Classifying numerical semigroups using polyhedral geometry

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Example:

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{c} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

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Multiplicity: $\mathrm{m}(S) =$ smallest nonzero element

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Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

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For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
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- $|\,\mathrm{Ap}(S)| = m$

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Theorem

If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup S with $\mathrm{Ap}(S) = A$ if and only if

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Big idea: the inequalities "$a_i + a_j \geq a_{i+j}$" to define a **cone** C_m.

Kunz cone

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The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

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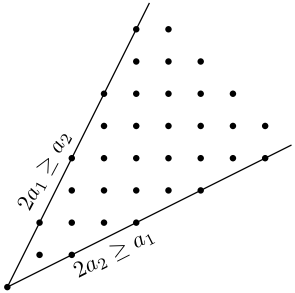
Example: C_3



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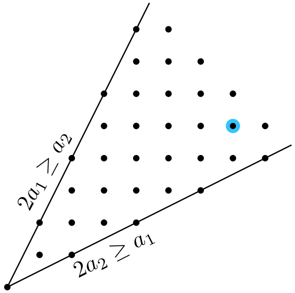

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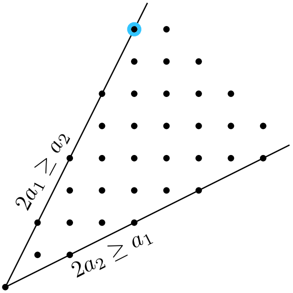

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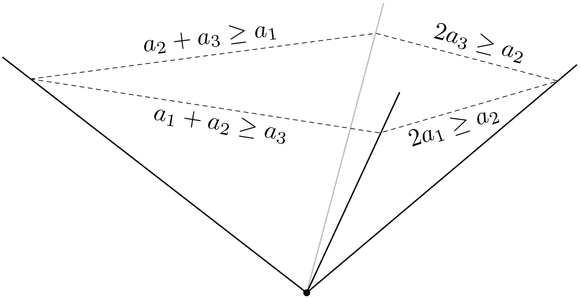

Faces of the Kunz cone

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When are numerical semigroups in (the relative interior of) the same face?

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Big picture: "moduli space" approach for studying XYZ's

- Define a space with XYZ's as points
  Small changes to an XYZ $\rightsquigarrow$ small movements in space
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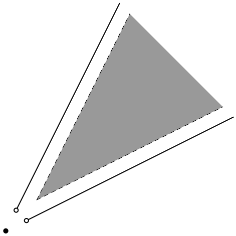
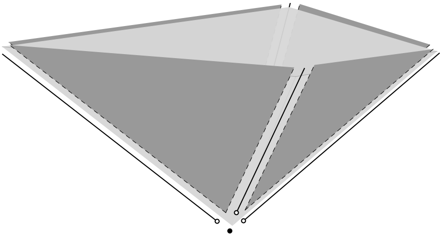
More interesting example: C_m





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Example: $S = \langle 4, 10, 11, 13 \rangle$

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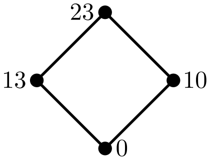

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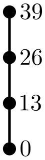

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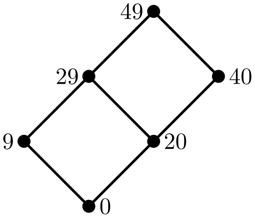

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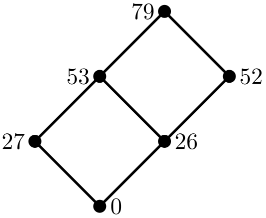

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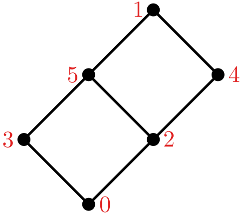

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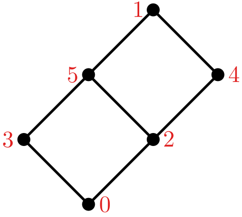

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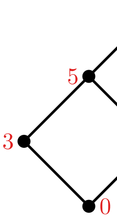

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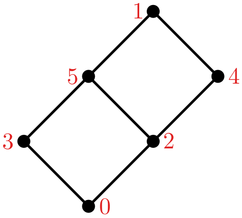

The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

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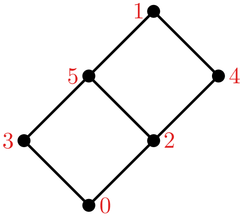

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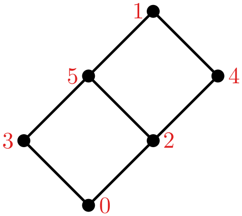
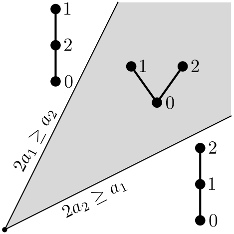


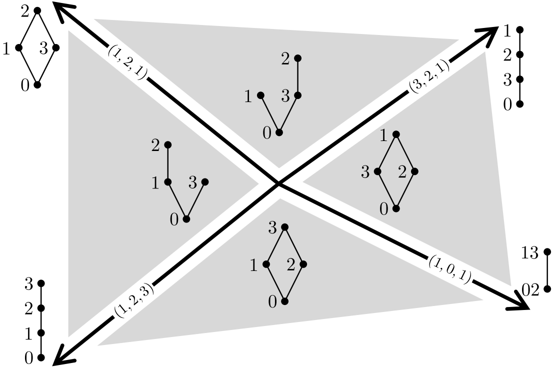
The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

Theorem (Bruns–García-Sánchez–O.–Wilburne)

Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

C_3 and C_4





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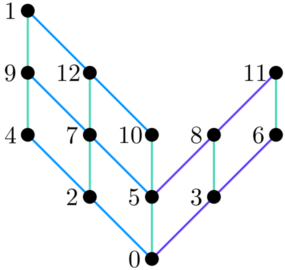

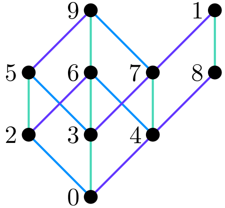

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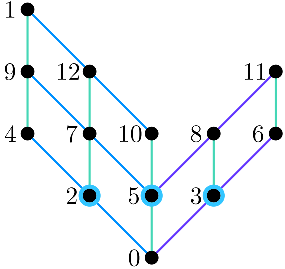

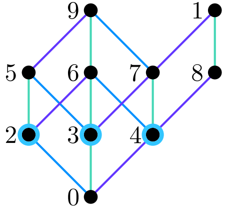

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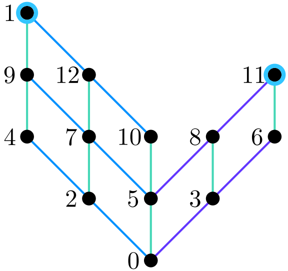

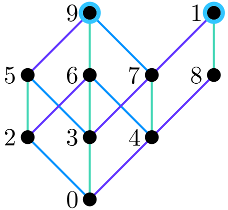

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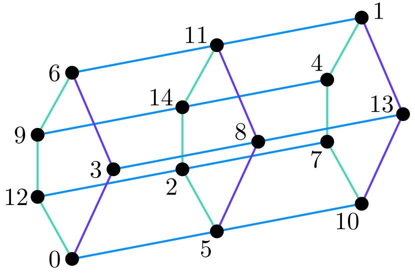

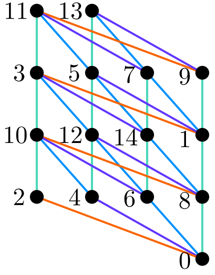

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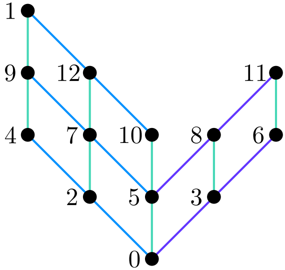

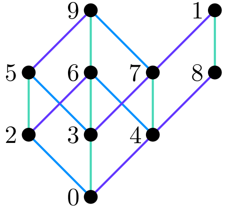

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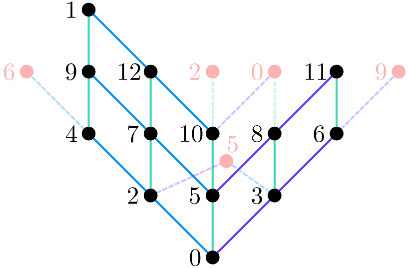

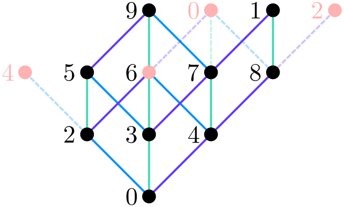

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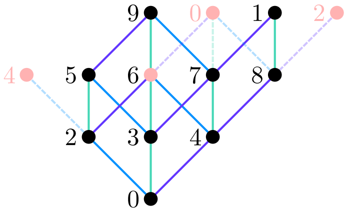
$$I_S = \langle\, x_1^2 - y^* x_3,\ x_1x_3 - x_2^2,\ x_2^2x_3 - y^*,\ x_3^3 - y^* x_1 \,\rangle \subseteq \Bbbk[y, x_1, x_2, x_3]$$



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- Betti numbers of I_S over $\mathbb{k}[\overline{x}]$

$$0 \leftarrow R \leftarrow R^6 \longleftarrow R^8 \longleftarrow R^3 \leftarrow 0$$

Map $R^6 \leftarrow R^8$:

	1,12	1,13	2,12	2,23	3,13	3,23	2,13	3,12
1,1	$-x_2$	$-x_3$					y^*	y^*
2,2	$-y^*$		x_1	$-x_3$		y^*		
3,3					x_1	x_2	$-y^*$	
2,1	x_1		$-x_2$	y^*	y^*		$-x_3$	
3,1	y^*	x_1			$-x_3$	$-y^*$		$-x_2$
3,2		$-y^*$	$-y^*$	x_2		$-x_3$	x_1	x_1

Map $R^8 \leftarrow R^3$:

	1,[3]	2,[3]	3,[3]
1,12	x_3	$-y^*$	
1,13	$-x_2$		y^*
2,12		x_3	$-y^*$
2,23	$-y^*$	x_1	
3,13	y^*		$-x_2$
3,23		$-y^*$	x_1
2,13	x_1	$-x_2$	
3,12	$-x_1$		x_3

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- Betti numbers of $\Bbbk$ over $\Bbbk[\overline{x}]/I_S$

$$0 \leftarrow R \xleftarrow{\begin{array}{c} \begin{array}{cccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \emptyset \, [\begin{array}{cccc} y & x_1 & x_2 & x_3 \end{array}] \end{array}} R^4 \xleftarrow{\begin{array}{c|cccccccccccc} & \mathbf{01} & \mathbf{02} & \mathbf{03} & \mathbf{11} & \mathbf{12} & \mathbf{13} & \mathbf{21} & \mathbf{22} & \mathbf{23} & \mathbf{31} & \mathbf{32} & \mathbf{33} \\ \hline \mathbf{0} & x_1 & x_2 & x_3 & & & {\color{red}-y^*} & & {\color{red}-y^*} & & {\color{red}-y^*} & & \\ \mathbf{1} & -y & & & x_1 & x_2 & x_3 & & & {\color{red}-y^*} & & {\color{red}-y^*} & \\ \mathbf{2} & & -y & & {\color{red}-y^*} & & & x_1 & x_2 & x_3 & & & {\color{red}-y^*} \\ \mathbf{3} & & & -y & & {\color{red}-y^*} & & {\color{red}-y^*} & & & x_1 & x_2 & x_3 \end{array}} R^{12}$$

$$\longleftarrow R^{36} \longleftarrow R^{108} \longleftarrow R^{324} \longleftarrow R^{972} \longleftarrow R^{2916} \longleftarrow \cdots$$

Gluing maps and complete intersections

Fact: for $x \in C_m$, the coordinates of 0's form a subgroup of $\mathbb{Z}_m$

Example: $\mathbb{Z}_{10}$: $(*, *, *, *, 0, *, *, *, *)$ $(*, 0, *, 0, *, 0, *, 0, *)$

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Theorem

If $d \mid m$, then there exists a map $C_d \hookrightarrow C_m$ that induces a dimension-preserving injection on face lattices.

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Theorem

If $d \mid m$, then there exists a map $C_d \hookrightarrow C_m$ that induces a dimension-preserving injection on face lattices.

Rays of $C_2 \subseteq \mathbb{R}^1$:

(1)

Rays of $C_4 \subseteq \mathbb{R}^3$:

$(1, 0, 1)$

$(1, 2, 1)$

$(1, 2, 3)$

$(3, 2, 1)$

Rays of $C_3 \subseteq \mathbb{R}^2$:

$(1, 2)$

$(2, 1)$

Rays of $C_{12} \subseteq \mathbb{R}^{11}$:

$(1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1)$

$(1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1)$

$(1, 2, 3, 0, 1, 2, 3, 0, 1, 2, 3)$

$(3, 2, 1, 0, 3, 2, 1, 0, 3, 2, 1)$

$\vdots$

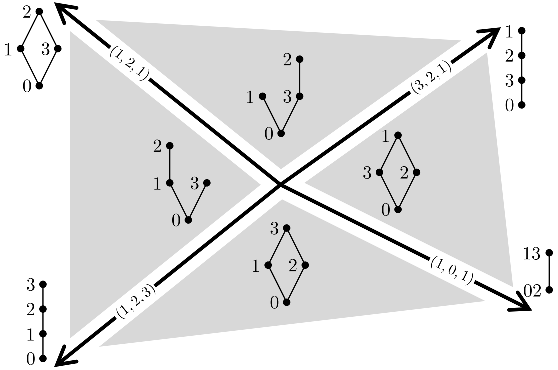
$\vdots$

$(1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2)$

$(2, 1, 0, 2, 1, 0, 2, 1, 0, 2, 1)$

$\vdots$

Gluing maps and complete intersections



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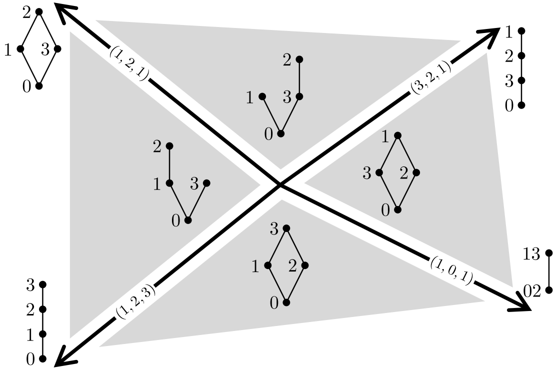

Complete intersections come from *gluings*:

$$S = \langle 4, 10, 21 \rangle$$
$$= 2\langle 2, 5 \rangle + \langle 21 \rangle$$

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References

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