Numerical semigroups and t-norms of factorizations

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A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**.

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Factorization length

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$ and an element $n \in S$.
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$$n = a_1 n_1 + \cdots + a_k n_k$$

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Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

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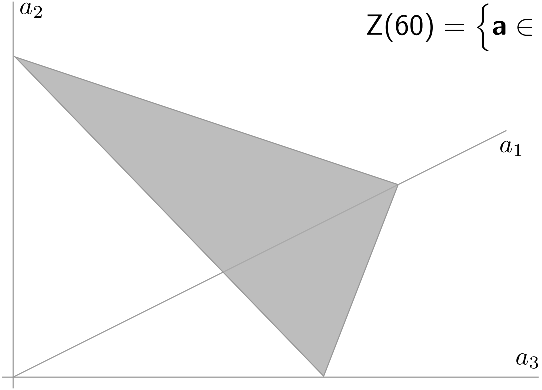

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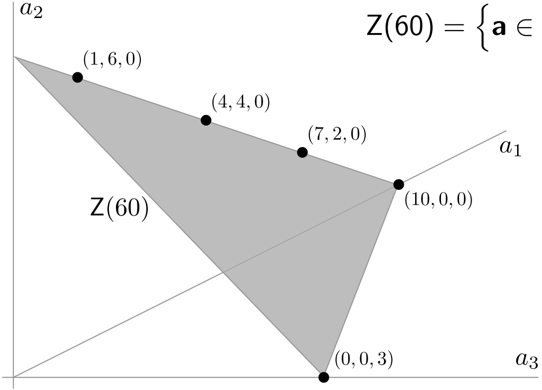

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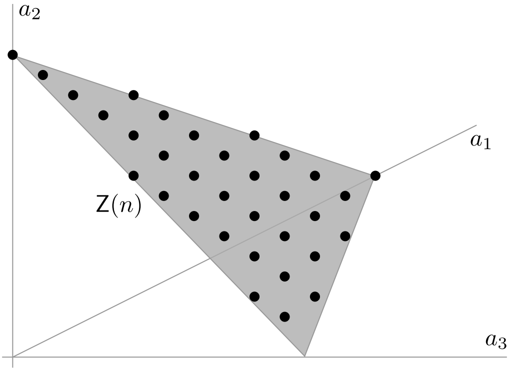

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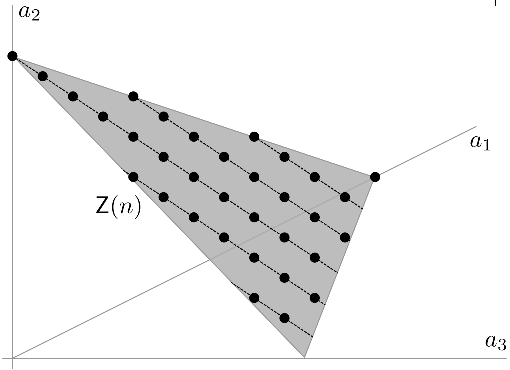
$$|\mathbf{a}| = \|\mathbf{a}\|_1 \quad \text{(the } \ell_1\text{-norm)}$$



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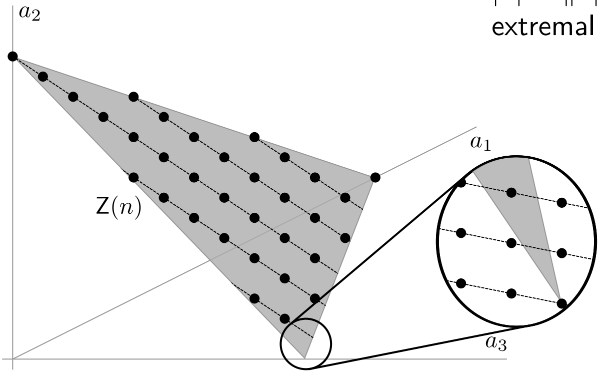
$|\mathbf{a}| = \|\mathbf{a}\|_1$ (the ℓ_1-norm)

extremal lengths near boundary



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Given $t \in [1, \infty)$, the *t-norm* of a factorization $\mathbf{a} \in \mathsf{Z}(n)$ is defined as

$$\|\mathbf{a}\|_t = (a_1^t + \cdots + a_k^t)^{1/t}$$

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REU Question

Which results for "classical" factorization length (i.e., for $t = 1$) extend/generalize to other t-norms?

Extremal factorization length

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$, let

$$\mathsf{L}(n) = \{a_1 + \cdots + a_k : n = a_1 n_1 + \cdots + a_k n_k\}$$

denotes the *length set* of n, and

$$\mathsf{M}(n) = \max \mathsf{L}(n) \qquad \text{and} \qquad \mathsf{m}(n) = \min \mathsf{L}(n)$$

denote the *maximum* and *minimum* factorization lengths of n.

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Example

$S = \langle 9, 10, 21 \rangle$:

$$\mathsf{M}(30) = 3 \quad \text{with} \quad 30 = 3(10)$$
$$\mathsf{M}(129) = 14 \quad \text{with} \quad 129 = 3(10) + 11(9)$$

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Theorem

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n \gg 0$ (i.e., for n sufficiently large),

$$\mathsf{M}(n + n_1) = 1 + \mathsf{M}(n)$$
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Equivalently, $\mathsf{M}(n)$, $\mathsf{m}(n)$ *are eventually quasilinear:*

$$\mathsf{M}(n) = \tfrac{1}{n_1} n + a_0(n)$$
$$\mathsf{m}(n) = \tfrac{1}{n_k} n + b_0(n)$$

for periodic functions $a_0(n)$, $b_0(n)$.

$$\mathsf{M}(n) = \begin{cases} \frac{1}{n_1} n + \underline{\qquad} & \text{if } n \equiv 0 \bmod n_1 \\ \frac{1}{n_1} n + \underline{\qquad} & \text{if } n \equiv 1 \bmod n_1 \\ \ldots & \end{cases}$$

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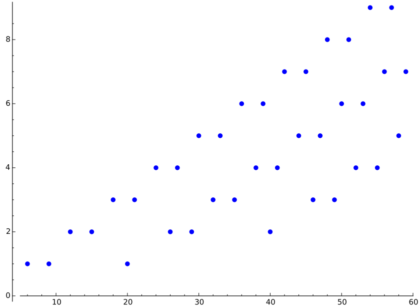
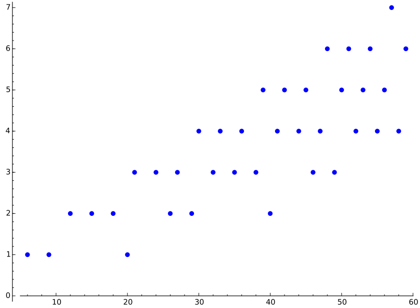


$\mathsf{M}(n) : S \to \mathbb{N}$



$\mathsf{m}(n) : S \to \mathbb{N}$

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Let $S = \langle n_1, \ldots, n_k \rangle$. For $n > n_k(n_{k-1} - 1)$,

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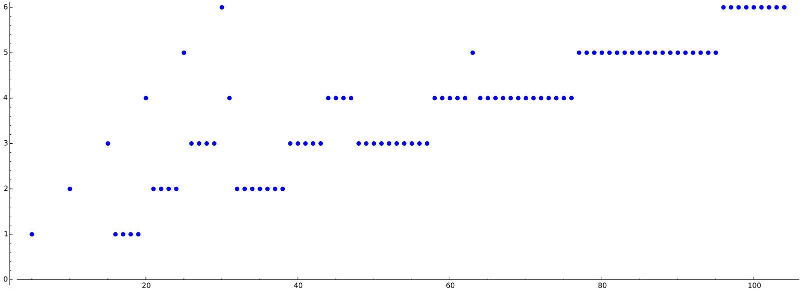

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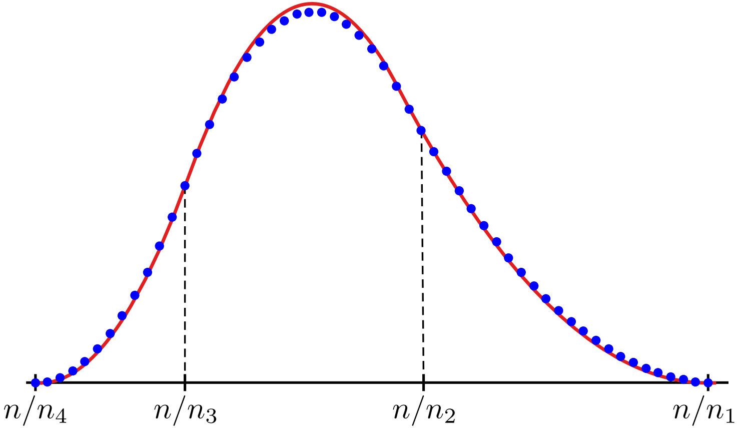
Example: $S = \langle 5, 6, 7, 8 \rangle$, $n = 500$



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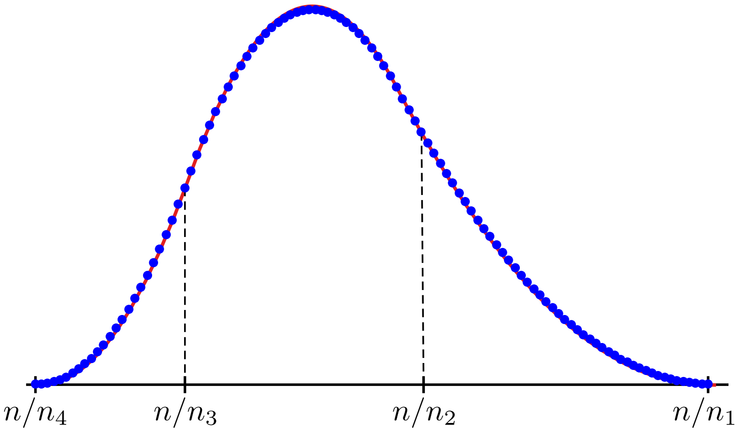

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$$\mathsf{L}(n) = \{a_1 + \cdots + a_k : n = a_1 n_1 + \cdots + a_k n_k\}$$

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  Example: in $S = \langle 5, 7, 9 \rangle$,
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Theorem (Structure theorem for sets of length)

There exist $d, t \in \mathbb{Z}_{\geq 1}$ where for each $n \gg 0$, there exist $A, A' \subseteq [1, t]$ so

$$\mathsf{L}(n) = \{\mathsf{m}(n), \mathsf{m}(n) + d, \ldots, \mathsf{M}(n)\} \setminus ((\mathsf{m}(n) + dA') \cup (\mathsf{M}(n) - dA)).$$

For numerical semigroups, $d = \gcd(n_2 - n_1, n_3 - n_2, \ldots, n_k - n_{k-1})$.

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Example: $S = \langle 5, 11, 12 \rangle$ and $n = 80, 81, 82, \ldots$

80	**81**	**82**	**83**	**84**	**85**	**86**	**87**	**88**	**89**	**90**	**91**	**92**
16	15	15	14	14	17	16	16	15	15	18	17	17
		14	13	13			15	14	14			16
				12					13			
	12		11			13		12			14	
12	11	11	10	10	13	12	12	11	11	14	13	13

The structure theorem for sets of length

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16	15	15	14	14	17	16	16	15	15	18	17	17
		14	13	13			15	14	14			16
				12					13			
	12		11			13		12			14	
12	11	11	10	10	13	12	12	11	11	14	13	13

Theorem

In the structure theorem, when we write

$$\mathsf{L}(n) = \{\mathsf{m}(n), \mathsf{m}(n) + d, \ldots, \mathsf{M}(n)\} \setminus ((\mathsf{m}(n) + dA') \cup (\mathsf{M}(n) - dA))$$

the sets A, A' depend only on the equivalence class of n modulo n_1, n_k.

Result 1: extremal t-norms

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$\mathsf{L}_t(n) = \{\|\mathbf{a}\|_t : n = a_1 n_1 + \cdots + a_k n_k\}$$

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Theorem

If $t \in [1, \infty]$, and $1/t + 1/q = 1$, then for $n \gg 0$, we have

$$\lim_{n \to \infty} \frac{\mathsf{M}_t(n)}{n} = \frac{1}{n_1},$$

$$\lim_{n \to \infty} \frac{\mathsf{m}_t(n)}{n} = \frac{1}{\|(n_1, \ldots, n_k)\|_q}$$

(in particular, $q = 1$ if $t = \infty$ and visa-versa).

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Theorem

For $n \gg 0$, we have

$$\mathsf{M}_\infty(n + n_1) = \mathsf{M}_\infty(n) + 1,$$
$$\mathsf{m}_\infty(n + n_1 + \cdots + n_k) = \mathsf{m}_\infty(n) + 1$$

Equivalently, $\mathsf{M}_\infty(n)$, $\mathsf{m}_\infty(n)$ *are eventually quasilinear:*

$$\mathsf{M}_\infty(n) = \begin{cases} \frac{1}{n_1} n + \underline{\qquad} & \text{if } n \equiv 0 \bmod n_1 \\ \frac{1}{n_1} n + \underline{\qquad} & \text{if } n \equiv 1 \bmod n_1 \\ \cdots \end{cases}$$

$$\mathsf{m}_\infty(n) = \begin{cases} \frac{1}{n_1 + \cdots + n_k} n + \underline{\qquad} & \text{if } n \equiv 0 \bmod (n_1 + \cdots + n_k) \\ \frac{1}{n_1 + \cdots + n_k} n + \underline{\qquad} & \text{if } n \equiv 1 \bmod (n_1 + \cdots + n_k) \\ \cdots \end{cases}$$

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Theorem

For $n \gg 0$, we have

$$\mathsf{m}_2(n + n_1^2 + \cdots + n_k^2)^2 = \mathsf{m}_2(n)^2 + 2n + n_1^2 + \cdots + n_k^2.$$

In particular,

$$\mathsf{m}_2(n)^2 = \begin{cases} \frac{1}{n_1^2 + \cdots + n_k^2} n^2 + \_\_\_\_ n + \_\_\_\_ & \text{if } n \equiv 0 \bmod (n_1^2 + \cdots + n_k^2) \\ \frac{1}{n_1^2 + \cdots + n_k^2} n^2 + \_\_\_\_ n + \_\_\_\_ & \text{if } n \equiv 1 \bmod (n_1^2 + \cdots + n_k^2) \\ \cdots \end{cases}$$

Result 2: a structure theorem for sets of ∞-length

Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subseteq \mathbb{Z}_{\geq 0}$.

$$\mathsf{L}_\infty(n) = \{\max(a_1, \ldots, a_k) : n = a_1 n_1 + \cdots + a_k n_k\}$$

Result 2: a structure theorem for sets of ∞-length

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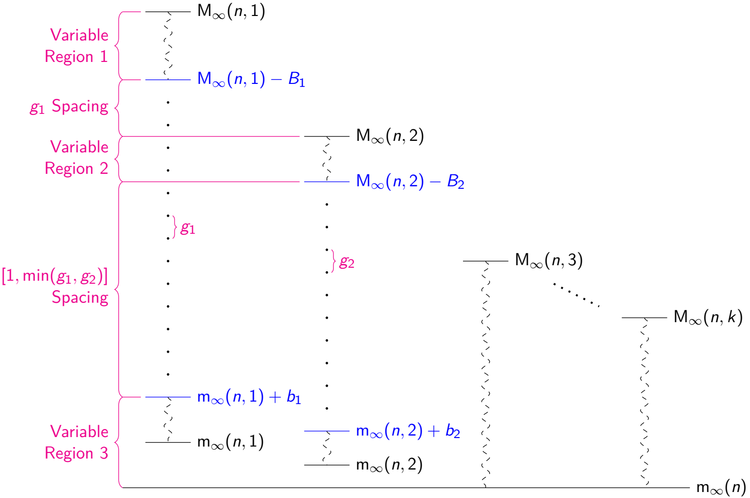

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Thanks!