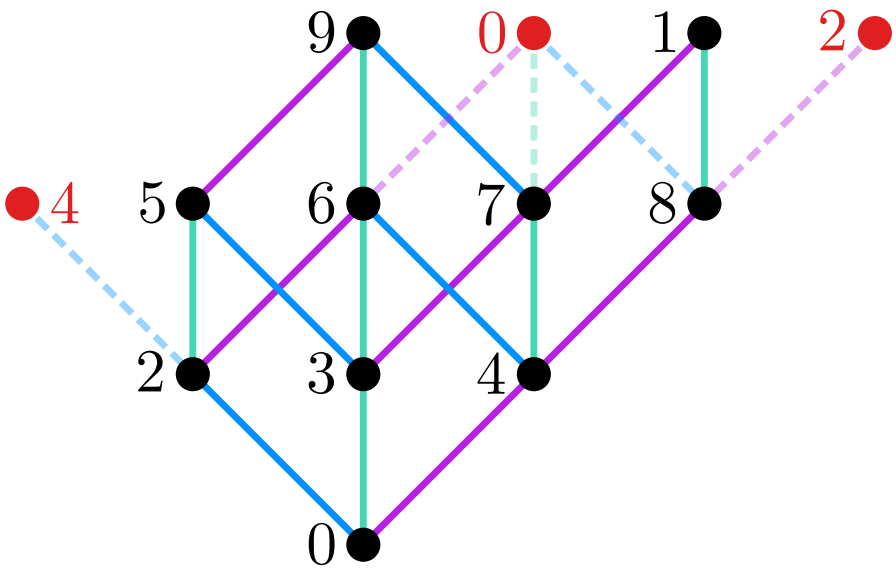

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Moral: use **sets** of factorizations, avoids overcounting minimal trades

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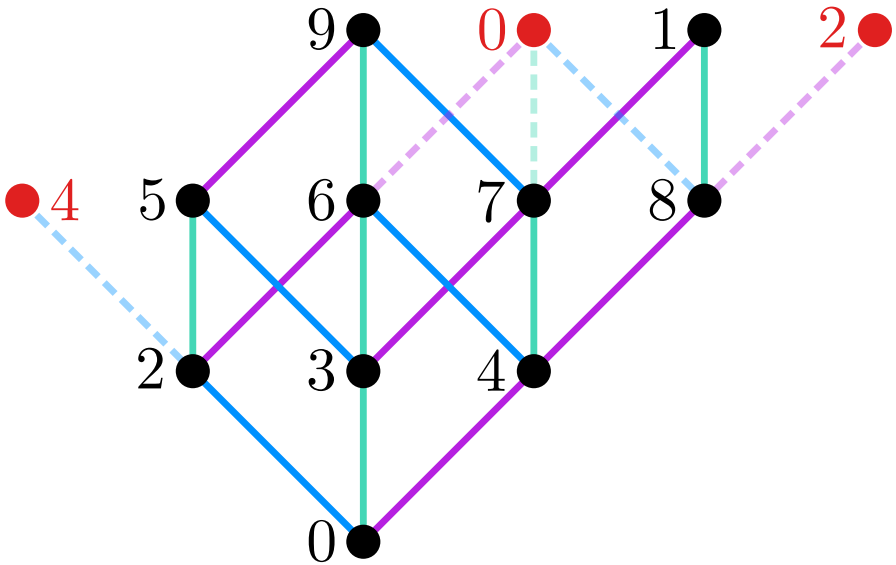

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Moral: use **sets** of factorizations, avoids overcounting minimal trades

0: $\{(0,0,2,1),(0,1,0,2)\}$
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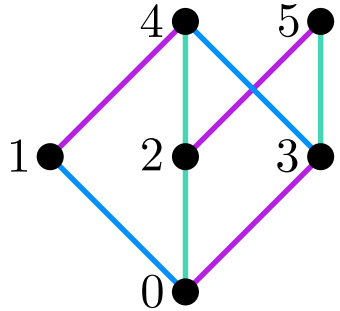

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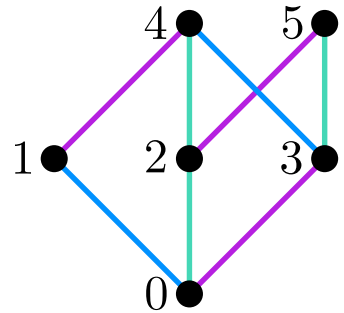

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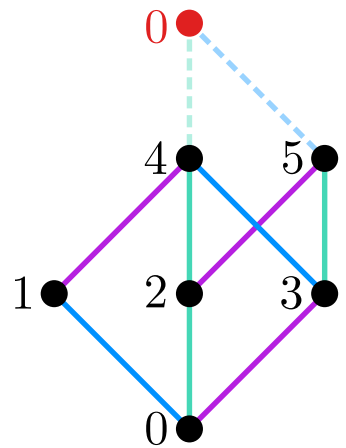

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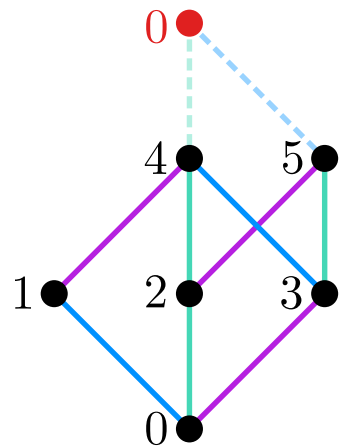

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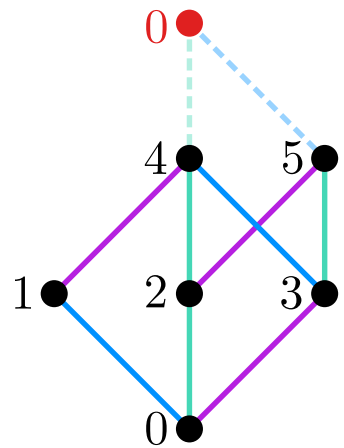

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The main theorem

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \operatorname{supp}(B)$.

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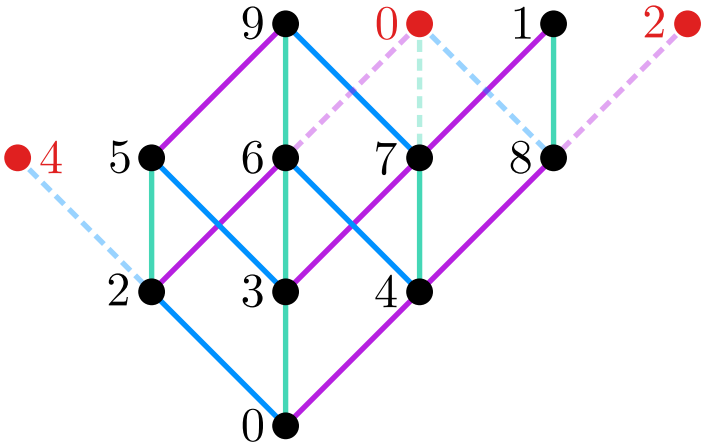

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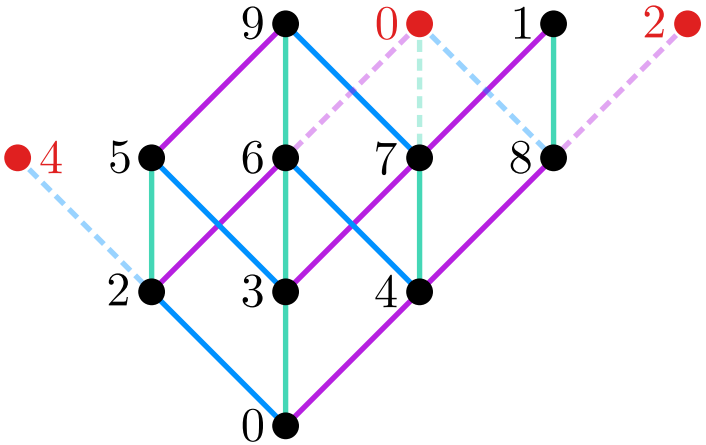

$B = \{(0,0,2,1),(0,1,0,2)\}$

$B - e_2 = \{(0,0,0,2)\} = \mathsf{Z}(a_8)$

$B - e_3 = \{(0,0,1,1)\} = \mathsf{Z}(a_7)$

$B - e_4 = \{(0,0,2,0),(0,1,0,1)\}$
$= \mathsf{Z}(a_6)$

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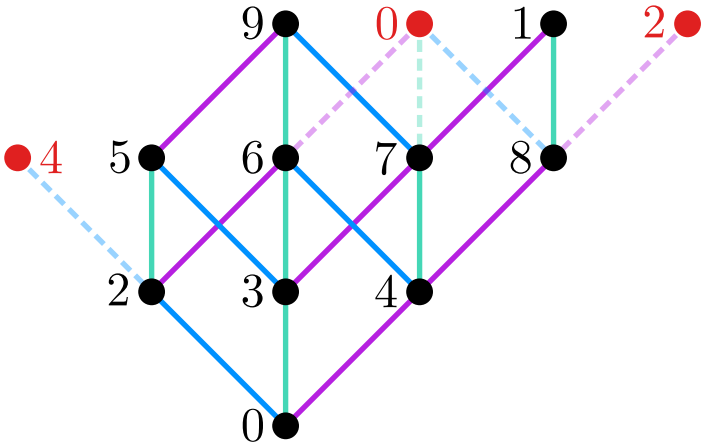

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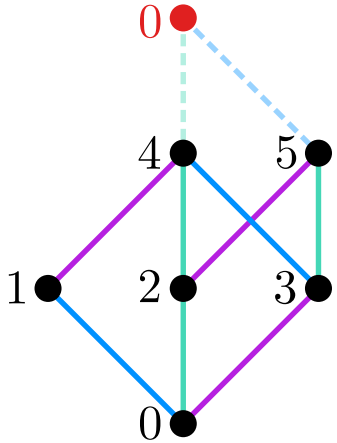

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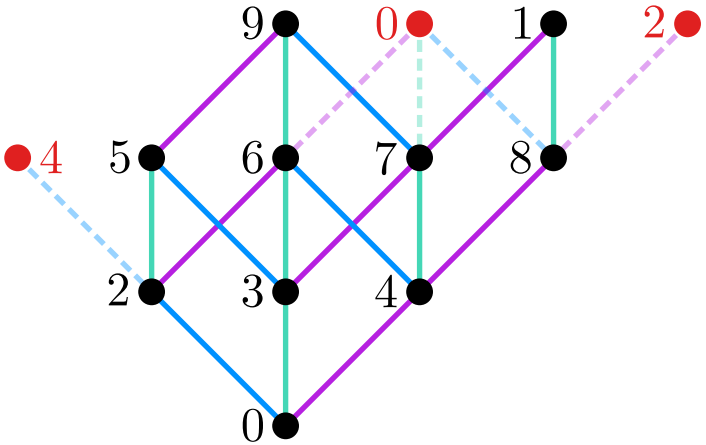

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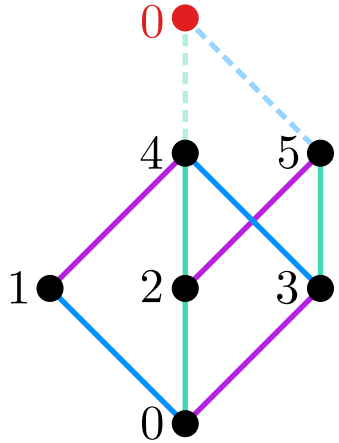

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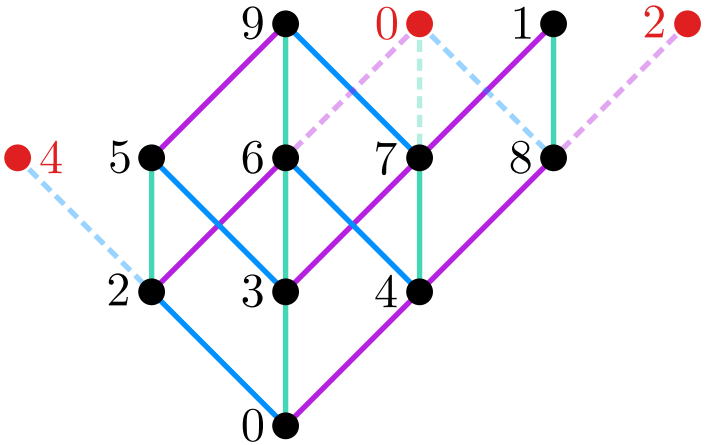

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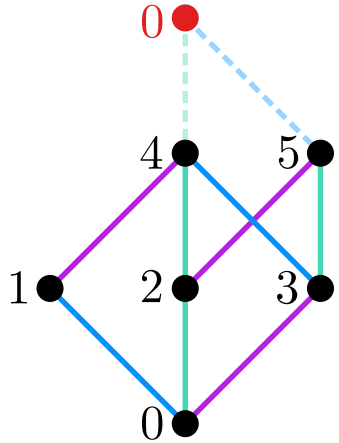

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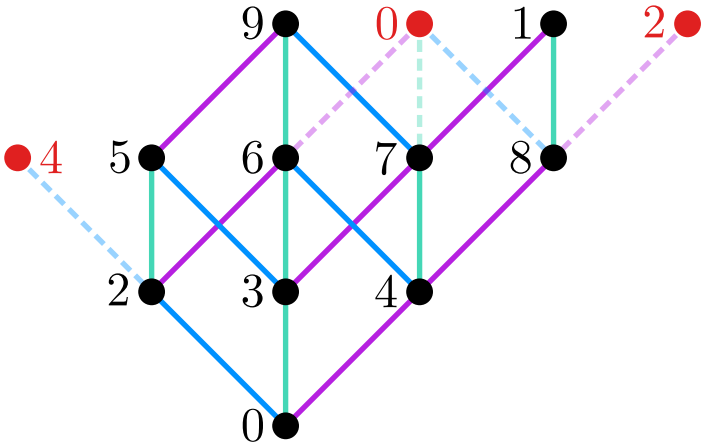

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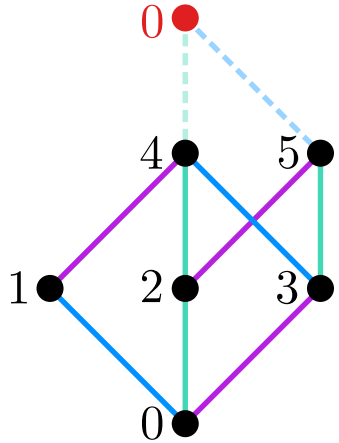

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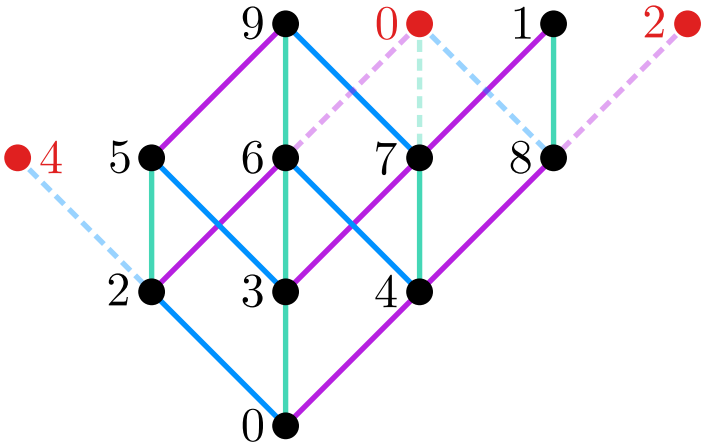

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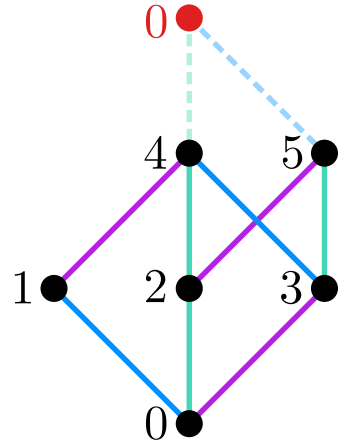

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If S has Kunz poset P, each minimal trade of S not ocurring in $\mathrm{Ap}(S)$ contains a factorization from a distinct outer Betti element of P.

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$\beta_1(I_S) = \#$ minimal trades $\leq \binom{m}{2}$

For $m = 4$, $\quad \beta_1(I_S) \in \{1, 2, 3, 6\}$

For $m = 5$, $\quad \beta_1(I_S) \in \{1, 2, 3, 5, 6, 10\}$

For $m = 6$, $\quad \beta_1(I_S) \in \{1, 2, 3, 4, 5, 6, 9, 10, 15\}$

Achievable minimal presentation cardinalities

Question

Given the multiplicity $m = \mathrm{m}(S)$ and $k = \#$ minimal generators of a numerical semigroup S, what can $\beta_1(I_S) = \#$ minimal trades be?

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m																								
3		3																						
4	2	3			6																			
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6	2	3	4	5	6			9	10					15										
7	2	3	4	5	6	7	8	9	10				14	15							21			
8	2	3	4	5	6	7	8	9	10	11		13	14	15					20	21				
9	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16			19	20	21				
10	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16		18	19	20	21	22			
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13	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
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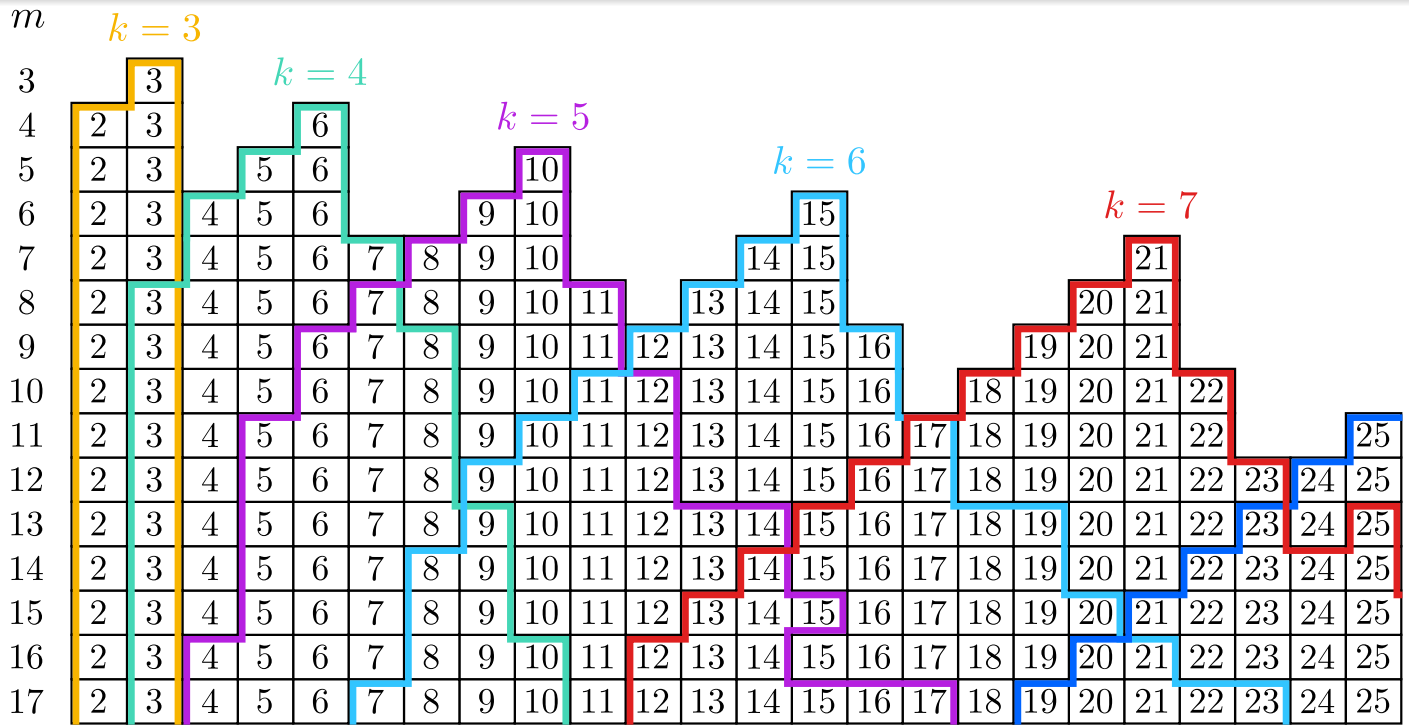
Given the multiplicity $m = \mathrm{m}(S)$ and $k = \#$ minimal generators of a numerical semigroup S, what can $\beta_1(I_S) = \#$ minimal trades be?



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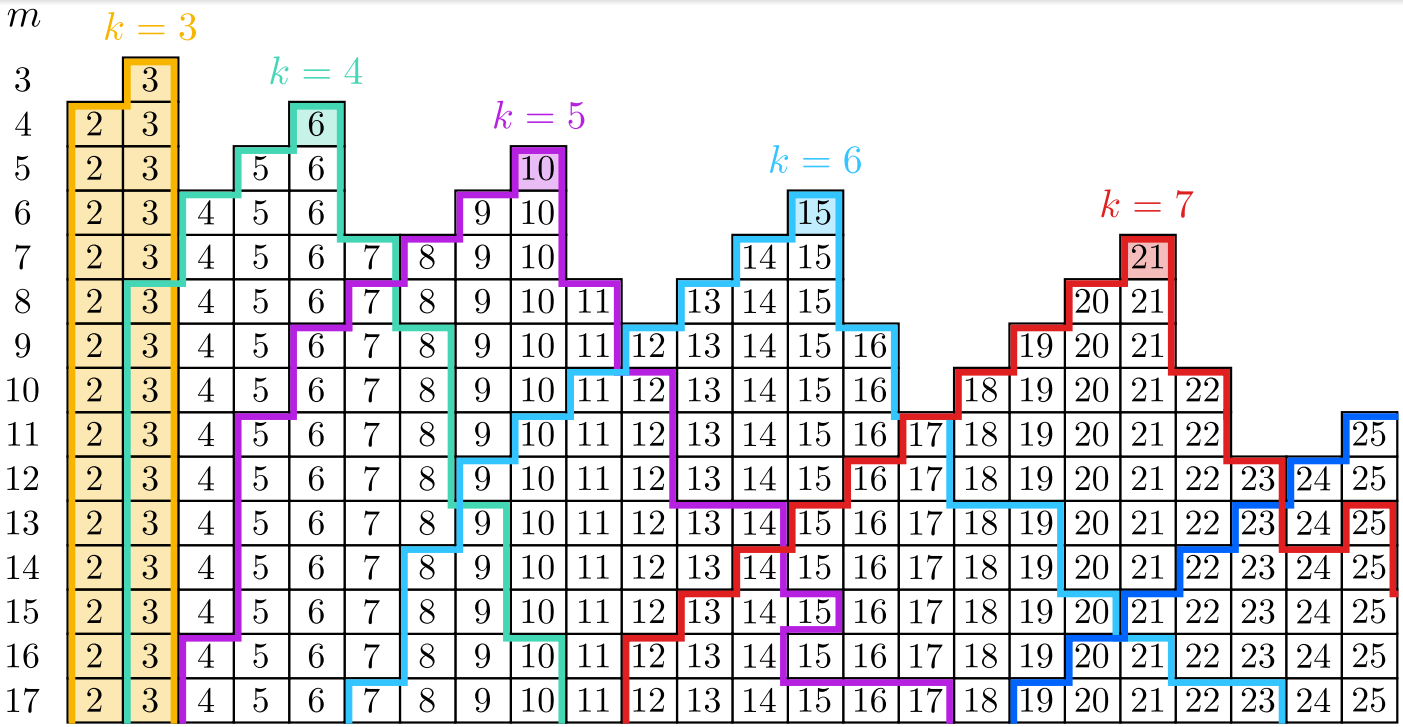

Well known: $\beta_1(S) \leq \binom{m}{2}$, with equality if and only if $k = m$

if $k = 3$, then $\beta_1(S) = 2, 3$

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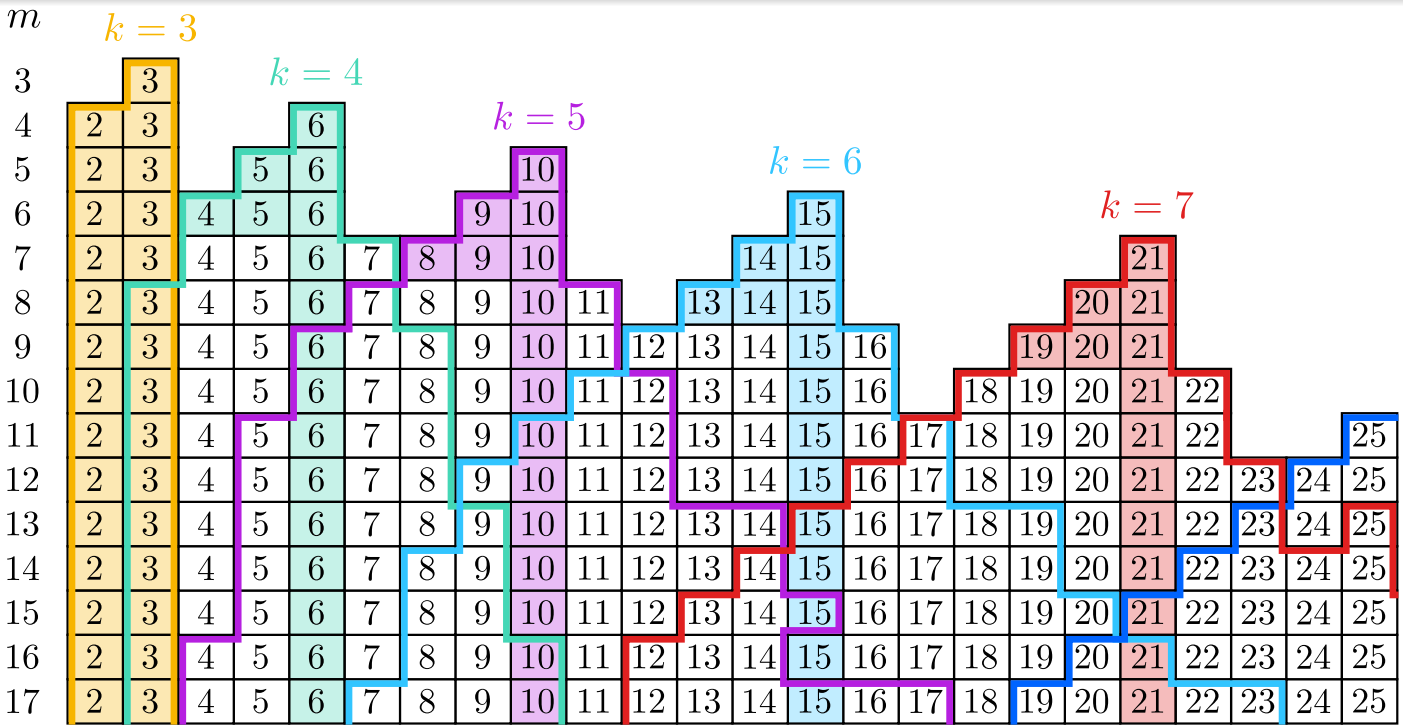

Prior work: a family has $\beta_1(S) = \binom{k}{2}$ for $3 \leq k \leq m$ (Rosales)

if $r = m - k \leq 2$, then $\beta_1(S) \in [\binom{k}{2} - r, \binom{k}{2}]$ (GS-R)

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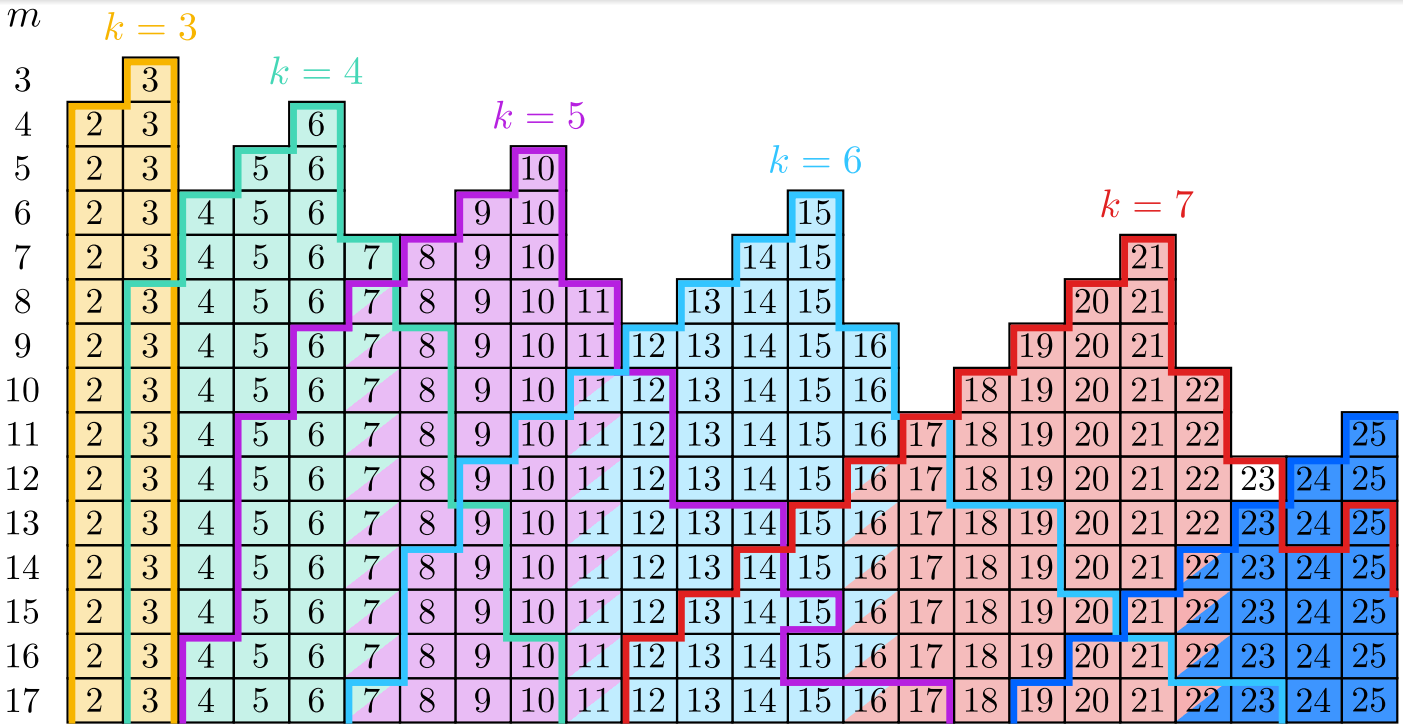

Using Kunz posets: a family hits each $\beta_1(S) \in [\binom{k}{2} - r, \binom{k}{2}]$ for $r = m - k \leq k - 2$

a family hits $\beta_1(S) = \binom{k}{2} + 1$ for each $m \geq k + 3$

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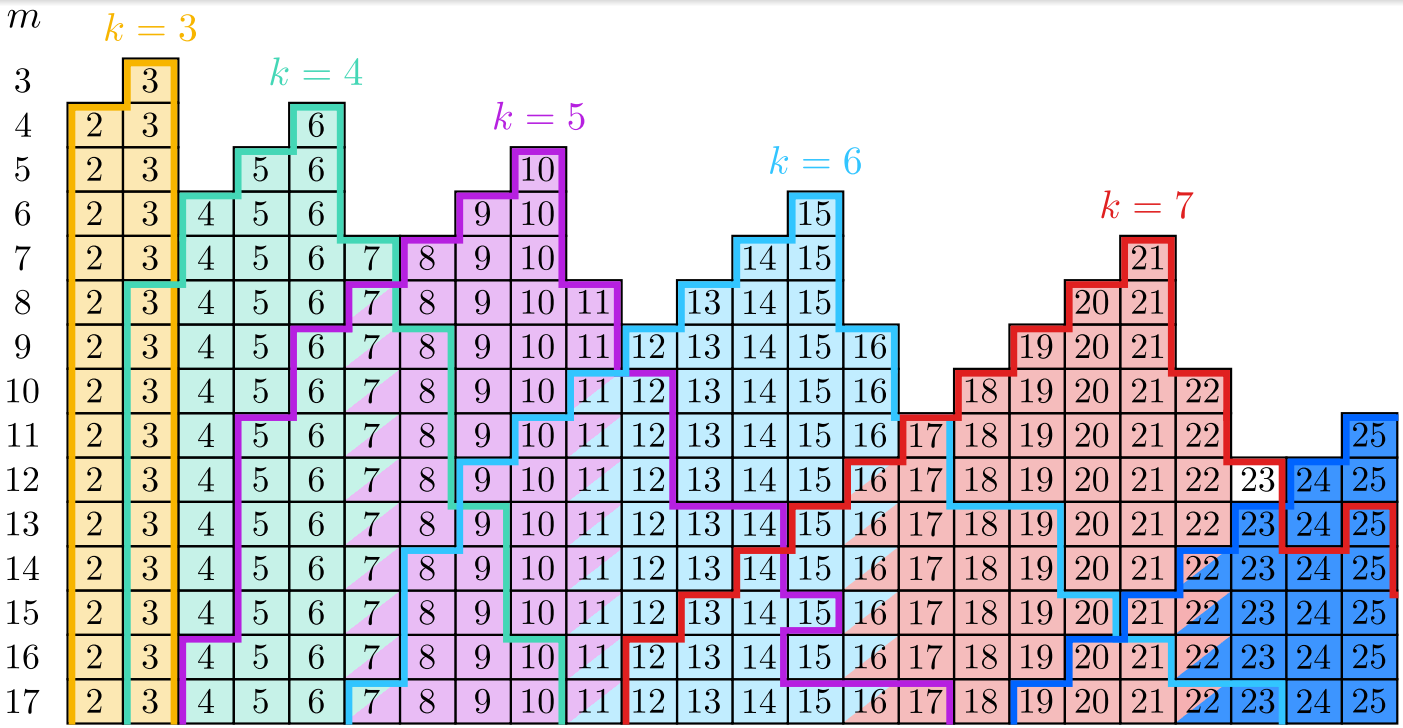

Bounds from Kunz posets: $\beta_1(S) \geq \binom{k}{2} - r$, where $r = m - k$

if $m - k = 3$, then $\beta_1(S) \in [\binom{k}{2} - 3, \binom{k}{2} + 1]$

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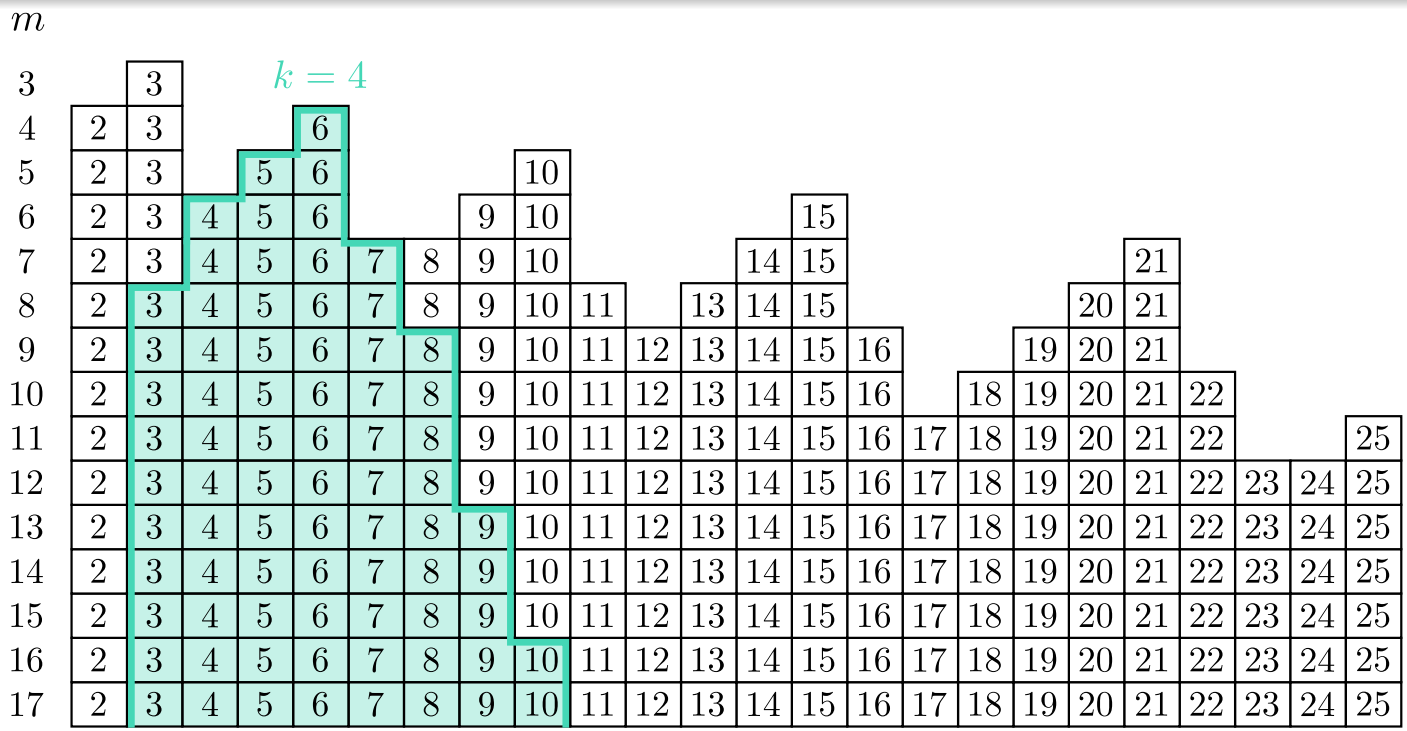

One more family: for $k = 4$, achieves each $\beta_1(S)$ with $(\beta_1(S) - 2)^2 \leq 4m$

conjectured to achieve every possible $\beta_1(S)$ for $k = 4$

References

T. Gomes, C. O'Neill, E. Torres Davila (2021)

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