Numerical semigroups, minimal presentations, and posets

Christopher O'Neill

San Diego State University

cdoneill@sdsu.edu

Joint with (i) Winfred Bruns, Pedro García-Sánchez, and Dane Wilbourne;
(ii) Nathan Kaplan; (iii) *Tara Gomes and *Eduardo Torres Davila;
(iv) *Ceyhun Elmacioglu, Kieran Hilmer, *Hannah Koufmann, and *Melin Okandan

* = undergraduate student

Slides available: `https://cdoneill.sdsu.edu/`

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$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

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Multiplicity: $\mathrm{m}(S) =$ smallest nonzero element

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

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For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
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Theorem

If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup S with $\mathrm{Ap}(S) = A$ if and only if

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Big idea: the inequalities "$a_i + a_j \geq a_{i+j}$" to define a **cone** C_m.

Kunz cone

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The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

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$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\ge 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

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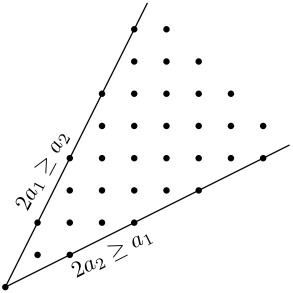
Example: C_3



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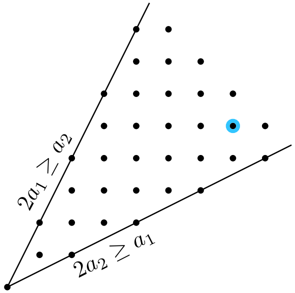

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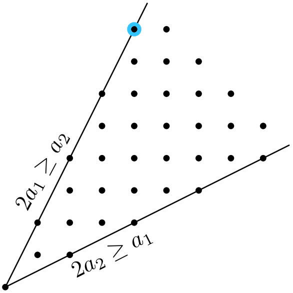

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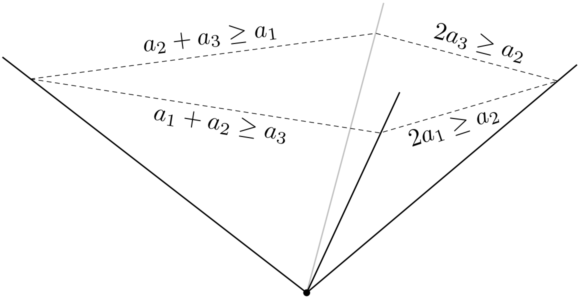

Faces of the Kunz cone

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When are numerical semigroups in (the relative interior of) the same face?

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Big picture: "moduli space" approach for studying XYZ's

- Define a space with XYZ's as points
  Small changes to an XYZ $\rightsquigarrow$ small movements in space
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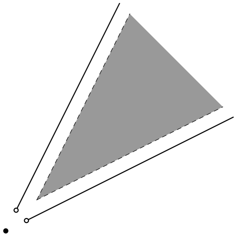
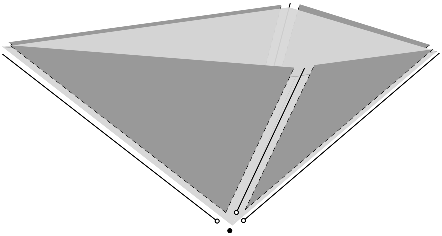
More interesting example: C_m





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$$S = \langle m, a_1, \ldots, a_{m-1} \rangle \quad \text{where} \quad \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\}$$

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What about the other faces?

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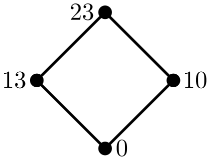
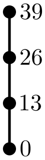


$\mathrm{Ap}(S) = \{0, 13, 26, 39\}$



Faces of the Kunz polyhedron

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$$S = \langle 6, 9, 20 \rangle$$
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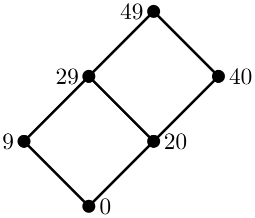

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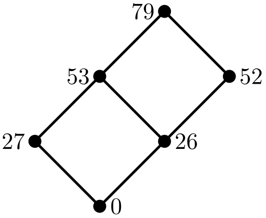

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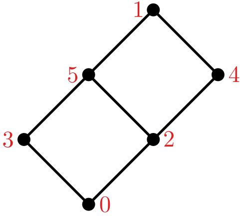

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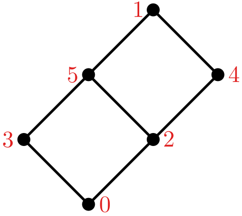

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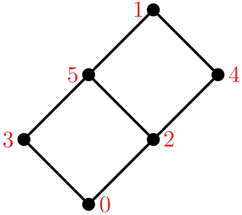

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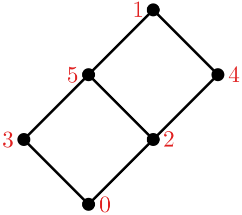

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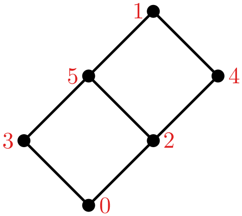

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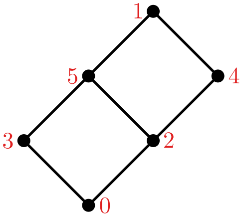

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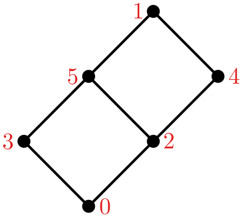

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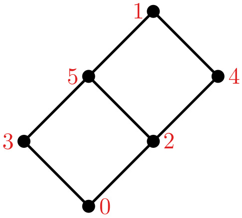

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$$2a_2 = a_4$$
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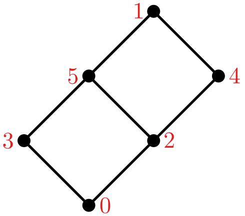

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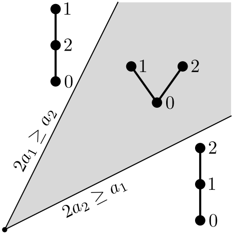

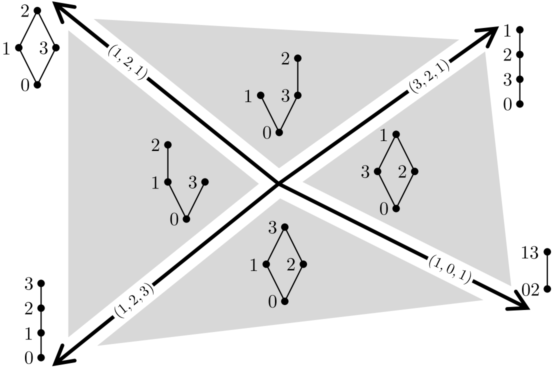

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Spoiler

If S, S' have identical Kunz posets, then S and S' have the same number of minimal trades.

Minimal presentations and Betti elements

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$\mathsf{Z}(n) = \left\{ \mathbf{a} \in \mathbb{Z}_{\geq 0}^k : n = a_1 n_1 + \cdots + a_k n_k \right\}$$

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that is closed under *translation*

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

Minimal presentations

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Definition

The *kernel* $\ker \pi$ is the relation $\sim$ on $\mathbb{Z}_{\geq 0}^k$ with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b}) \qquad\qquad x^{\mathbf{a}} - x^{\mathbf{b}} \in I_S = \ker \varphi$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a} \qquad\qquad x^{\mathbf{a}} - x^{\mathbf{a}} = 0 \in I_S$$

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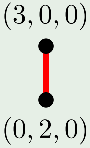

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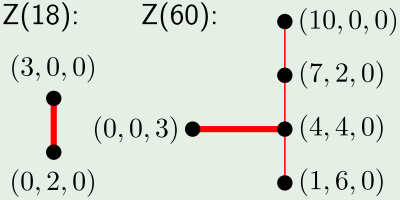

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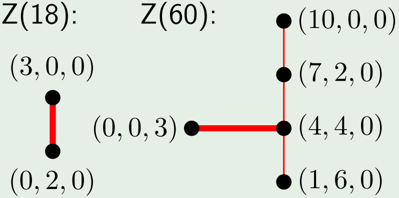

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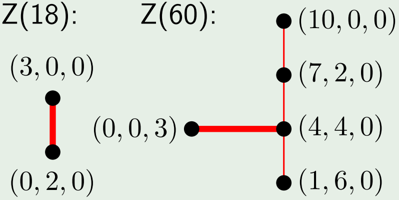

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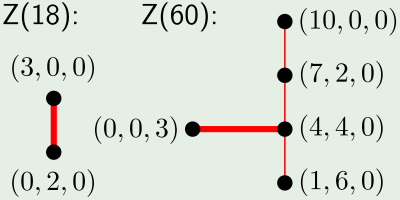

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Generating set for I_S $\Leftrightarrow$ $\mathsf{Z}(n)$ connected for all $n \in S$

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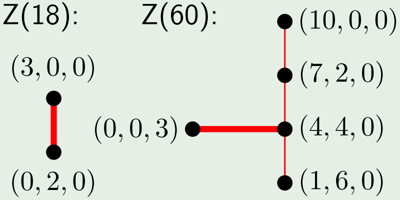

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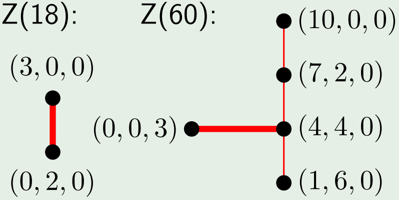

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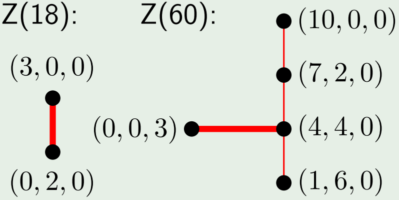

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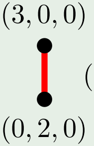

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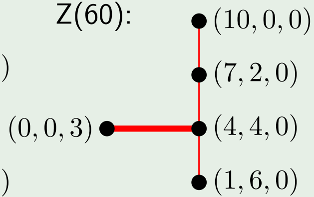

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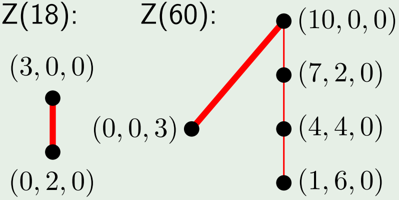

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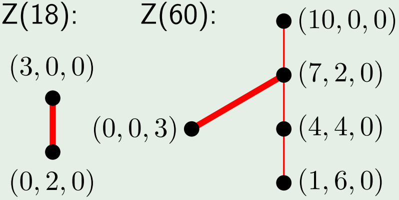

All minimal generating sets of I_S:

$$\begin{aligned} I_S &= \langle x^3 - y^2, \, x^{10} - z^3 \rangle \\ &= \langle x^3 - y^2, \, x^7 y^2 - z^3 \rangle \\ &= \langle x^3 - y^2, \, x^4 y^4 - z^3 \rangle \\ &= \langle x^3 - y^2, \, x^6 y - z^3 \rangle \end{aligned}$$

Minimal presentations

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$n = a_1 n_1 + \cdots + a_k n_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$$

Factorization homomorphism:

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minimal presentation of S $\leftrightsquigarrow$ minimal generating set of I_S

Example

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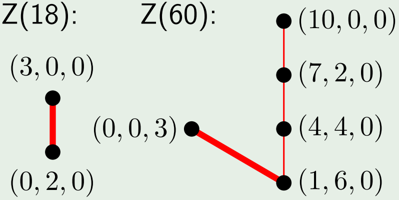

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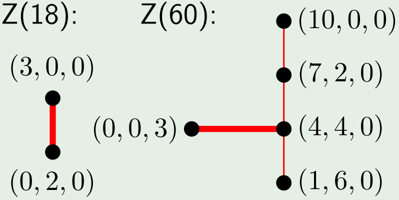

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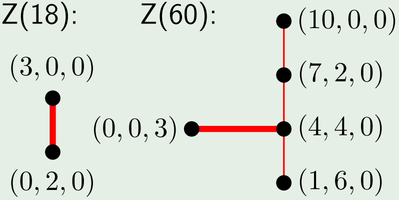

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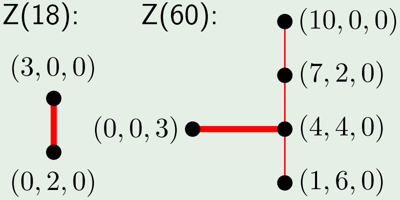

All minimal generating sets of $\sim$:

$(3,0,0) \sim (0,2,0)$, $(10,0,0) \sim (0,0,3)$

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$(3,0,0) \sim (0,2,0)$, $(\ 1,6,0) \sim (0,0,3)$

Minimal presentations and Betti elements

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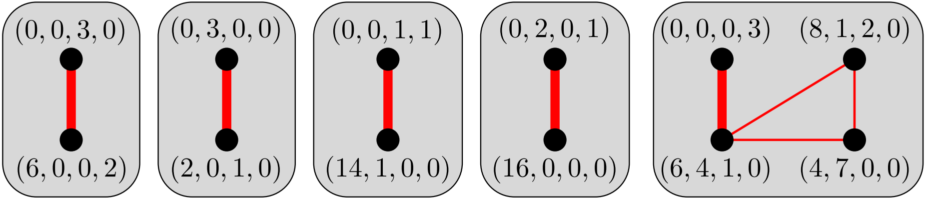

Z(132) Z(318) Z(226) Z(208) Z(360)

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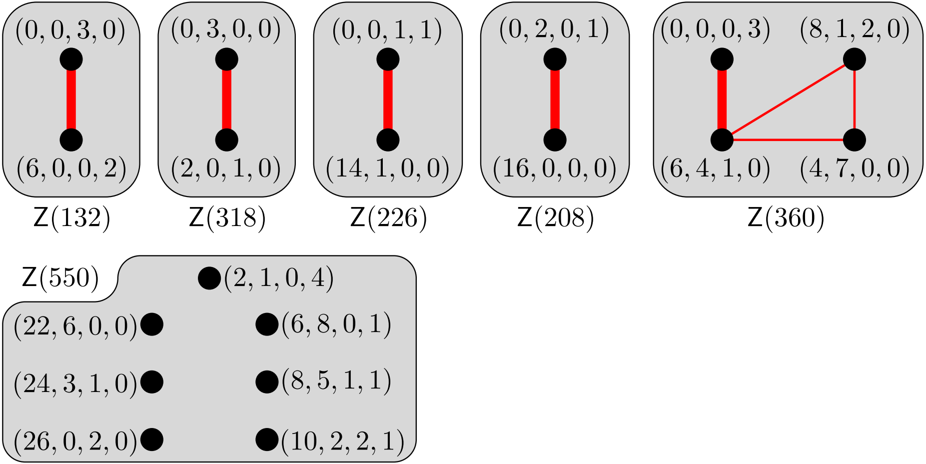

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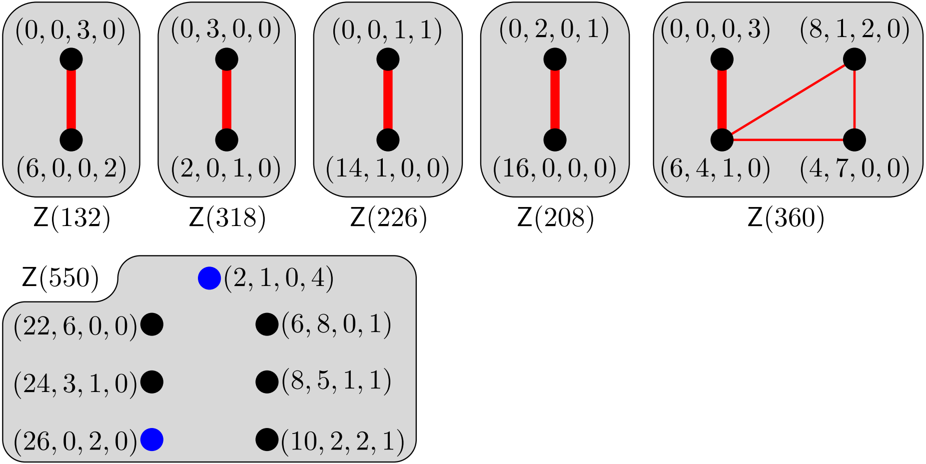

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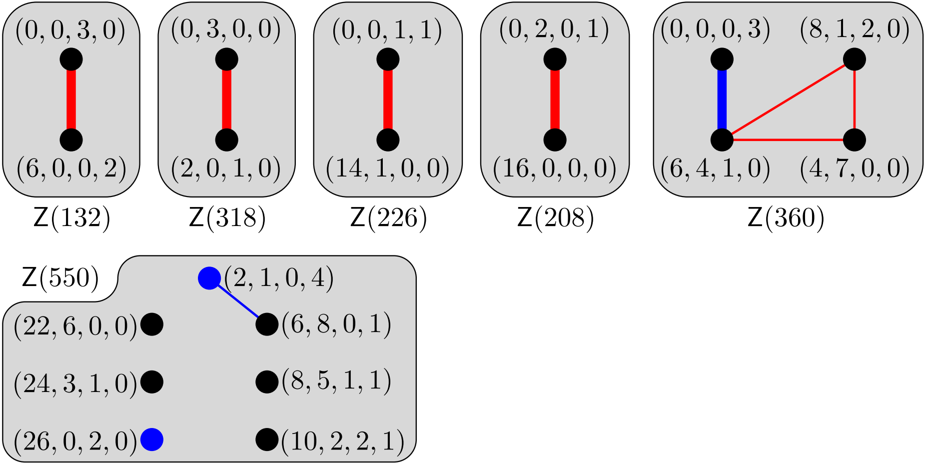

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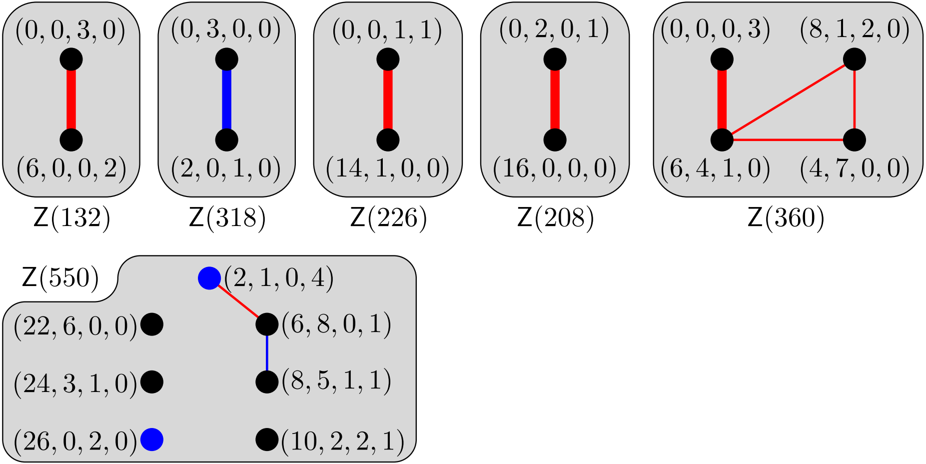

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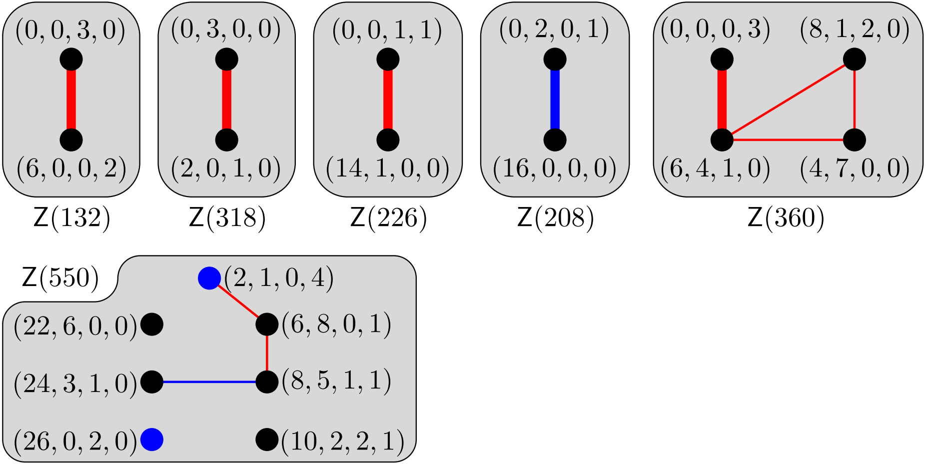

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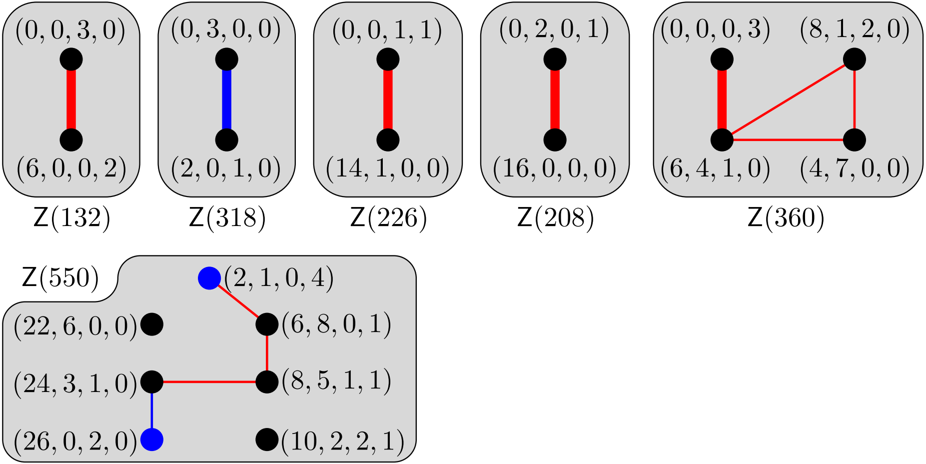

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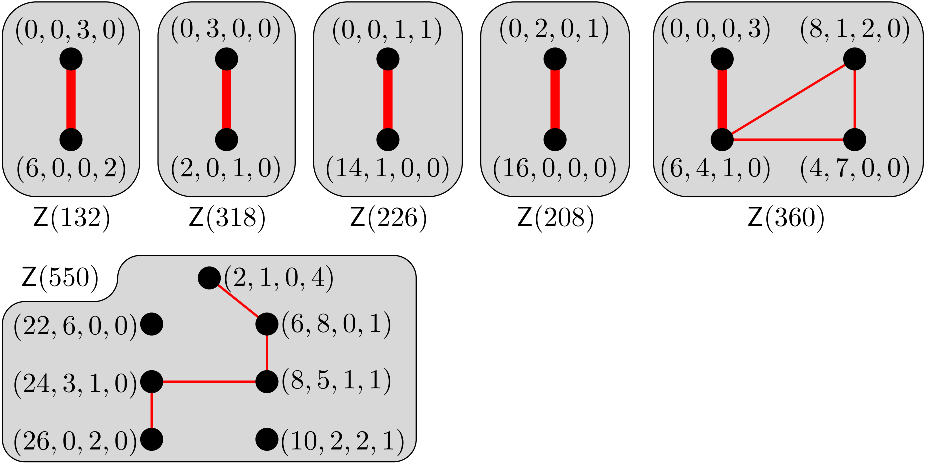

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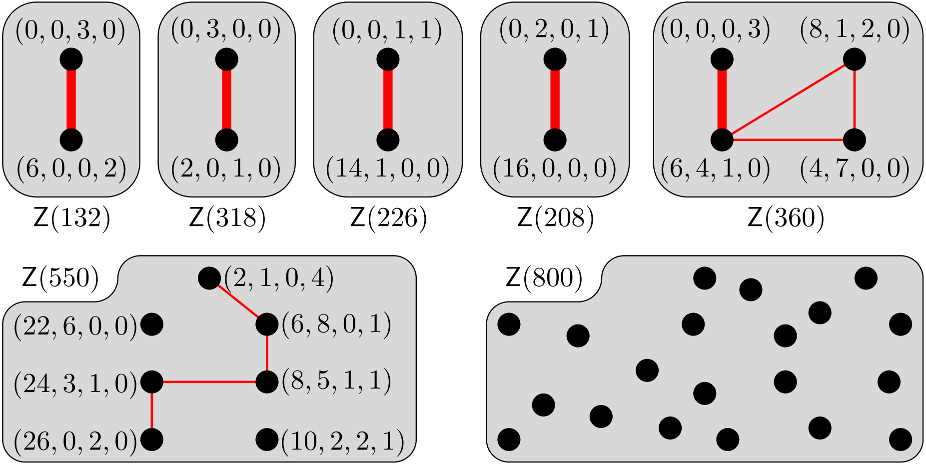

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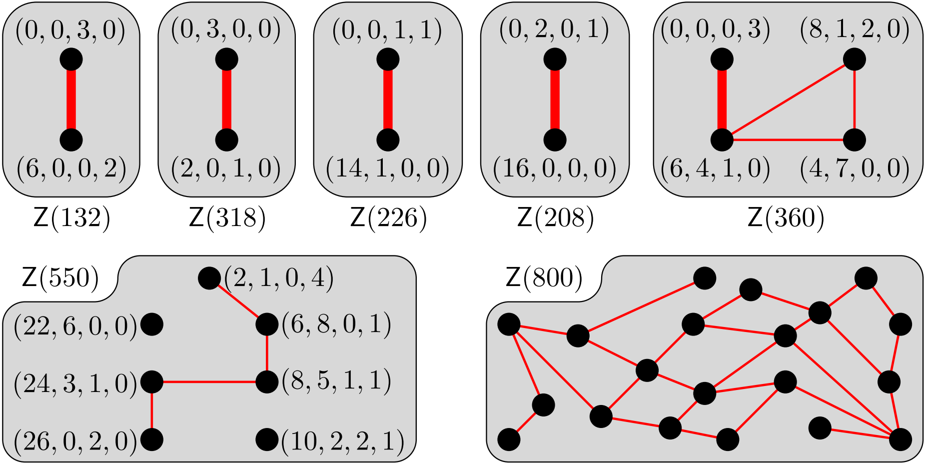

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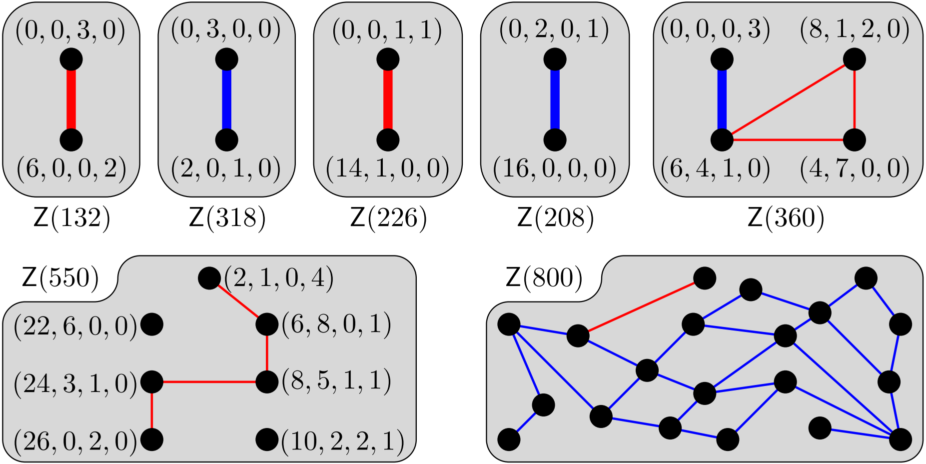

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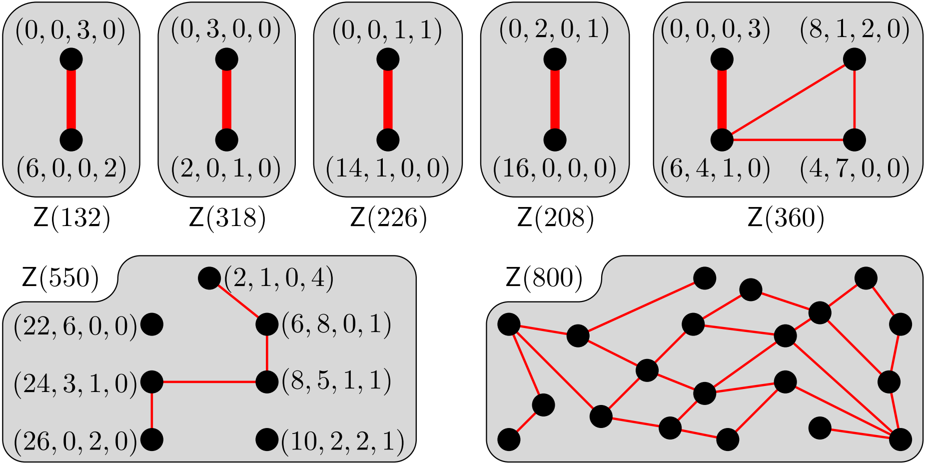

Minimal trades and Kunz posets

Question

How can one recover minimal trade structure from the Kunz poset?

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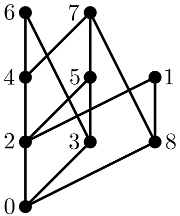

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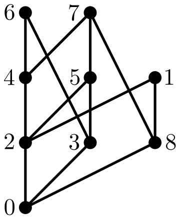
$$\mathrm{Ap}(S) = \{0, a_1, a_2, \ldots, a_8\}$$



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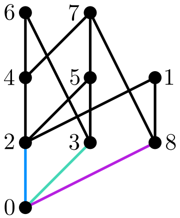

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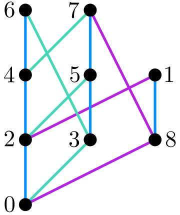

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Cover relations: add a generator

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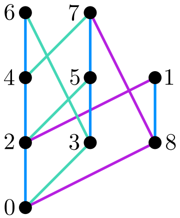

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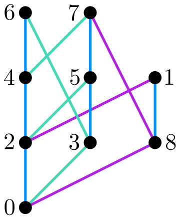

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2 "inner" minimal trades:
$(0,3,0,0) \sim (0,0,2,0)$ (at a_6)
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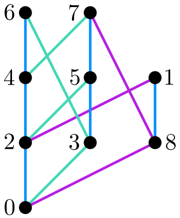

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Moral: can recover

- factorizations of $a \in \mathrm{Ap}(S)$
- (minimal) trades at $a \in \mathrm{Ap}(S)$

Minimal trades and Kunz posets

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Minimal trades and Kunz posets

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How can one recover minimal trade structure from the Kunz poset?

Key fact: each trade occurs at $a_i + n_j$ for some $a_i \in \mathrm{Ap}(S)$, generator n_j

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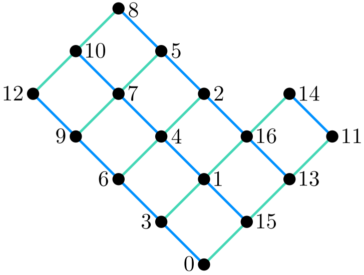

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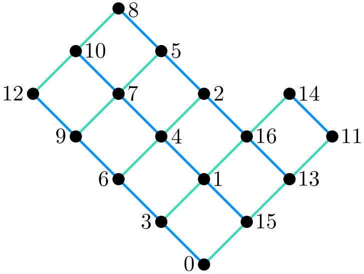

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3 minimal trades, none in $\mathrm{Ap}(S)$

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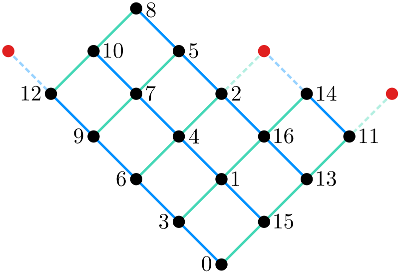

$S = \langle 17, a_3, a_{15} \rangle$

3 minimal trades, none in $\mathrm{Ap}(S)$

Minimal trades and Kunz posets

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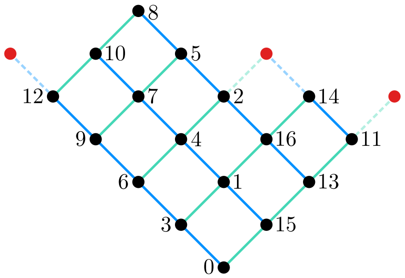

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$a_{12} + a_3$: $(0,5,0) \sim (\ ,\ ,\ )$

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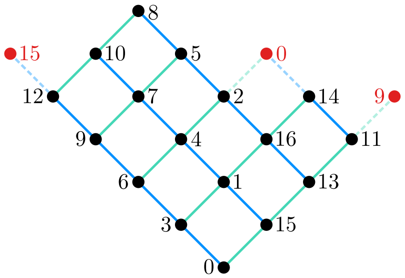

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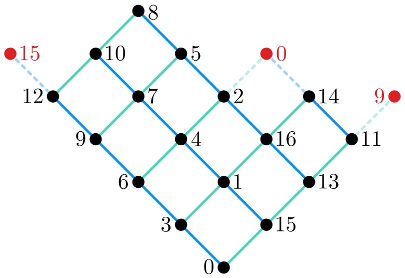

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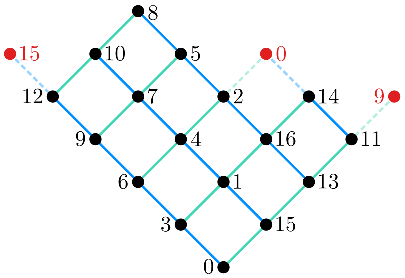

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Possible method to locate the "outer" trades:

- factorizations of $a \in \mathrm{Ap}(S)$ form a monomial staircase
- one "outer" minimal trade for each monomial generator

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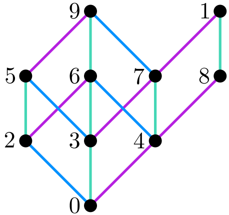

$$S = \langle 10, a_2, a_3, a_4 \rangle$$

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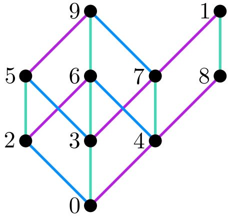

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"inner" trade at a_6:
$(0,0,2,0) \sim (0,1,0,1)$

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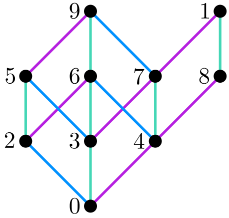

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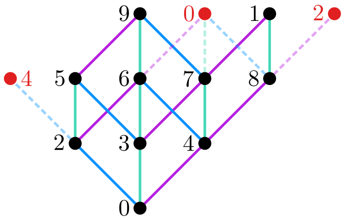

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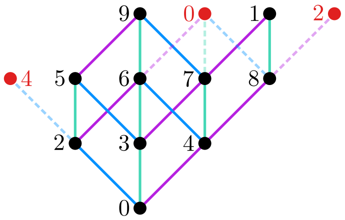

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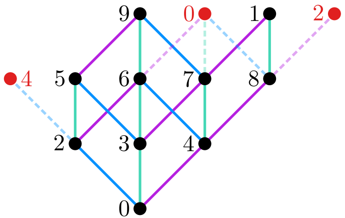

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Moral: use **sets** of factorizations, avoids overcounting minimal trades

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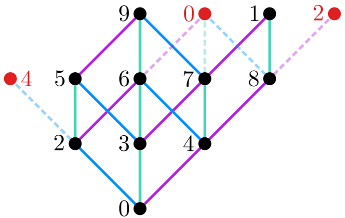

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$(0, 0, 2, 0) \sim (0, 1, 0, 1)$

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Moral: use **sets** of factorizations, avoids overcounting minimal trades

0: $\{(0, 0, 2, 1), (0, 1, 0, 2)\}$
2: $\{(0, 0, 0, 3)\}$, 4: $\{(0, 2, 0, 0)\}$

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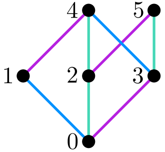

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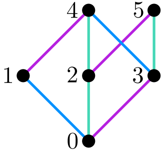

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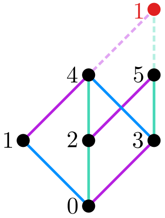

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"inner" trade at a_4:
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Minimal trades and Kunz posets

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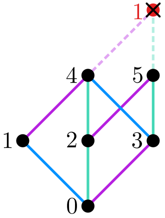

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No trades in $\mathsf{Z}(25)$:
$\{(0,0,2,1),(0,1,0,2),(3,1,0,0)\}$

Minimal trades and Kunz posets

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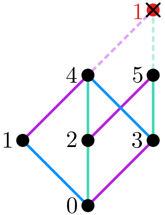

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A technical definition

Definition

An *outer Betti element* of a Kunz poset P is a set B of factorizations with connected factorization graph and $B - e_i = \mathsf{Z}(a_i)$ for each $i \in \operatorname{supp}(B)$.

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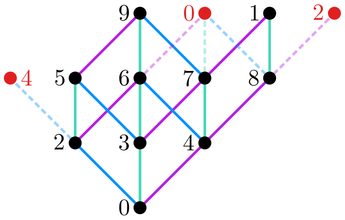

$B = \{(0,0,2,1),(0,1,0,2)\}$
$B - e_2 = \{(0,0,0,2)\} = \mathsf{Z}(a_8)$
$B - e_3 = \{(0,0,1,1)\} = \mathsf{Z}(a_7)$
$B - e_4 = \{(0,0,2,0),(0,1,0,1)\}$
$= \mathsf{Z}(a_6)$

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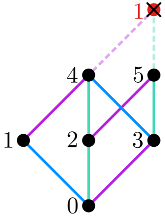

$B = \{(0,0,2,1)\}$?
$B - e_4 = \{(0,0,2,0)\} \subsetneq \mathsf{Z}(a_4)$

$B = \{(0,0,2,1),(0,1,0,2)\}$?
$B - e_3 = \{(0,0,1,1)\} \not\subseteq \mathsf{Z}(a_i)$

The main theorem

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Recovering minimal presentation from the Kunz poset P of S:

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- trades occuring at $a \in \mathrm{Ap}(S)$ recovered from factorizations of $\overline{a} \in P$

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- each trade occuring outside $\mathrm{Ap}(S)$ corresponds to an outer Betti element of P

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Theorem (Gomes–O.–Torres Davila)

If S has Kunz poset P, each minimal trade of S not ocurring in $\mathrm{Ap}(S)$ *contains a factorization from a distinct outer Betti element of P.*

In particular, if S, S' have identical Kunz poset, then S and S' have the same number of minimal trades.

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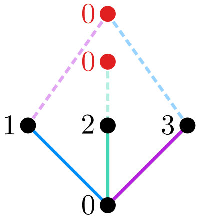

$B_1 = \{(0, 0, 2, 0)\}$

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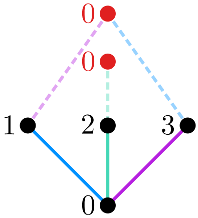

$S = \langle 4, 9, 14, 11 \rangle$

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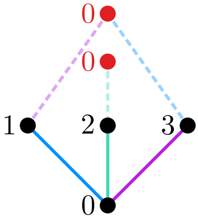

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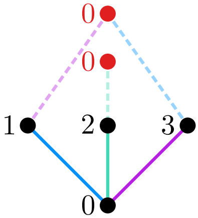

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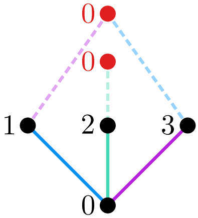

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For $m = 6$: $\quad\#$ minimal trades $\in \{1, 2, 3, 4, 5, 6, 9, 10, 15\}$

Application: classifying minimal trades

Question

Given the multiplicity $m = \mathrm{m}(S)$ and # minimal generators k of a numerical semigroup S, what can $\beta_1(I_S) = \#$ minimal trades be?

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m																								
3		3																						
4	2	3			6																			
5	2	3		5	6				10															
6	2	3	4	5	6			9	10					15										
7	2	3	4	5	6	7	8	9	10				14	15						21				
8	2	3	4	5	6	7	8	9	10	11		13	14	15					20	21				
9	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16			19	20	21				
10	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16		18	19	20	21	22			
11	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22			25
12	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
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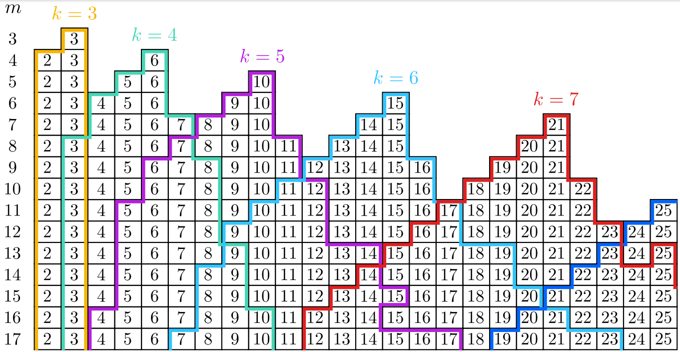
14	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
15	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
16	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
17	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

Application: classifying minimal trades

Question

Given the multiplicity $m = \mathrm{m}(S)$ and # minimal generators k of a numerical semigroup S, what can $\beta_1(I_S) =$ # minimal trades be?



Application: classifying minimal trades

Question

Given the multiplicity $m = \mathsf{m}(S)$ and # minimal generators k of a numerical semigroup S, what can $\beta_1(I_S) = \#$ minimal trades be?

$k = 3$, $k = 4$, $k = 5$, $k = 6$, $k = 7$

m																								
3		3																						
4	2	3			6																			
5	2	3		5	6				10															
6	2	3	4	5	6			9	10					15										
7	2	3	4	5	6	7	8	9	10				14	15						21				
8	2	3	4	5	6	7	8	9	10	11		13	14	15					20	21				
9	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16			19	20	21				
10	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16		18	19	20	21	22			
11	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22			25
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15	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
16	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
17	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

Well known: $\beta_1(S) \le \binom{m}{2}$, with equality if and only if $k = m$

if $k = 3$, then $\beta_1(S) = 2, 3$

Application: classifying minimal trades

Question

Given the multiplicity $m = \mathrm{m}(S)$ and # minimal generators k of a numerical semigroup S, what can $\beta_1(I_S) =$ # minimal trades be?

$k = 3$ $k = 4$ $k = 5$ $k = 6$ $k = 7$

m																								
3		3																						
4	2	3			6																			
5	2	3		5	6				10															
6	2	3	4	5	6			9	10					15										
7	2	3	4	5	6	7	8	9	10				14	15						21				
8	2	3	4	5	6	7	8	9	10	11		13	14	15					20	21				
9	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16			19	20	21				
10	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16		18	19	20	21	22			
11	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22			25
12	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
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14	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
15	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
16	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
17	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

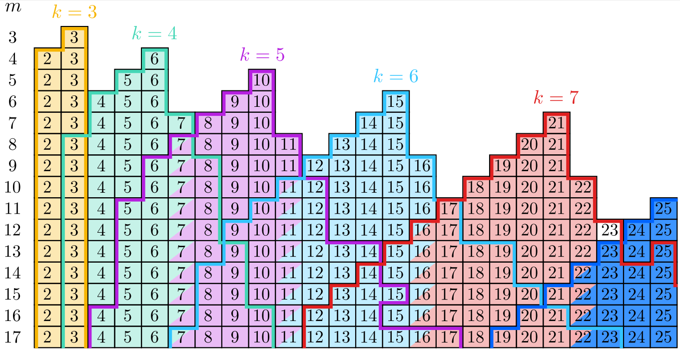
Prior work: a family has $\beta_1(S) = \binom{k}{2}$ for $3 \le k \le m$ (Rosales)

if $r = m - k \le 2$, then $\beta_1(S) \in [\binom{k}{2} - r, \binom{k}{2}]$ (GS-R)

Application: classifying minimal trades

Question

Given the multiplicity $m = \mathrm{m}(S)$ and # minimal generators k of a numerical semigroup S, what can $\beta_1(I_S) =$ # minimal trades be?



Using Kunz posets: a family hits each $\beta_1(S) \in [\binom{k}{2} - r, \binom{k}{2}]$

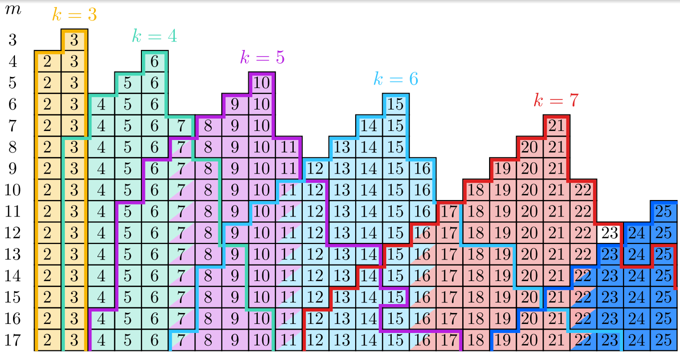
for $r = m - k \leq k - 2$

a family hits $\beta_1(S) = \binom{k}{2} + 1$ for each $m \geq k + 3$

Application: classifying minimal trades

Question

Given the multiplicity $m = \mathrm{m}(S)$ and # minimal generators k of a numerical semigroup S, what can $\beta_1(I_S) =$ # minimal trades be?



Bounds from Kunz posets: $\beta_1(S) \geq \binom{k}{2} - r$, where $r = m - k$

if $m - k = 3$, then $\beta_1(S) \in [\binom{k}{2} - 3, \binom{k}{2} + 1]$

Application: classifying minimal trades

Question

Given the multiplicity $m = \mathrm{m}(S)$ and # minimal generators k of a numerical semigroup S, what can $\beta_1(I_S) =$ # minimal trades be?

$k = 4$

m																								
3		3																						
4	2	3			6																			
5	2	3		5	6				10															
6	2	3	4	5	6			9	10					15										
7	2	3	4	5	6	7	8	9	10				14	15						21				
8	2	3	4	5	6	7	8	9	10	11		13	14	15					20	21				
9	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16			19	20	21				
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12	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
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17	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

One more family: for $k = 4$, achieves each $\beta_1(S)$ with $(\beta_1(S) - 2)^2 \leq 4m$

conjectured to achieve every possible $\beta_1(S)$ for $k = 4$

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Thanks!