Classifying numerical semigroups using polyhedral geometry

Christopher O'Neill

San Diego State University

cdoneill@sdsu.edu

Joint with (i) Winfred Bruns, Pedro García-Sánchez, Dane Wilbourne; (ii) Nathan Kaplan;
(iii) J. Autry, *A. Ezell, *T. Gomes, *C. Preuss, *T. Saluja, *E. Torres Davila
(iv) B. Braun, T. Gomes, E. Miller, C. O'Neill, and A. Sobieska

* = undergraduate student

Slides available: `https://cdoneill.sdsu.edu/`

March 12, 2024

Numerical semigroups

Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Numerical semigroups

Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example:

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Numerical semigroups

Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example: "McNugget Semigroup"

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Numerical semigroups

Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example: "McNugget Semigroup"

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Example: $S = \langle 6, 9, 18, 20, 32 \rangle$

Numerical semigroups

Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example: "McNugget Semigroup"

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Example: $S = \langle 6, 9, \cancel{18}, 20, \cancel{32} \rangle$

Numerical semigroups

Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example: "McNugget Semigroup"

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Example: $S = \langle 6, 9, \cancel{18}, 20, \cancel{32} \rangle = McN$

Numerical semigroups

Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example: "McNugget Semigroup"

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Example: $S = \langle 6, 9, \cancel{18}, 20, \cancel{32} \rangle = McN$

Fact

Every numerical semigroup has a unique minimal generating set.

Numerical semigroups

Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example: "McNugget Semigroup"

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Example: $S = \langle 6, 9, \cancel{18}, 20, \cancel{32} \rangle = McN$

Fact

Every numerical semigroup has a unique minimal generating set.

Multiplicity: $\mathrm{m}(S) =$ smallest nonzero element

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

Observations:

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

Observations:

- The elements of $\mathrm{Ap}(S)$ are distinct modulo m

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

Observations:

- The elements of $\mathrm{Ap}(S)$ are distinct modulo m
- $|\mathrm{Ap}(S)| = m$

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

Many things can be easily recovered from the Apéry set.

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

Many things can be easily recovered from the Apéry set.

- Fast membership test:

$$n \in S \text{ if } n \geq a \text{ for } a \in \mathrm{Ap}(S) \text{ with } a \equiv n \bmod m$$

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

Many things can be easily recovered from the Apéry set.

- Fast membership test:

$$n \in S \text{ if } n \geq a \text{ for } a \in \mathrm{Ap}(S) \text{ with } a \equiv n \bmod m$$

- Frobenius number: $\mathrm{F}(S) = \max(\mathrm{Ap}(S)) - m$

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

Many things can be easily recovered from the Apéry set.

- Fast membership test:

$$n \in S \text{ if } n \geq a \text{ for } a \in \mathrm{Ap}(S) \text{ with } a \equiv n \bmod m$$

- Frobenius number: $\mathrm{F}(S) = \max(\mathrm{Ap}(S)) - m$
- Number of gaps (the *genus*):

$$\mathrm{g}(S) = |\mathbb{N} \setminus S| = \sum_{a \in \mathrm{Ap}(S)} \left\lfloor \frac{a}{m} \right\rfloor$$

Apéry sets

Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

Definition

The *Apéry set* of S is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

Many things can be easily recovered from the Apéry set.

- Fast membership test:

$$n \in S \text{ if } n \geq a \text{ for } a \in \mathrm{Ap}(S) \text{ with } a \equiv n \bmod m$$

- Frobenius number: $\mathrm{F}(S) = \max(\mathrm{Ap}(S)) - m$
- Number of gaps (the *genus*):

$$\mathrm{g}(S) = |\mathbb{N} \setminus S| = \sum_{a \in \mathrm{Ap}(S)} \left\lfloor \frac{a}{m} \right\rfloor$$

The Apéry set is a "one stop shop" for computation.

Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ the Apéry set of some numerical semigroup?

Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 5, \quad a_1 = 11,\ a_2 = 7,\ a_3 = 23,\ a_4 = 19$$

Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 5, \quad a_1 = 11,\ a_2 = 7,\ a_3 = 23,\ a_4 = 19$$

but $a_1 + a_2 \equiv 3 \bmod 5$ and $a_1 + a_2 < a_3$.

Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 5, \quad a_1 = 11,\ a_2 = 7,\ a_3 = 23,\ a_4 = 19$$

but $a_1 + a_2 \equiv 3 \bmod 5$ and $a_1 + a_2 < a_3$.

Is $\{0, 13, 14, 27, 10, 11\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 6, \quad a_1 = 13,\ a_2 = 14,\ a_3 = 27,\ a_4 = 10,\ a_5 = 11$$

Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 5, \quad a_1 = 11,\ a_2 = 7,\ a_3 = 23,\ a_4 = 19$$

but $a_1 + a_2 \equiv 3 \bmod 5$ and $a_1 + a_2 < a_3$.

Is $\{0, 13, 14, 27, 10, 11\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 6, \quad a_1 = 13,\ a_2 = 14,\ a_3 = 27,\ a_4 = 10,\ a_5 = 11$$

but $a_4 + a_5 \equiv 3 \bmod 6$ and $a_4 + a_5 < a_3$.

Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 5, \quad a_1 = 11,\ a_2 = 7,\ a_3 = 23,\ a_4 = 19$$

but $a_1 + a_2 \equiv 3 \bmod 5$ and $a_1 + a_2 < a_3$.

Is $\{0, 13, 14, 27, 10, 11\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 6, \quad a_1 = 13,\ a_2 = 14,\ a_3 = 27,\ a_4 = 10,\ a_5 = 11$$

but $a_4 + a_5 \equiv 3 \bmod 6$ and $a_4 + a_5 < a_3$.

Theorem

If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup S with $\mathrm{Ap}(S) = A$ if and only if

$$a_i + a_j \geq a_{i+j} \qquad \textit{whenever} \qquad i + j \neq 0.$$

Is $A = \{0, 11, 7, 23, 19\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 5, \quad a_1 = 11, \; a_2 = 7, \; a_3 = 23, \; a_4 = 19$$

but $a_1 + a_2 \equiv 3 \bmod 5$ and $a_1 + a_2 < a_3$.

Is $\{0, 13, 14, 27, 10, 11\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 6, \quad a_1 = 13, \; a_2 = 14, \; a_3 = 27, \; a_4 = 10, \; a_5 = 11$$

but $a_4 + a_5 \equiv 3 \bmod 6$ and $a_4 + a_5 < a_3$.

Theorem

If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup S with $\mathrm{Ap}(S) = A$ if and only if

$$a_i + a_j \geq a_{i+j} \qquad \textit{whenever} \qquad i + j \neq 0.$$

Big idea: the inequalities "$a_i + a_j \geq a_{i+j}$" to define a **cone** C_m.

Kunz cone

Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

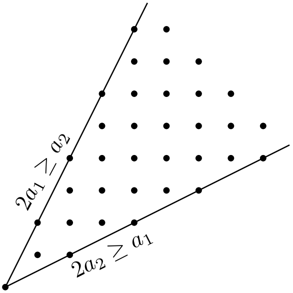
Kunz cone

Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities
$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

Example: C_3



Kunz cone

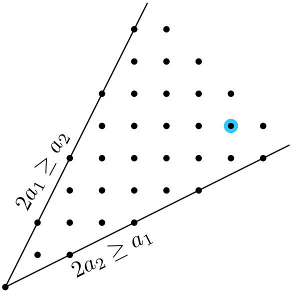
Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

Example: C_3



$S = \langle 3, 5, 7 \rangle$
$\mathrm{Ap}(S) = \{0, 7, 5\}$

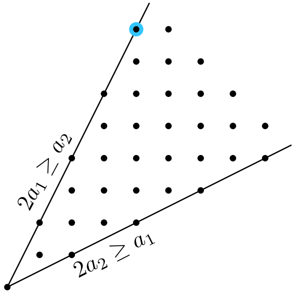
Kunz cone

Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities
$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

Example: C_3



$S = \langle 3, 5, 7 \rangle$
$\mathrm{Ap}(S) = \{0, 7, 5\}$

$S = \langle 3, 4 \rangle$
$\mathrm{Ap}(S) = \{0, 4, 8\}$

Kunz cone

Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities
$$a_i + a_j \ge a_{i+j} \qquad \text{whenever} \qquad i + j \ne 0.$$

$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\ge 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

Kunz cone

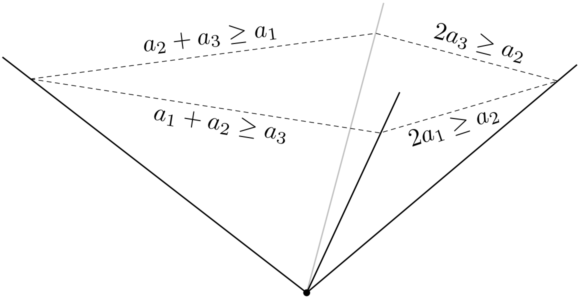
Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

Example: C_4



Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Big picture: "moduli space" approach for studying XYZ's

- Define a space with XYZ's as points
  Small changes to an XYZ $\rightsquigarrow$ small movements in space
- Let geometric/topological structure inform study of XYZ's

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Big picture: "moduli space" approach for studying XYZ's

- Define a space with XYZ's as points
  Small changes to an XYZ $\rightsquigarrow$ small movements in space
- Let geometric/topological structure inform study of XYZ's

Basic example: $\mathrm{GL}_n(\mathbb{R}) \hookrightarrow \mathbb{R}^{n^2}$

Faces of the Kunz cone

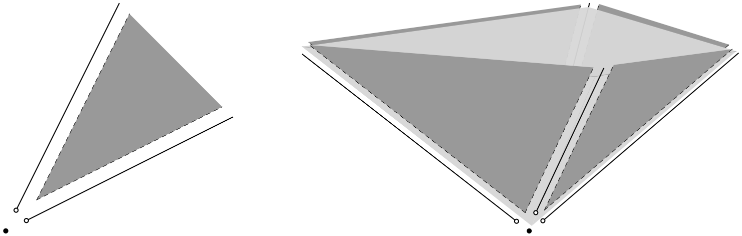
Question

When are numerical semigroups in (the relative interior of) the same face?

Big picture: "moduli space" approach for studying XYZ's

- Define a space with XYZ's as points
  Small changes to an XYZ $\rightsquigarrow$ small movements in space
- Let geometric/topological structure inform study of XYZ's

Basic example: $\mathrm{GL}_n(\mathbb{R}) \hookrightarrow \mathbb{R}^{n^2}$
More interesting example: C_m



Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Motivation: $S \in \operatorname{Int}(C_m)$ if and only if S has *max embedding dimension*

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Motivation: $S \in \mathrm{Int}(C_m)$ if and only if S has *max embedding dimension*

If $S = \langle n_1, \ldots, n_k \rangle$, then

$$n_i \not\equiv n_j \bmod n_1 \quad \Longrightarrow \quad k \le \mathrm{m}(S)$$

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Motivation: $S \in \mathrm{Int}(C_m)$ if and only if S has *max embedding dimension*

If $S = \langle n_1, \ldots, n_k \rangle$, then

$$n_i \not\equiv n_j \bmod n_1 \quad \Longrightarrow \quad k \le \mathrm{m}(S)$$

If $k = \mathrm{m}(S)$, then S has *max embedding dimension*

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Motivation: $S \in \mathrm{Int}(C_m)$ if and only if S has *max embedding dimension*

If $S = \langle n_1, \ldots, n_k \rangle$, then

$$n_i \not\equiv n_j \bmod n_1 \quad \Longrightarrow \quad k \le \mathrm{m}(S)$$

If $k = \mathrm{m}(S)$, then S has *max embedding dimension*

$$S = \langle m, a_1, \ldots, a_{m-1} \rangle \quad \text{where} \quad \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\}$$

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Motivation: $S \in \mathrm{Int}(C_m)$ if and only if S has *max embedding dimension*

If $S = \langle n_1, \ldots, n_k \rangle$, then

$$n_i \not\equiv n_j \bmod n_1 \quad \Longrightarrow \quad k \le \mathrm{m}(S)$$

If $k = \mathrm{m}(S)$, then S has *max embedding dimension*

$$S = \langle m, a_1, \ldots, a_{m-1} \rangle \quad \text{where} \quad \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\}$$

Geometrically: "most" numerical semigroups with $\mathrm{m}(S) = m$ are MED

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Motivation: $S \in \mathrm{Int}(C_m)$ if and only if S has *max embedding dimension*

If $S = \langle n_1, \ldots, n_k \rangle$, then

$$n_i \not\equiv n_j \bmod n_1 \quad \Longrightarrow \quad k \le \mathrm{m}(S)$$

If $k = \mathrm{m}(S)$, then S has *max embedding dimension*

$$S = \langle m, a_1, \ldots, a_{m-1} \rangle \quad \text{where} \quad \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\}$$

Geometrically: "most" numerical semigroups with $\mathrm{m}(S) = m$ are MED

What about the other faces?

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Example: $S = \langle 4, 10, 11, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 11\}$$
$$a_1 = 13, \quad a_2 = 10, \quad a_3 = 11$$

$$2a_1 > a_2 \qquad a_1 + a_2 > a_3$$
$$2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Example: $S = \langle 4, 10, 11, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 11\} \qquad 2a_1 > a_2 \qquad a_1 + a_2 > a_3$$
$$a_1 = 13, \quad a_2 = 10, \quad a_3 = 11 \qquad 2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

Example: $S = \langle 4, 10, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 23\} \qquad 2a_1 > a_2 \qquad a_1 + a_2 = a_3$$
$$a_1 = 13, \quad a_2 = 10, \quad a_3 = 23 \qquad 2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Example: $S = \langle 4, 10, 11, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 11\} \qquad a_1 = 13, \quad a_2 = 10, \quad a_3 = 11$$

$$2a_1 > a_2 \qquad a_1 + a_2 > a_3$$
$$2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

Example: $S = \langle 4, 10, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 23\} \qquad a_1 = 13, \quad a_2 = 10, \quad a_3 = 23$$

$$2a_1 > a_2 \qquad a_1 + a_2 = a_3$$
$$2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

Example: $S = \langle 4, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 26, 39\} \qquad a_1 = 13, \quad a_2 = 26, \quad a_3 = 39$$

$$2a_1 = a_2 \qquad a_1 + a_2 = a_3$$
$$2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Faces of the Kunz cone

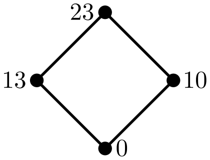
Question

When are numerical semigroups in (the relative interior of) the same face?

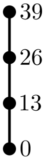
Definition

The *Apéry poset* of S: define $a \preceq a'$ whenever $a' - a \in S$.

$\mathrm{Ap}(S) = \{0, 13, 10, 23\}$



$\mathrm{Ap}(S) = \{0, 13, 26, 39\}$



Faces of the Kunz polyhedron

Question

When are numerical semigroups in (the relative interior of) the same face?

Faces of the Kunz polyhedron

Question

When are numerical semigroups in (the relative interior of) the same face?

$$S = \langle 6, 9, 20 \rangle$$
$$\mathrm{Ap}(S) = \{0, 49, 20, \ \ 9, 40, 29\}$$

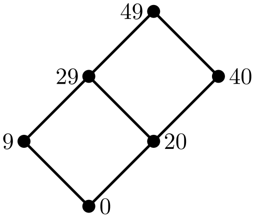
$$S' = \langle 6, 26, 27 \rangle$$
$$\mathrm{Ap}(S') = \{0, 79, 26, 27, 52, 53\}$$

Question

When are numerical semigroups in (the relative interior of) the same face?

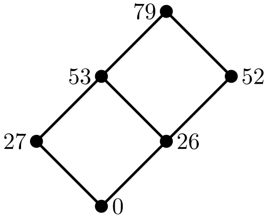
$S = \langle 6, 9, 20 \rangle$

$\mathrm{Ap}(S) = \{0, 49, 20, \ \ 9, 40, 29\}$



$S' = \langle 6, 26, 27 \rangle$

$\mathrm{Ap}(S') = \{0, 79, 26, 27, 52, 53\}$



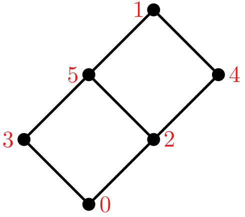
Faces of the Kunz polyhedron

Question

When are numerical semigroups in (the relative interior of) the same face?

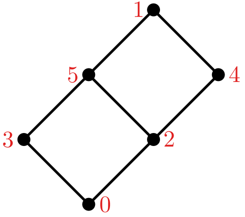
$$S = \langle 6, 9, 20 \rangle$$
$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

1 2 3 4 5



$$S' = \langle 6, 26, 27 \rangle$$
$$\mathrm{Ap}(S') = \{0, 79, 26, 27, 52, 53\}$$

1 2 3 4 5



Faces of the Kunz polyhedron

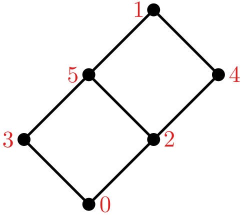
Question

When are numerical semigroups in (the relative interior of) the same face?

$S = \langle 6, 9, 20 \rangle$

$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$

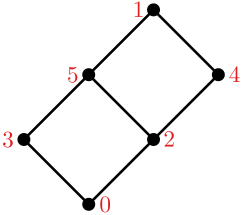
1 2 3 4 5



$S' = \langle 6, 26, 27 \rangle$

$\mathrm{Ap}(S') = \{0, 79, 26, 27, 52, 53\}$

1 2 3 4 5



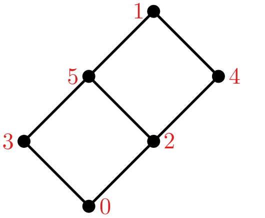
The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

Faces of the Kunz polyhedron

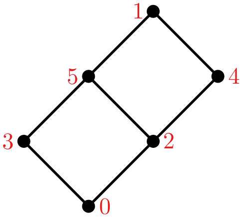
Question

When are numerical semigroups in (the relative interior of) the same face?

$$S = \langle 6, 9, 20 \rangle$$
$$\mathrm{Ap}(S) = \{0, \underset{1}{49}, \underset{2}{20}, \underset{3}{9}, \underset{4}{40}, \underset{5}{29}\}$$



$$S' = \langle 6, 26, 27 \rangle$$
$$\mathrm{Ap}(S') = \{0, \underset{1}{79}, \underset{2}{26}, \underset{3}{27}, \underset{4}{52}, \underset{5}{53}\}$$



The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

Theorem (Bruns–García-Sánchez–O.–Wilburne)

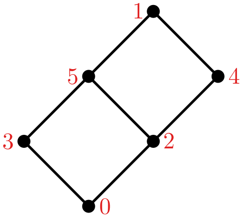
Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

Faces of the Kunz polyhedron

Question

When are numerical semigroups in (the relative interior of) the same face?

$$S = \langle 6, 9, 20 \rangle$$
$$\mathrm{Ap}(S) = \{0, \underset{1}{49}, \underset{2}{20}, \underset{3}{9}, \underset{4}{40}, \underset{5}{29}\}$$



The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

Theorem (Bruns–García-Sánchez–O.–Wilburne)

Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

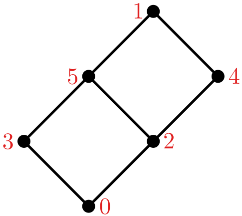
Faces of the Kunz polyhedron

Question

When are numerical semigroups in (the relative interior of) the same face?

$S = \langle 6, 9, 20 \rangle$

$\mathrm{Ap}(S) = \{0, \underset{1}{49}, \underset{2}{20}, \underset{3}{9}, \underset{4}{40}, \underset{5}{29}\}$



Defining facet equations:

$$2a_2 = a_4$$
$$a_2 + a_3 = a_5$$
$$a_2 + a_5 = a_1$$
$$a_3 + a_4 = a_1$$

The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

Theorem (Bruns–García-Sánchez–O.–Wilburne)

Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

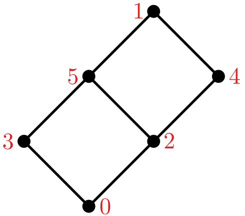
Faces of the Kunz polyhedron

Question

When are numerical semigroups in (the relative interior of) the same face?

$$S = \langle 6, 9, 20 \rangle$$

$$\mathrm{Ap}(S) = \{0, \underset{1}{49}, \underset{2}{20}, \underset{3}{9}, \underset{4}{40}, \underset{5}{29}\}$$



Defining facet equations:

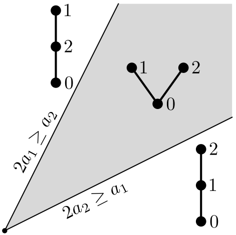
$2a_2 = a_4$	$2 \preceq 4$
$a_2 + a_3 = a_5$	$2 \preceq 5$
	$3 \preceq 5$
$a_2 + a_5 = a_1$	$2 \preceq 1$
	$5 \preceq 1$
$a_3 + a_4 = a_1$	$3 \preceq 1$
	$4 \preceq 1$

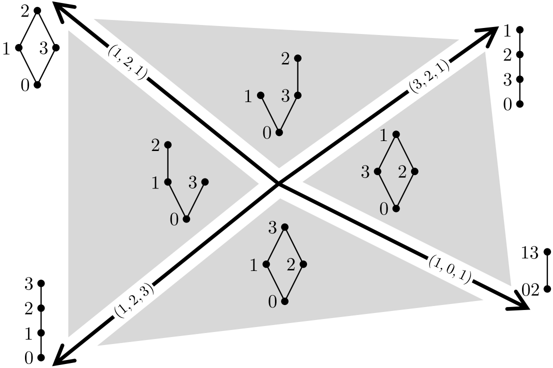
The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

Theorem (Bruns–García-Sánchez–O.–Wilburne)

Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

C_3 and C_4





A couple of long-standing (**hard**) conjectures

Genus $g = \mathrm{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$: number of “gaps” of S.

A couple of long-standing (**hard**) conjectures

Genus $g = \mathrm{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$: number of “gaps” of S.
$n_g = \#$ of numerical semigroups with genus g.

A couple of long-standing (**hard**) conjectures

Genus $g = \mathrm{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$: number of "gaps" of S.
$n_g = \#$ of numerical semigroups with genus g.
Example: $n_3 = 4$

$$\begin{aligned}
\langle 2, 7 \rangle &= \{0, \quad 2, \quad 4, \quad 6, 7, 8, \ldots\} \\
\langle 3, 4 \rangle &= \{0, \quad\quad 3, 4, \quad 6, 7, 8, \ldots\} \\
\langle 3, 5, 7 \rangle &= \{0, \quad\quad 3, \quad 5, 6, 7, 8, \ldots\} \\
\langle 4, 5, 6, 7 \rangle &= \{0, \quad\quad\quad 4, 5, 6, 7, 8, \ldots\}
\end{aligned}$$

A couple of long-standing (**hard**) conjectures

Genus $g = \mathrm{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$: number of "gaps" of S.
$n_g = \#$ of numerical semigroups with genus g.
Example: $n_3 = 4$

$$\begin{aligned} \langle 2,7 \rangle &= \{0, \quad 2, \quad 4, \quad 6,7,8,\ldots\} \\ \langle 3,4 \rangle &= \{0, \quad\quad 3,4, \quad 6,7,8,\ldots\} \\ \langle 3,5,7 \rangle &= \{0, \quad\quad 3, \quad 5,6,7,8,\ldots\} \\ \langle 4,5,6,7 \rangle &= \{0, \quad\quad\quad 4,5,6,7,8,\ldots\} \end{aligned}$$

Suspected: $n_g \geq n_{g-1} + n_{g-2}$ for all g (verified for $g \leq 70$)

A couple of long-standing (**hard**) conjectures

Genus $g = \mathrm{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$: number of "gaps" of S.
$n_g = \#$ of numerical semigroups with genus g.
Example: $n_3 = 4$

$$\begin{aligned}
\langle 2,7 \rangle &= \{0, \quad 2, \quad 4, \quad 6,7,8,\ldots\} \\
\langle 3,4 \rangle &= \{0, \qquad 3,4, \quad 6,7,8,\ldots\} \\
\langle 3,5,7 \rangle &= \{0, \qquad 3, \quad 5,6,7,8,\ldots\} \\
\langle 4,5,6,7 \rangle &= \{0, \qquad\quad 4,5,6,7,8,\ldots\}
\end{aligned}$$

Suspected: $n_g \geq n_{g-1} + n_{g-2}$ for all g (verified for $g \leq 70$)
Known: $\lim_{g\to\infty} \frac{n_{g+1}}{n_g}$ = the golden ratio

A couple of long-standing (**hard**) conjectures

Genus $g = \mathrm{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$: number of "gaps" of S.
$n_g = \#$ of numerical semigroups with genus g.
Example: $n_3 = 4$

$$\begin{aligned} \langle 2,7 \rangle &= \{0, \quad 2, \quad 4, \quad 6,7,8,\ldots\} \\ \langle 3,4 \rangle &= \{0, \quad\; 3,4, \quad 6,7,8,\ldots\} \\ \langle 3,5,7 \rangle &= \{0, \quad\; 3, \quad 5,6,7,8,\ldots\} \\ \langle 4,5,6,7 \rangle &= \{0, \qquad 4,5,6,7,8,\ldots\} \end{aligned}$$

Suspected: $n_g \geq n_{g-1} + n_{g-2}$ for all g (verified for $g \leq 70$)

Known: $\lim_{g\to\infty} \frac{n_{g+1}}{n_g} =$ the golden ratio

Conjecture (Bras-Amoros, 2008)

For all g, we have $n_g \geq n_{g-1}$.

A couple of long-standing (**hard**) conjectures

Genus $g = \mathrm{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$: number of "gaps" of S.
$n_g = \#$ of numerical semigroups with genus g.
Example: $n_3 = 4$

$$\begin{aligned} \langle 2,7\rangle &= \{0, \quad 2, \quad 4, \quad 6,7,8,\ldots\} \\ \langle 3,4\rangle &= \{0, \quad 3,4, \quad 6,7,8,\ldots\} \\ \langle 3,5,7\rangle &= \{0, \quad 3, \quad 5,6,7,8,\ldots\} \\ \langle 4,5,6,7\rangle &= \{0, \quad 4,5,6,7,8,\ldots\} \end{aligned}$$

Suspected: $n_g \geq n_{g-1} + n_{g-2}$ for all g (verified for $g \leq 70$)
Known: $\lim_{g\to\infty} \frac{n_{g+1}}{n_g} =$ the golden ratio

Conjecture (Bras-Amoros, 2008)

For all g, we have $n_g \geq n_{g-1}$.

Not true for $n'_f = \#$ of numerical semigroups with Frobenius number f

$$n'_{11} = 51 \qquad n'_{12} = 40 \qquad n'_{13} = 106$$

A couple of long-standing (**hard**) conjectures

Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathsf{F}(S) + 1 \leq k(\mathsf{F}(S) + 1 - \mathsf{g}(S))$.

A couple of long-standing (**hard**) conjectures

Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathsf{F}(S) + 1 \le k(\mathsf{F}(S) + 1 - \mathsf{g}(S))$.

Equivalently,

$$\frac{1}{k} \le \underbrace{\frac{\mathsf{F}(S) + 1 - \mathsf{g}(S)}{\mathsf{F}(S) + 1}}_{\text{\% of } [0, F(S)] \text{ in } S}$$

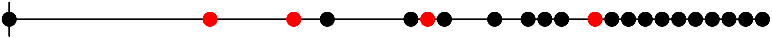
A couple of long-standing (**hard**) conjectures

Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathsf{F}(S) + 1 \le k(\mathsf{F}(S) + 1 - \mathsf{g}(S))$.

Equivalently,

$$\frac{1}{k} \le \underbrace{\frac{\mathsf{F}(S) + 1 - \mathsf{g}(S)}{\mathsf{F}(S) + 1}}_{\%\text{ of } [0, F(S)] \text{ in } S}$$



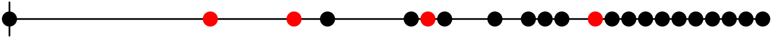
A couple of long-standing (**hard**) conjectures

Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \leq k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

Equivalently,

$$\frac{1}{k} \leq \underbrace{\frac{\mathrm{F}(S) + 1 - \mathrm{g}(S)}{\mathrm{F}(S) + 1}}_{\text{\% of } [0, F(S)] \text{ in } S}$$



Equality holds when:

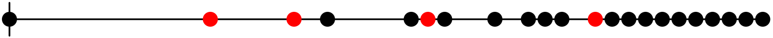
A couple of long-standing (**hard**) conjectures

Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathsf{F}(S) + 1 \le k(\mathsf{F}(S) + 1 - \mathsf{g}(S))$.

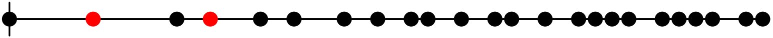
Equivalently,

$$\frac{1}{k} \le \underbrace{\frac{\mathsf{F}(S) + 1 - \mathsf{g}(S)}{\mathsf{F}(S) + 1}}_{\%\text{ of } [0, F(S)] \text{ in } S}$$



Equality holds when:

- $S = \langle a, b \rangle$



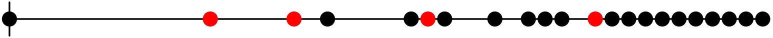
A couple of long-standing (**hard**) conjectures

Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \le k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

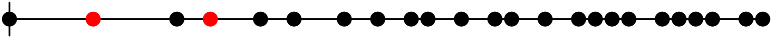
Equivalently,

$$\frac{1}{k} \le \underbrace{\frac{\mathrm{F}(S) + 1 - \mathrm{g}(S)}{\mathrm{F}(S) + 1}}_{\%\text{ of } [0, F(S)] \text{ in } S}$$



Equality holds when:

- $S = \langle a, b \rangle$



- $S = \langle m, m+1, \ldots, 2m-1 \rangle$

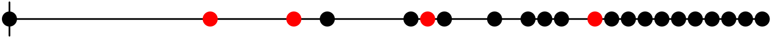
A couple of long-standing (**hard**) conjectures

Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathsf{F}(S) + 1 \le k(\mathsf{F}(S) + 1 - \mathsf{g}(S))$.

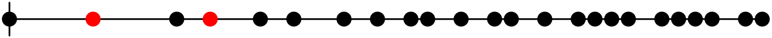
Equivalently,

$$\frac{1}{k} \le \underbrace{\frac{\mathsf{F}(S) + 1 - \mathsf{g}(S)}{\mathsf{F}(S) + 1}}_{\%\text{ of } [0, F(S)] \text{ in } S}$$

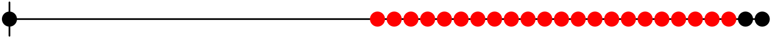


Equality holds when:

- $S = \langle a, b \rangle$



- $S = \langle m, m+1, \ldots, 2m-1 \rangle$



Proved in many special cases, including $\mathsf{g}(S) \le 60$.

A couple of long-standing (**hard**) conjectures

A couple of long-standing (**hard**) conjectures

Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \le k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

Bras-Amoros Conjecture

For all g, we have $n_g \ge n_{g-1}$.

A couple of long-standing (**hard**) conjectures

Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \le k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

Bras-Amoros Conjecture

For all g, we have $n_g \ge n_{g-1}$.

Direct ties to geometry: if S corresponds to $x = (a_1, \ldots, a_{m-1}) \in C_m$,

$$g(S) = \|x\|_1 - \tfrac{1}{2}m(m-1), \qquad F(S) = \|x\|_\infty - m,$$

and $\#$ generators k is determined by the face $F \subseteq \mathcal{C}_m$ containing x.

A couple of long-standing (**hard**) conjectures

Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \leq k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

Bras-Amoros Conjecture

For all g, we have $n_g \geq n_{g-1}$.

Direct ties to geometry: if S corresponds to $x = (a_1, \ldots, a_{m-1}) \in C_m$,

$$g(S) = \|x\|_1 - \tfrac{1}{2}m(m-1), \qquad F(S) = \|x\|_\infty - m,$$

and $\#$ generators k is determined by the face $F \subseteq \mathcal{C}_m$ containing x.

Theorem (Bruns-García-Sánchez-O.-Wilburne, 2020)

Wilf's conjecture holds for all numerical semigroups S with $m \leq 18$.

A couple of long-standing (**hard**) conjectures

Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathsf{F}(S) + 1 \le k(\mathsf{F}(S) + 1 - \mathsf{g}(S))$.

Bras-Amoros Conjecture

For all g, we have $n_g \ge n_{g-1}$.

Direct ties to geometry: if S corresponds to $x = (a_1, \ldots, a_{m-1}) \in C_m$,

$$g(S) = \|x\|_1 - \tfrac{1}{2}m(m-1), \qquad F(S) = \|x\|_\infty - m,$$

and $\#$ generators k is determined by the face $F \subseteq \mathcal{C}_m$ containing x.

Theorem (Bruns-García-Sánchez-O.-Wilburne, 2020)

Wilf's conjecture holds for all numerical semigroups S with $m \le 18$.

Conjecture (Kaplan)

For fixed m, the number of numerical semigroups g gaps is non-decreasing.

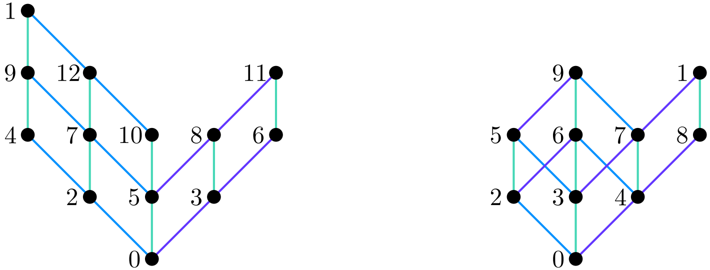
Shared properties within a face

What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

Shared properties within a face

What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

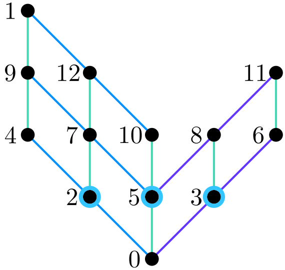
- $k = 1 + \#$ atoms of P

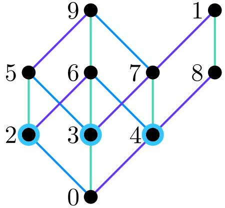


Shared properties within a face

What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of P

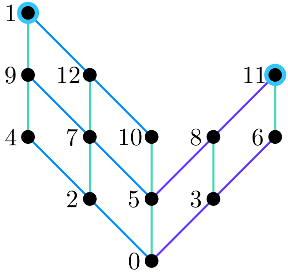


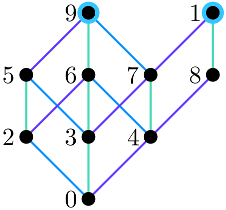


Shared properties within a face

What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of P
- $\mathsf{t}(S) = \#$ maximal elements
  (Cohen-Macaulay type of S)





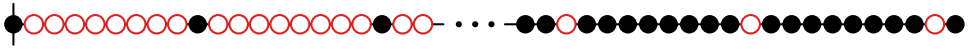
Shared properties within a face

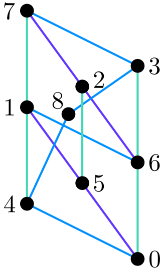
What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of P
- $\mathsf{t}(S) = \#$ maximal elements
  (Cohen-Macaulay type of S)
- Symmetric/Gorenstein?

$S = \langle 4, 7 \rangle$

$S = \langle 9, 40, 50, 60 \rangle$

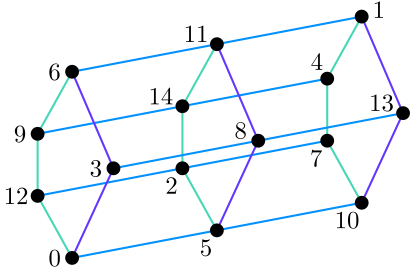


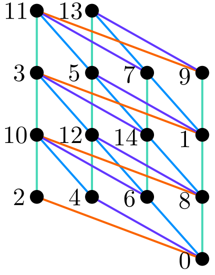


Shared properties within a face

What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of P
- $\mathrm{t}(S) = \#$ maximal elements (Cohen-Macaulay type of S)
- Symmetric/Gorenstein?
- Complete intersection?
- Generalized arithmetical?





Shared properties within a face

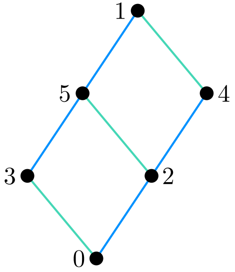
What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of P
- $\mathrm{t}(S) = \#$ maximal elements (Cohen-Macaulay type of S)
- Symmetric/Gorenstein?
- Complete intersection?
- Generalized arithmetical?
- Minimal binomial generators of the *defining toric ideal* of S:

$$I_S = \ker\left(\Bbbk[\overline{x}] \to \Bbbk[t]\right)$$
$$x_i \mapsto t^{n_i}$$

$S = \langle 6, 9, 20 \rangle$

$$I_S = \langle x^3 - y^2, x^4y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$$



Shared properties within a face

What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

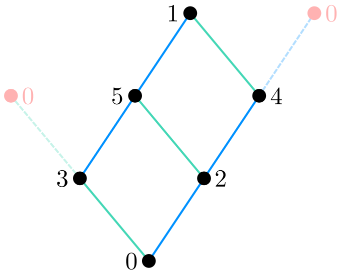
- $k = 1 + \#$ atoms of P
- $\mathrm{t}(S) = \#$ maximal elements (Cohen-Macaulay type of S)
- Symmetric/Gorenstein?
- Complete intersection?
- Generalized arithmetical?

- Minimal binomial generators of the *defining toric ideal* of S:

$$I_S = \ker\left(\Bbbk[\overline{x}] \to \Bbbk[t]\right) \qquad x_i \mapsto t^{n_i}$$

$$S = \langle 6, 9, 20 \rangle$$

$$I_S = \langle x^3 - y^2, x^4y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$$

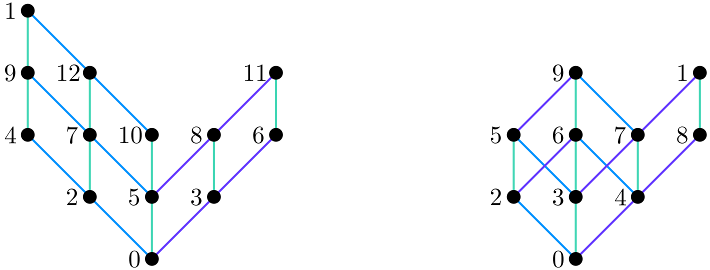


Shared properties within a face

What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of P
- $t(S) = \#$ maximal elements (Cohen-Macaulay type of S)
- Symmetric/Gorenstein?
- Complete intersection?
- Generalized arithmetical?
- Minimal binomial generators of the *defining toric ideal* of S:

$$I_S = \ker\left(\Bbbk[\overline{x}] \to \Bbbk[t]\right) \quad x_i \mapsto t^{n_i}$$

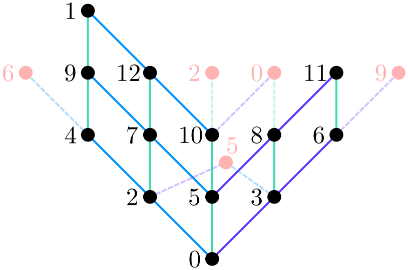


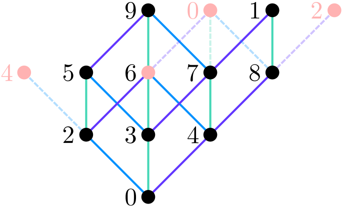
Shared properties within a face

What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of P
- $\mathrm{t}(S) = \#$ maximal elements (Cohen-Macaulay type of S)
- Symmetric/Gorenstein?
- Complete intersection?
- Generalized arithmetical?
- Minimal binomial generators of the *defining toric ideal* of S:

$$I_S = \ker\left(\Bbbk[\overline{x}] \to \Bbbk[t]\right)$$
$$x_i \mapsto t^{n_i}$$





Shared properties within a face

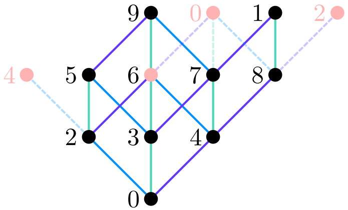
What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of P
- $\mathrm{t}(S) = \#$ maximal elements (Cohen-Macaulay type of S)
- Symmetric/Gorenstein?
- Complete intersection?
- Generalized arithmetical?
- Minimal binomial generators of the *defining toric ideal* of S:

$$I_S = \ker\left(\Bbbk[\overline{x}] \to \Bbbk[t]\right) \quad x_i \mapsto t^{n_i}$$

$$S = \langle 10, a_2, a_3, a_4 \rangle$$

$$I_S = \langle\, x_2^2 - y^* x_4,\ x_2x_4 - x_3^2,\ x_3^2 x_4 - y^*,\ x_4^3 - y^* x_2 \,\rangle \subseteq \Bbbk[y, x_2, x_3, x_4]$$



Shared properties within a face

What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of P
- $\mathrm{t}(S) = \#$ maximal elements (Cohen-Macaulay type of S)
- Symmetric/Gorenstein?
- Complete intersection?
- Generalized arithmetical?
- Minimal binomial generators of the *defining toric ideal* of S:

$$I_S = \ker\left(\Bbbk[\overline{x}] \to \Bbbk[t]\right)$$
$$x_i \mapsto t^{n_i}$$

- Betti numbers of I_S over $\Bbbk[\overline{x}]$

$$0 \leftarrow R \leftarrow R^6 \longleftarrow R^8 \longleftarrow R^3 \leftarrow 0$$

	1,12	1,13	2,12	2,23	3,13	3,23	2,13	3,12
1,1	$-x_2$	$-x_3$					y^*	y^*
2,2	$-y^*$		x_1	$-x_3$		y^*		
3,3					x_1	x_2	$-y^*$	
2,1	x_1		$-x_2$	y^*	y^*		$-x_3$	
3,1	y^*	x_1			$-x_3$	$-y^*$		$-x_2$
3,2		$-y^*$	$-y^*$	x_2		$-x_3$	x_1	x_1

	1,[3]	2,[3]	3,[3]
1,12	x_3	$-y^*$	
1,13	$-x_2$		y^*
2,12		x_3	$-y^*$
2,23	$-y^*$	x_1	
3,13	y^*		$-x_2$
3,23		$-y^*$	x_1
2,13	x_1	$-x_2$	
3,12	$-x_1$		x_3

Shared properties within a face

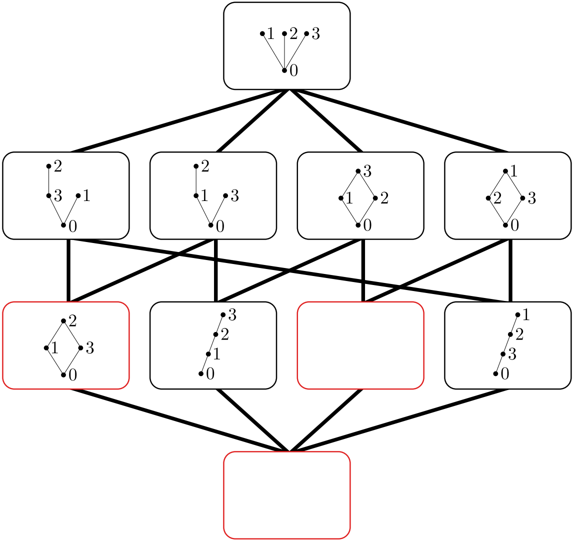
What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of P
- $\mathrm{t}(S) = \#$ maximal elements (Cohen-Macaulay type of S)
- Symmetric/Gorenstein?
- Complete intersection?
- Generalized arithmetical?
- Minimal binomial generators of the *defining toric ideal* of S:
$$I_S = \ker\left(\mathbb{k}[\overline{x}] \to \mathbb{k}[t]\right) \qquad x_i \mapsto t^{n_i}$$
- Betti numbers of I_S over $\mathbb{k}[\overline{x}]$
- Betti numbers of $\mathbb{k}$ over $\mathbb{k}[\overline{x}]/I_S$

$$0 \leftarrow R \xleftarrow{\begin{array}{c} \begin{array}{cccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \emptyset \; [\, y \;\; x_1 \;\; x_2 \;\; x_3 \,] \end{array}} R^4 \xleftarrow{\begin{array}{c|cccccccccccc} & \mathbf{01} & \mathbf{02} & \mathbf{03} & \mathbf{11} & \mathbf{12} & \mathbf{13} & \mathbf{21} & \mathbf{22} & \mathbf{23} & \mathbf{31} & \mathbf{32} & \mathbf{33} \\ \hline \mathbf{0} & x_1 & x_2 & x_3 & & & -y^* & & -y^* & & -y^* & & \\ \mathbf{1} & -y & & & x_1 & x_2 & x_3 & & & -y^* & & -y^* & \\ \mathbf{2} & & -y & & -y^* & & & x_1 & x_2 & x_3 & & & -y^* \\ \mathbf{3} & & & -y & & -y^* & & -y^* & & & x_1 & x_2 & x_3 \end{array}} R^{12}$$

$$\longleftarrow R^{36} \longleftarrow R^{108} \longleftarrow R^{324} \longleftarrow R^{972} \longleftarrow R^{2916} \longleftarrow \cdots$$

Posets for the poset-less faces



Face lattice of C_4

Posets for the poset-less faces

Theorem

If $d \mid m$, then there exists a map $C_d \hookrightarrow C_m$ that induces a dimension-preserving injection on face lattices.

Posets for the poset-less faces

Theorem

If $d \mid m$, then there exists a map $C_d \hookrightarrow C_m$ that induces a dimension-preserving injection on face lattices.

For $x \in C_m$, the coordinates of 0's form a subgroup of $\mathbb{Z}_m$

Example: $\mathbb{Z}_{10}$ $(*, *, *, *, 0, *, *, *, *)$ $(*, 0, *, 0, *, 0, *, 0, *)$
$(0, 0, 0, 0, 0, 0, 0, 0, 0)$ $(*, *, *, *, *, *, *, *, *)$

Posets for the poset-less faces

Theorem

If $d \mid m$, then there exists a map $C_d \hookrightarrow C_m$ that induces a dimension-preserving injection on face lattices.

For $x \in C_m$, the coordinates of 0's form a subgroup of $\mathbb{Z}_m$

Example: $\mathbb{Z}_{10}$ $\quad (*,*,*,*,0,*,*,*,*) \quad (*,0,*,0,*,0,*,0,*)$

$(0,0,0,0,0,0,0,0,0) \quad (*,*,*,*,*,*,*,*,*)$

Maps are induced by $\mathbb{Z}_m/\langle d\rangle \cong \mathbb{Z}_d$

$C_5 \hookrightarrow C_{10}$: $\quad (x_1,x_2,x_3,x_4) \mapsto (x_1,x_2,x_3,x_4,0,x_1,x_2,x_3,x_4)$

$C_4 \hookrightarrow C_{12}$: $\quad (x_1,x_2,x_3) \mapsto (x_1,x_2,x_3,0,x_1,x_2,x_3,0,x_1,x_2,x_3)$

Posets for the poset-less faces

Theorem

If $d \mid m$, then there exists a map $C_d \hookrightarrow C_m$ that induces a dimension-preserving injection on face lattices.

Posets for the poset-less faces

Theorem

If $d \mid m$, then there exists a map $C_d \hookrightarrow C_m$ that induces a dimension-preserving injection on face lattices.

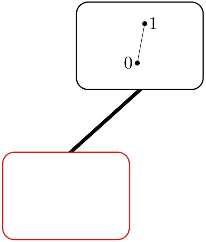
Rays of $C_2 \subseteq \mathbb{R}^1$:
(1)

Rays of $C_4 \subseteq \mathbb{R}^3$:
$(1,0,1)$
$(1,2,1)$
$(1,2,3)$
$(3,2,1)$

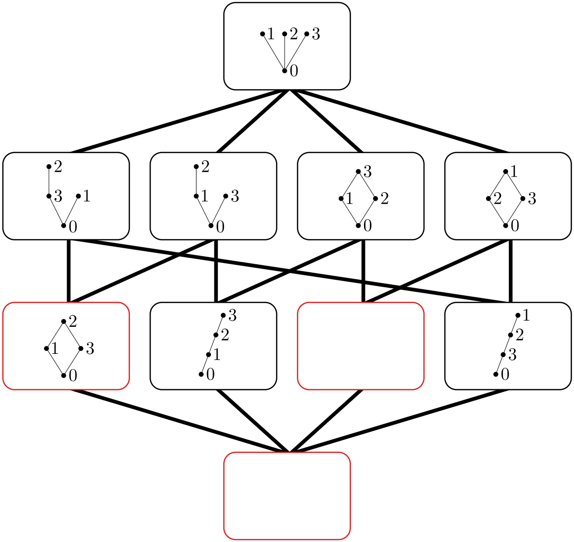
Rays of $C_3 \subseteq \mathbb{R}^2$:
$(1,2)$
$(2,1)$

Rays of $C_{12} \subseteq \mathbb{R}^{11}$:
$(1,0,1,0,1,0,1,0,1,0,1)$
$(1,2,1,0,1,2,1,0,1,2,1)$
$(1,2,3,0,1,2,3,0,1,2,3)$
$(3,2,1,0,3,2,1,0,3,2,1)$
$\vdots$
$\vdots$
$(1,2,0,1,2,0,1,2,0,1,2)$
$(2,1,0,2,1,0,2,1,0,2,1)$
$\vdots$

Posets for the poset-less faces

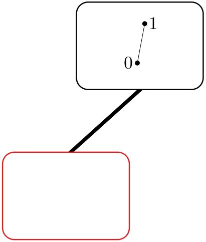


Face lattice of C_2

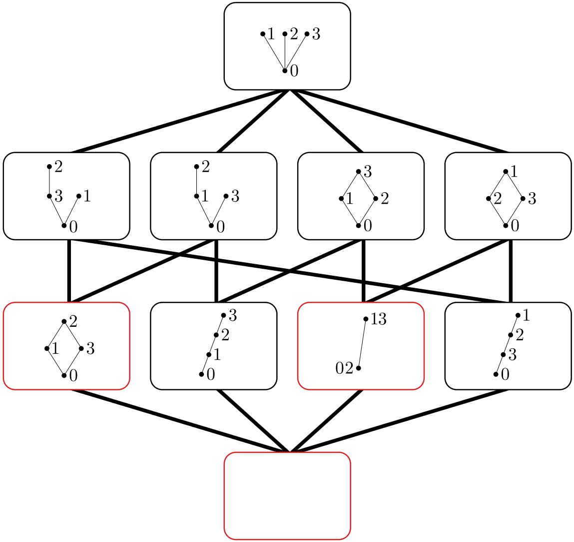


Face lattice of C_4

Posets for the poset-less faces

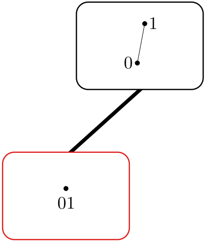


Face lattice of C_2

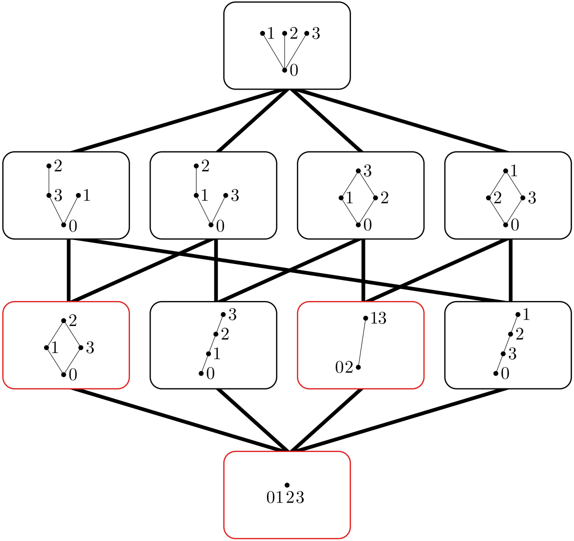


Face lattice of C_4

Posets for the poset-less faces

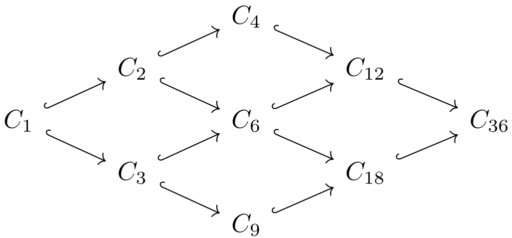


Face lattice of C_2

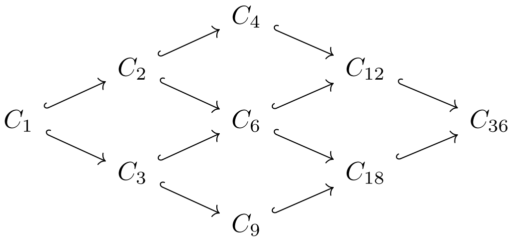


Face lattice of C_4

Posets for the poset-less faces



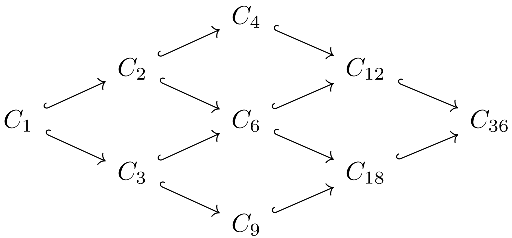
Posets for the poset-less faces



Takeaways:

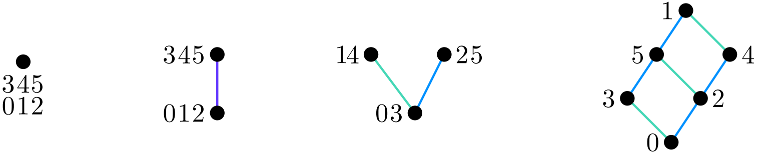
- Define $C_1 = \mathbb{R}^0 = \{\bullet\}$, and for $m \geq 1$, define $C_1 \hookrightarrow C_m$ with $\bullet \mapsto 0$

Posets for the poset-less faces

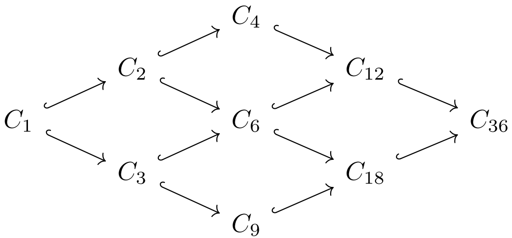


Takeaways:

- Define $C_1 = \mathbb{R}^0 = \{\bullet\}$, and for $m \geq 1$, define $C_1 \hookrightarrow C_m$ with $\bullet \mapsto 0$
- Every $F \subseteq C_m$ has a Kunz poset; elements are *cosets* of $H \subseteq \mathbb{Z}_m$

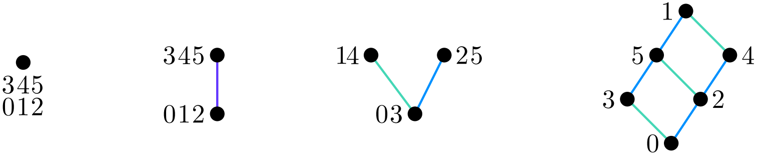


Posets for the poset-less faces



Takeaways:

- Define $C_1 = \mathbb{R}^0 = \{\bullet\}$, and for $m \geq 1$, define $C_1 \hookrightarrow C_m$ with $\bullet \mapsto 0$
- Every $F \subseteq C_m$ has a Kunz poset; elements are *cosets* of $H \subseteq \mathbb{Z}_m$



- Categorical limit: polyhedral complex C_∞ with C_m as subcomplexes

Gluings and complete intersections

The *gluing* of $S = \langle n_1, \ldots, n_k \rangle$, $S' = \langle n'_1, \ldots, n'_\ell \rangle$ by $a, b \in \mathbb{Z}_{\geq 0}$:

$$T = aS + bS' = \langle an_1, \ldots, an_k, bn'_1, \ldots, bn'_\ell \rangle$$

Requirements: $a \in S', b \in S$ non-generators with $\gcd(a, b) = 1$.

Gluings and complete intersections

The *gluing* of $S = \langle n_1, \ldots, n_k \rangle$, $S' = \langle n'_1, \ldots, n'_\ell \rangle$ by $a, b \in \mathbb{Z}_{\geq 0}$:

$$T = aS + bS' = \langle an_1, \ldots, an_k, bn'_1, \ldots, bn'_\ell \rangle$$

Requirements: $a \in S', b \in S$ non-generators with $\gcd(a, b) = 1$.

$$\langle 55, 66, 77, 100, 150 \rangle$$

Gluings and complete intersections

The *gluing* of $S = \langle n_1, \ldots, n_k \rangle$, $S' = \langle n'_1, \ldots, n'_\ell \rangle$ by $a, b \in \mathbb{Z}_{\geq 0}$:

$$T = aS + bS' = \langle an_1, \ldots, an_k, bn'_1, \ldots, bn'_\ell \rangle$$

Requirements: $a \in S', b \in S$ non-generators with $\gcd(a, b) = 1$.

$$\langle 55, 66, 77, 100, 150 \rangle = 11\langle 5, 6, 7 \rangle + 50\langle 2, 3 \rangle$$

Gluings and complete intersections

The *gluing* of $S = \langle n_1, \ldots, n_k \rangle$, $S' = \langle n'_1, \ldots, n'_\ell \rangle$ by $a, b \in \mathbb{Z}_{\geq 0}$:

$$T = aS + bS' = \langle an_1, \ldots, an_k, bn'_1, \ldots, bn'_\ell \rangle$$

Requirements: $a \in S', b \in S$ non-generators with $\gcd(a, b) = 1$.

$$\langle 55, 66, 77, 100, 150 \rangle = 11\langle 5, 6, 7 \rangle + 50\langle 2, 3 \rangle$$

Monoscopic gluings: $S' = \langle 1 \rangle$

Gluings and complete intersections

The *gluing* of $S = \langle n_1, \ldots, n_k \rangle$, $S' = \langle n'_1, \ldots, n'_\ell \rangle$ by $a, b \in \mathbb{Z}_{\geq 0}$:

$$T = aS + bS' = \langle an_1, \ldots, an_k, bn'_1, \ldots, bn'_\ell \rangle$$

Requirements: $a \in S', b \in S$ non-generators with $\gcd(a, b) = 1$.

$$\langle 55, 66, 77, 100, 150 \rangle = 11\langle 5, 6, 7 \rangle + 50\langle 2, 3 \rangle$$

Monoscopic gluings: $S' = \langle 1 \rangle$

$$\langle 10, 12, 14, 15 \rangle = 2\langle 5, 6, 7 \rangle + \langle 15 \rangle = 2\langle 5, 6, 7 \rangle + 15\langle 1 \rangle$$

Gluings and complete intersections

The *gluing* of $S = \langle n_1, \ldots, n_k \rangle$, $S' = \langle n'_1, \ldots, n'_\ell \rangle$ by $a, b \in \mathbb{Z}_{\geq 0}$:

$$T = aS + bS' = \langle an_1, \ldots, an_k, bn'_1, \ldots, bn'_\ell \rangle$$

Requirements: $a \in S', b \in S$ non-generators with $\gcd(a, b) = 1$.

$$\langle 55, 66, 77, 100, 150 \rangle = 11\langle 5, 6, 7 \rangle + 50\langle 2, 3 \rangle$$

Monoscopic gluings: $S' = \langle 1 \rangle$

$$\langle 10, 12, 14, 15 \rangle = 2\langle 5, 6, 7 \rangle + \langle 15 \rangle = 2\langle 5, 6, 7 \rangle + 15\langle 1 \rangle$$

Complete intersections: gluing from the ground up

Gluings and complete intersections

The *gluing* of $S = \langle n_1, \ldots, n_k \rangle$, $S' = \langle n'_1, \ldots, n'_\ell \rangle$ by $a, b \in \mathbb{Z}_{\geq 0}$:

$$T = aS + bS' = \langle an_1, \ldots, an_k, bn'_1, \ldots, bn'_\ell \rangle$$

Requirements: $a \in S', b \in S$ non-generators with $\gcd(a, b) = 1$.

$$\langle 55, 66, 77, 100, 150 \rangle = 11\langle 5, 6, 7 \rangle + 50\langle 2, 3 \rangle$$

Monoscopic gluings: $S' = \langle 1 \rangle$

$$\langle 10, 12, 14, 15 \rangle = 2\langle 5, 6, 7 \rangle + \langle 15 \rangle = 2\langle 5, 6, 7 \rangle + 15\langle 1 \rangle$$

Complete intersections: gluing from the ground up

$\langle 70, 105, 112, 150, 200 \rangle$

Gluings and complete intersections

The *gluing* of $S = \langle n_1, \ldots, n_k \rangle$, $S' = \langle n'_1, \ldots, n'_\ell \rangle$ by $a, b \in \mathbb{Z}_{\geq 0}$:

$$T = aS + bS' = \langle an_1, \ldots, an_k, bn'_1, \ldots, bn'_\ell \rangle$$

Requirements: $a \in S', b \in S$ non-generators with $\gcd(a, b) = 1$.

$$\langle 55, 66, 77, 100, 150 \rangle = 11\langle 5, 6, 7 \rangle + 50\langle 2, 3 \rangle$$

Monoscopic gluings: $S' = \langle 1 \rangle$

$$\langle 10, 12, 14, 15 \rangle = 2\langle 5, 6, 7 \rangle + \langle 15 \rangle = 2\langle 5, 6, 7 \rangle + 15\langle 1 \rangle$$

Complete intersections: gluing from the ground up

$$\langle 70, 105, 112, 150, 200 \rangle = 7\langle 10, 15, 16 \rangle + 50\langle 3, 4 \rangle$$

Gluings and complete intersections

The *gluing* of $S = \langle n_1, \ldots, n_k \rangle$, $S' = \langle n'_1, \ldots, n'_\ell \rangle$ by $a, b \in \mathbb{Z}_{\geq 0}$:

$$T = aS + bS' = \langle an_1, \ldots, an_k, bn'_1, \ldots, bn'_\ell \rangle$$

Requirements: $a \in S', b \in S$ non-generators with $\gcd(a, b) = 1$.

$$\langle 55, 66, 77, 100, 150 \rangle = 11\langle 5, 6, 7 \rangle + 50\langle 2, 3 \rangle$$

Monoscopic gluings: $S' = \langle 1 \rangle$

$$\langle 10, 12, 14, 15 \rangle = 2\langle 5, 6, 7 \rangle + \langle 15 \rangle = 2\langle 5, 6, 7 \rangle + 15\langle 1 \rangle$$

Complete intersections: gluing from the ground up

$$\begin{aligned}\langle 70, 105, 112, 150, 200 \rangle &= 7\langle 10, 15, 16 \rangle + 50\langle 3, 4 \rangle \\ &= 7\Big(5\langle 2, 3 \rangle + \langle 16 \rangle\Big) + 50\langle 3, 4 \rangle\end{aligned}$$

Gluings and complete intersections

The *gluing* of $S = \langle n_1, \ldots, n_k \rangle$, $S' = \langle n'_1, \ldots, n'_\ell \rangle$ by $a, b \in \mathbb{Z}_{\geq 0}$:

$$T = aS + bS' = \langle an_1, \ldots, an_k, bn'_1, \ldots, bn'_\ell \rangle$$

Requirements: $a \in S', b \in S$ non-generators with $\gcd(a, b) = 1$.

$$\langle 55, 66, 77, 100, 150 \rangle = 11\langle 5, 6, 7 \rangle + 50\langle 2, 3 \rangle$$

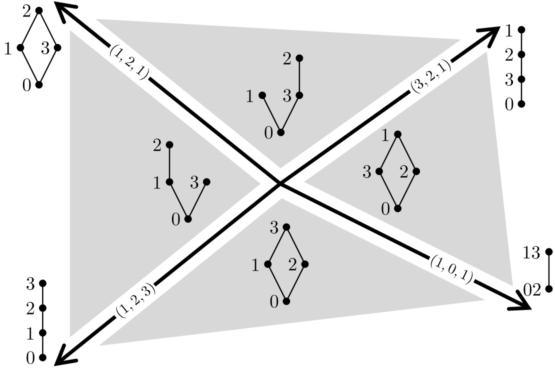
Monoscopic gluings: $S' = \langle 1 \rangle$

$$\langle 10, 12, 14, 15 \rangle = 2\langle 5, 6, 7 \rangle + \langle 15 \rangle = 2\langle 5, 6, 7 \rangle + 15\langle 1 \rangle$$

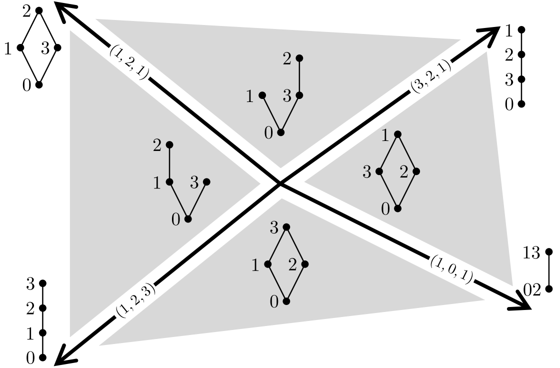
Complete intersections: gluing from the ground up

$$\begin{aligned}\langle 70, 105, 112, 150, 200 \rangle &= 7\langle 10, 15, 16 \rangle + 50\langle 3, 4 \rangle \\ &= 7\Big(5\langle 2, 3 \rangle + \langle 16 \rangle\Big) + 50\langle 3, 4 \rangle \\ &= 7\Big(5\Big(\langle 2 \rangle + \langle 3 \rangle\Big) + \langle 16 \rangle\Big) + 50\Big(\langle 3 \rangle + \langle 4 \rangle\Big)\end{aligned}$$

Gluings and complete intersections



Gluings and complete intersections

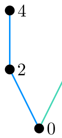


$$S = \langle 4, 10, 21 \rangle$$
$$= 2\langle 2, 5 \rangle + \langle 21 \rangle$$

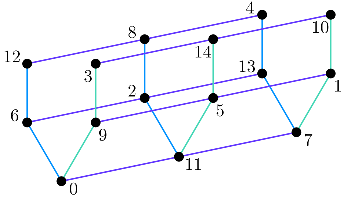
$$S = \langle 4, 10, 23 \rangle$$
$$= 2\langle 2, 5 \rangle + \langle 23 \rangle$$

Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$



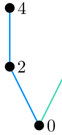
$T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$

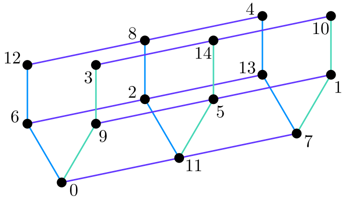


Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$

$T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$

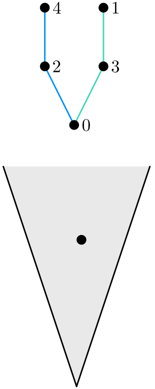


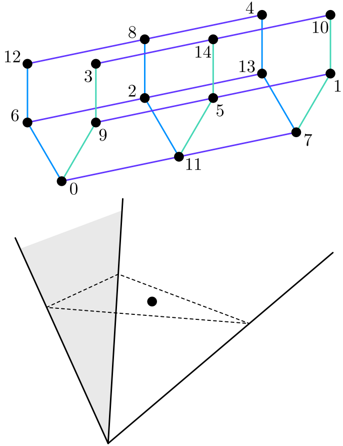


Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$

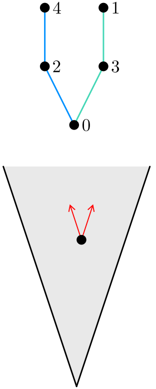
$T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$

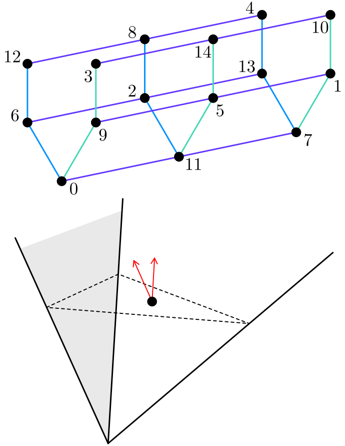




Gluings and complete intersections

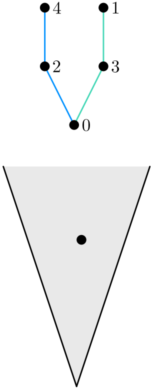
$S = \langle 5, {\color{red}12}, {\color{red}13} \rangle$ $\qquad$ $T = 3{\color{red}S} + \langle 41 \rangle = \langle 15, {\color{red}36}, {\color{red}39}, 41 \rangle$

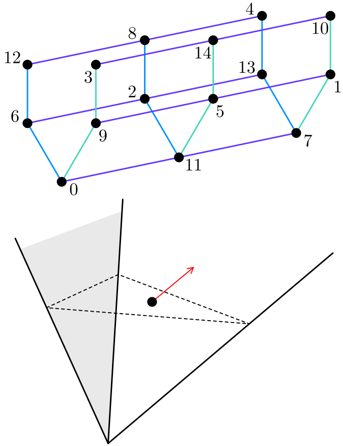




Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$ $\qquad$ $T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$

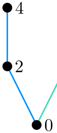


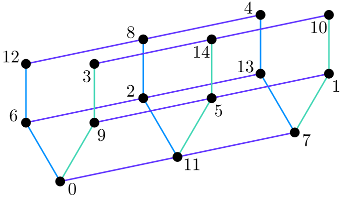


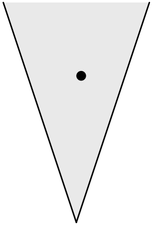
Gluings and complete intersections

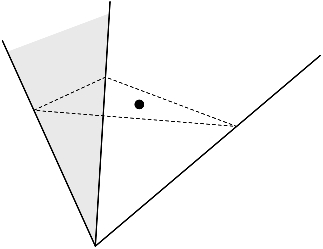
$S = \langle 5, 12, 13 \rangle$

$T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$



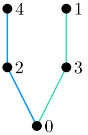


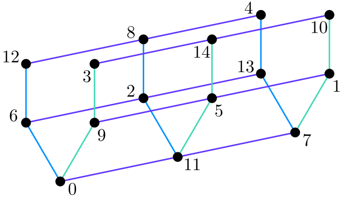


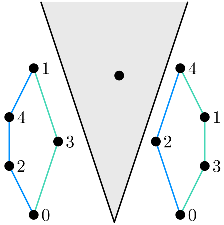


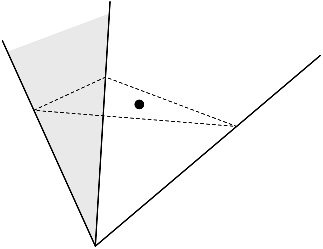
Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$ $\qquad$ $T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$



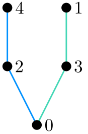


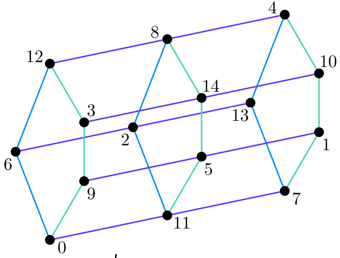


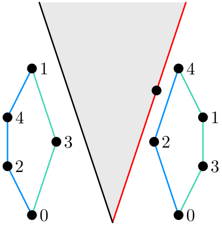


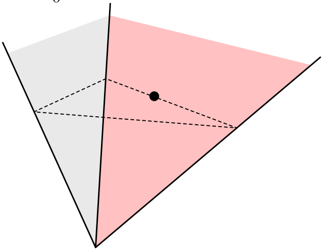
Gluings and complete intersections

$S = \langle 5, 12, {\color{red}8} \rangle$ $\quad$ $T = 3{\color{red}S} + \langle 41 \rangle = \langle 15, 36, {\color{red}24}, 41 \rangle$





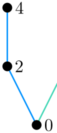


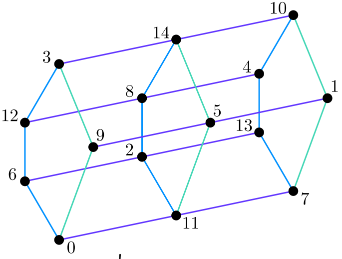


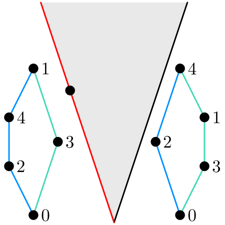
Gluings and complete intersections

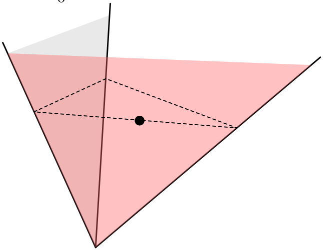
$S = \langle 5, 12, \color{red}{18} \rangle$

$T = 3\color{red}{S} + \langle 41 \rangle = \langle 15, 36, \color{red}{54}, 41 \rangle$



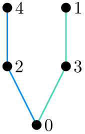


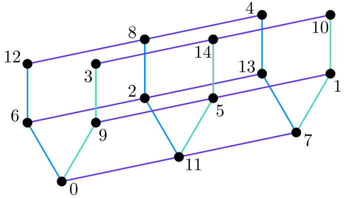


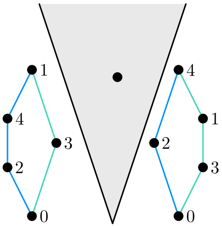


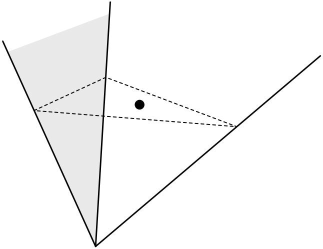
Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$ $\qquad$ $T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$





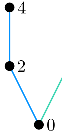


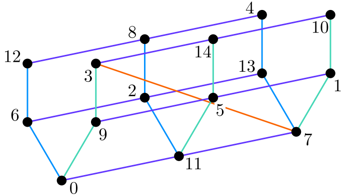


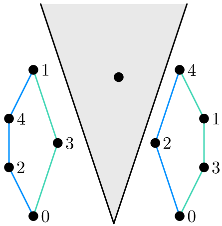
Gluings and complete intersections

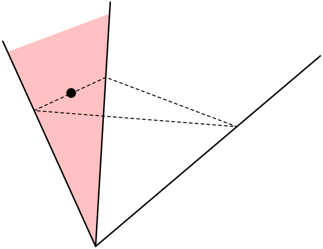
$S = \langle 5, 12, 13 \rangle$

$T = 3S + \langle 26 \rangle = \langle 15, 36, 39, 26 \rangle$





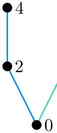


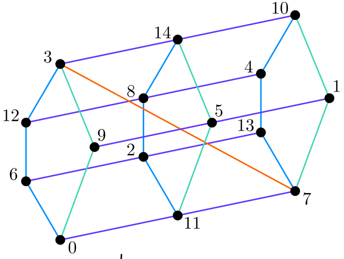


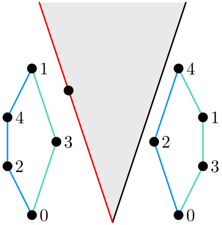
Gluings and complete intersections

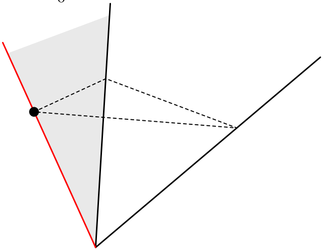
$S = \langle 5, 12, 13 \rangle$

$T = 3S + \langle 26 \rangle = \langle 15, 36, 39, 26 \rangle$



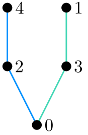


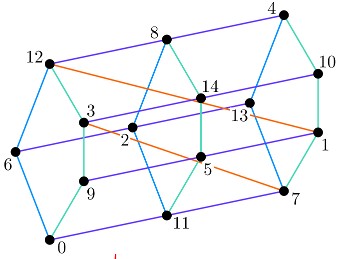


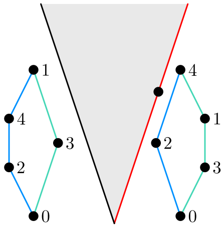


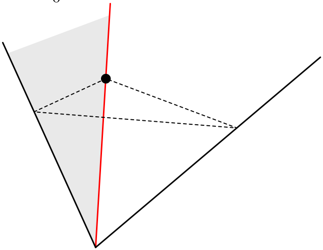
Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$ $\quad$ $T = 3S + \langle 26 \rangle = \langle 15, 36, 39, 26 \rangle$





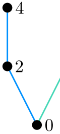


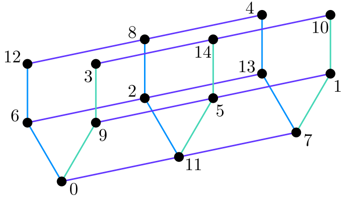


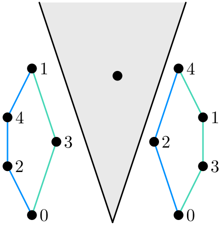
Gluings and complete intersections

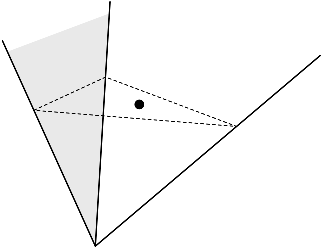
$S = \langle 5, 12, 13 \rangle$

$T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$



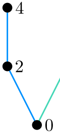


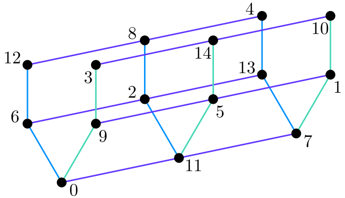


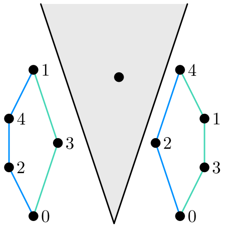


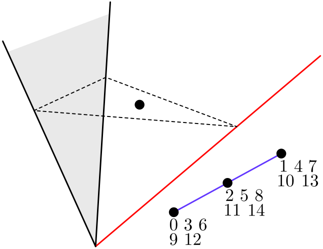
Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$ $\qquad$ $T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$









References

W. Bruns, P. García-Sánchez, C. O'Neill, D. Wilburne (2020)
Wilf's conjecture in fixed multiplicity
International Journal of Algebra and Computation **30** (2020), no. 4, 861–882. (arXiv:1903.04342)

N. Kaplan, C. O'Neill, (2021)
Numerical semigroups, polyhedra, and posets I: the group cone
Combinatorial Theory **1** (2021), #19. (arXiv:1912.03741)

J. Autry, A. Ezell, T. Gomes, C. O'Neill, C. Preuss, T. Saluja, E. Torres Davila (2022)
Numerical semigroups, polyhedra, and posets II: locating certain families of semigroups.
Advances in Geometry **22** (2022), no. 1, 33–48. (arXiv:1912.04460)

T. Gomes, C. O'Neill, E. Torres Davila (2023)
Numerical semigroups, polyhedra, and posets III: minimal presentations and face dimension.
Electronic Journal of Combinatorics **30** (2023), no. 2, #P2.5. (arXiv:2009.05921)

B. Braun, T. Gomes, E. Miller, C. O'Neill, and A. Sobieska (2023)
Minimal free resolutions of numerical semigroup algebras via Apéry specialization
under review. (arXiv:2310.03612)

References

W. Bruns, P. García-Sánchez, C. O'Neill, D. Wilburne (2020)
Wilf's conjecture in fixed multiplicity
International Journal of Algebra and Computation **30** (2020), no. 4, 861–882. (arXiv:1903.04342)

N. Kaplan, C. O'Neill, (2021)
Numerical semigroups, polyhedra, and posets I: the group cone
Combinatorial Theory **1** (2021), #19. (arXiv:1912.03741)

J. Autry, A. Ezell, T. Gomes, C. O'Neill, C. Preuss, T. Saluja, E. Torres Davila (2022)
Numerical semigroups, polyhedra, and posets II: locating certain families of semigroups.
Advances in Geometry **22** (2022), no. 1, 33–48. (arXiv:1912.04460)

T. Gomes, C. O'Neill, E. Torres Davila (2023)
Numerical semigroups, polyhedra, and posets III: minimal presentations and face dimension.
Electronic Journal of Combinatorics **30** (2023), no. 2, #P2.5. (arXiv:2009.05921)

B. Braun, T. Gomes, E. Miller, C. O'Neill, and A. Sobieska (2023)
Minimal free resolutions of numerical semigroup algebras via Apéry specialization
under review. (arXiv:2310.03612)

Thanks!