Classifying numerical semigroups using polyhedral geometry

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Example:

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

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Every numerical semigroup has a unique minimal generating set.

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Multiplicity: $\mathrm{m}(S) =$ smallest nonzero element

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Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

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- $|\mathrm{Ap}(S)| = m$

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The Apéry set is a "one stop shop" for computation.

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Theorem

If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup S with $\mathrm{Ap}(S) = A$ if and only if

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Big idea: the inequalities "$a_i + a_j \geq a_{i+j}$" to define a **cone** C_m.

Kunz cone

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The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

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$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

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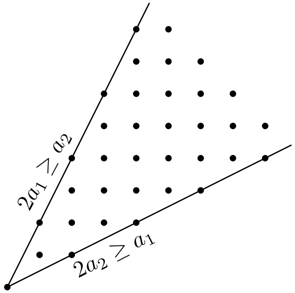

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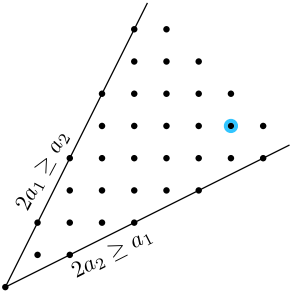

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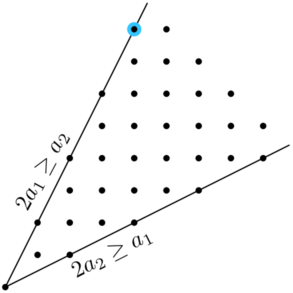

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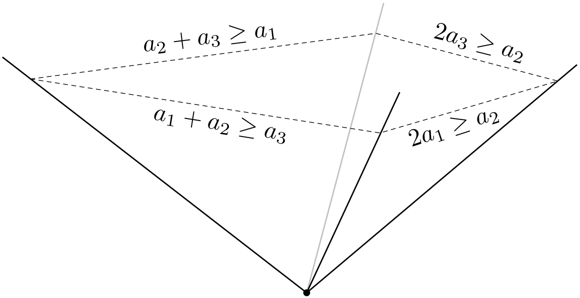

Faces of the Kunz cone

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When are numerical semigroups in (the relative interior of) the same face?

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Big picture: "moduli space" approach for studying XYZ's

- Define a space with XYZ's as points
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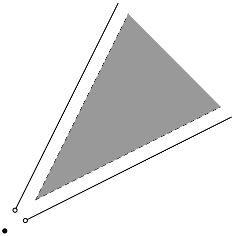
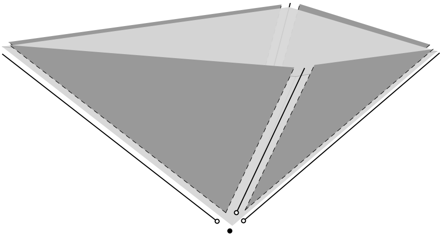
More interesting example: C_m

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Example: $S = \langle 4, 10, 11, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 11\}$$
$$a_1 = 13, \quad a_2 = 10, \quad a_3 = 11$$

$$2a_1 > a_2 \qquad a_1 + a_2 > a_3$$
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Faces of the Kunz cone

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Definition

The *Apéry poset* of S: define $a \preceq a'$ whenever $a' - a \in S$.

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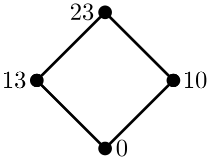

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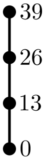

Faces of the Kunz polyhedron

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$$S = \langle 6, 9, 20 \rangle \qquad S' = \langle 6, 26, 27 \rangle$$

$$\mathrm{Ap}(S) = \{0, 49, 20, \ \ 9, 40, 29\} \qquad \mathrm{Ap}(S') = \{0, 79, 26, 27, 52, 53\}$$

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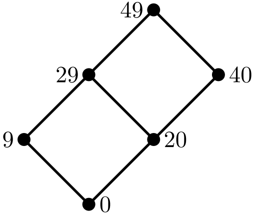

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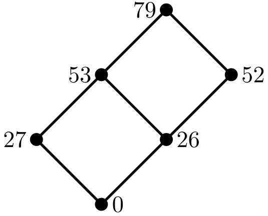

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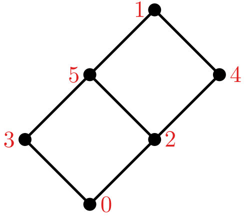

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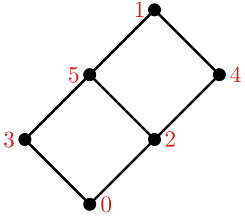

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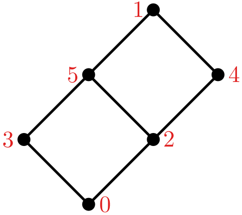

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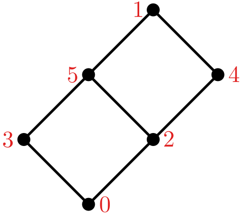

The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\text{Ap}(S)$.

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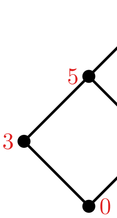
1 2 3 4 5



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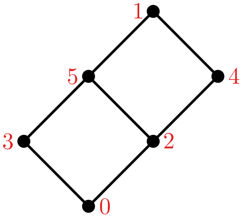

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Theorem (Bruns–García-Sánchez–O.–Wilburne)

Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

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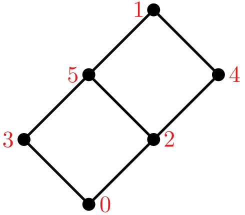

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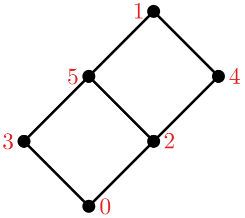

Defining facet equations:

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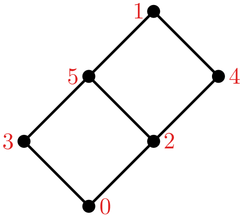

Defining facet equations:

$2a_2 = a_4$	$2 \preceq 4$
$a_2 + a_3 = a_5$	$2 \preceq 5$
	$3 \preceq 5$
$a_2 + a_5 = a_1$	$2 \preceq 1$
	$5 \preceq 1$
$a_3 + a_4 = a_1$	$3 \preceq 1$
	$4 \preceq 1$

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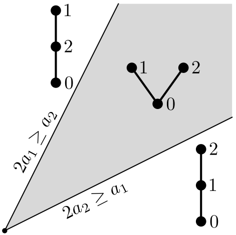
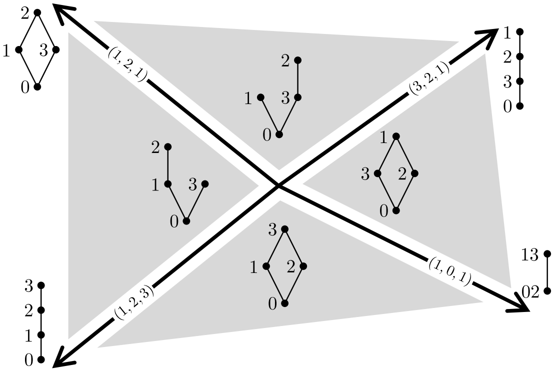
C_3 and C_4



1
2
0
1
2
0
$2a_1 \geq a_2$
$2a_2 \geq a_1$
2
1
0



2
1
3
0
(1, 2, 1)
(3, 2, 1)
(1, 2, 3)
(1, 0, 1)
1
2
3
0
13
02
3
2
1
0

A couple of long-standing (**hard**) conjectures

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Example: $n_3 = 4$

$$\begin{aligned} \langle 2,7 \rangle &= \{0, \quad 2, \quad 4, \quad 6,7,8,\ldots\} \\ \langle 3,4 \rangle &= \{0, \quad\quad 3,4, \quad 6,7,8,\ldots\} \\ \langle 3,5,7 \rangle &= \{0, \quad\quad 3, \quad 5,6,7,8,\ldots\} \\ \langle 4,5,6,7 \rangle &= \{0, \quad\quad\quad 4,5,6,7,8,\ldots\} \end{aligned}$$

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Not true for $n'_f = \#$ of numerical semigroups with Frobenius number f

$$n'_{11} = 51 \qquad n'_{12} = 40 \qquad n'_{13} = 106$$

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Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \le k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

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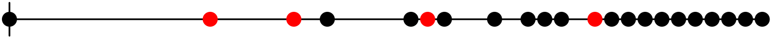

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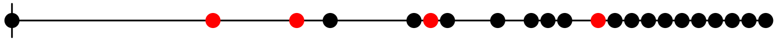

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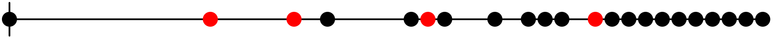

Equality holds when:

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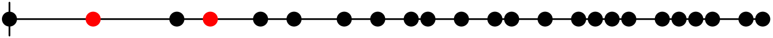

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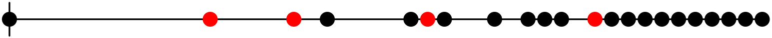

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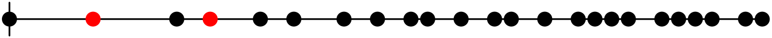
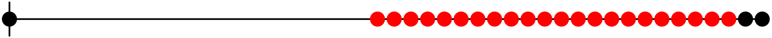


- $S = \langle m, m+1, \ldots, 2m-1 \rangle$



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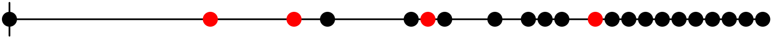

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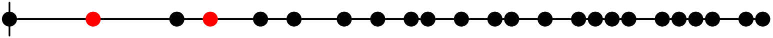

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Proved in many special cases, including $\mathrm{g}(S) \leq 60$.

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Direct ties to geometry: if S corresponds to $x = (a_1, \ldots, a_{m-1}) \in C_m$,

$$g(S) = \|x\|_1 - \tfrac{1}{2}m(m-1), \qquad F(S) = \|x\|_\infty - m,$$

and # generators k is determined by the face $F \subseteq \mathcal{C}_m$ containing x.

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Theorem (Bruns-García-Sánchez-O.-Wilburne, 2020)

Wilf's conjecture holds for all numerical semigroups S with $m \le 18$.

A couple of long-standing (**hard**) conjectures

Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \le k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

Bras-Amoros Conjecture

For all g, we have $n_g \ge n_{g-1}$.

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Wilf's conjecture holds for all numerical semigroups S with $m \le 18$.

Conjecture (Kaplan)

For fixed m, the number of numerical semigroups g gaps is non-decreasing.

Shared properties within a face

What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

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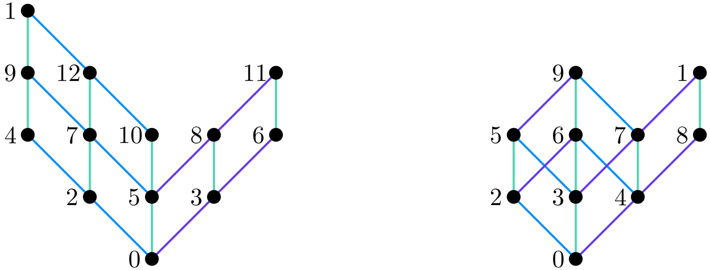

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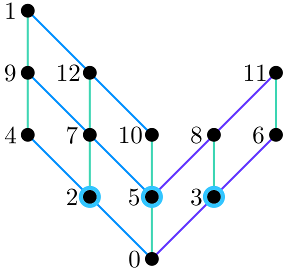

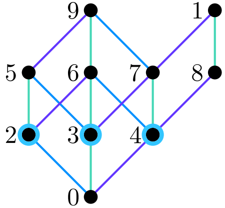

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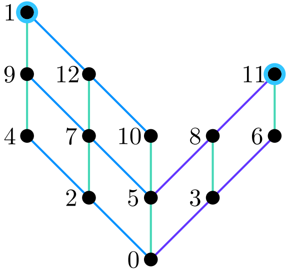

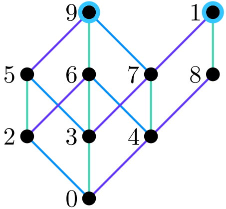

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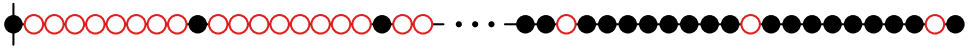
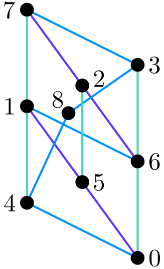
$S = \langle 4, 7 \rangle$

$S = \langle 9, 40, 50, 60 \rangle$





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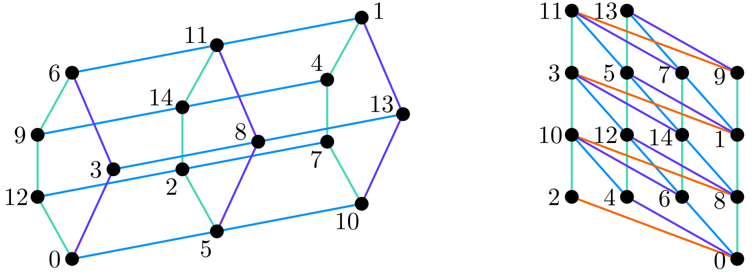

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$$I_S = \ker\left(\Bbbk[\overline{x}] \to \Bbbk[t]\right)$$
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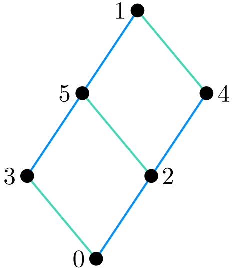
$S = \langle 6, 9, 20 \rangle$

$$I_S = \langle x^3 - y^2, x^4y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$$



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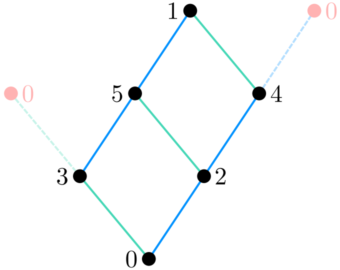

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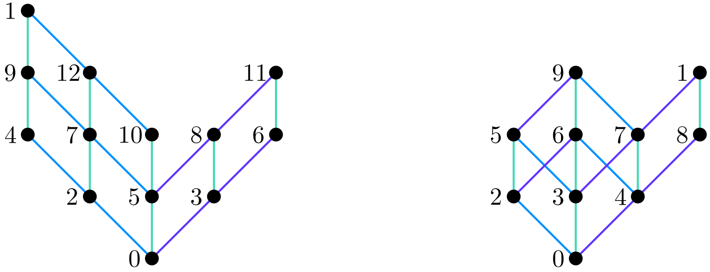

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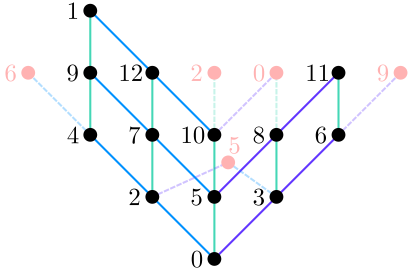

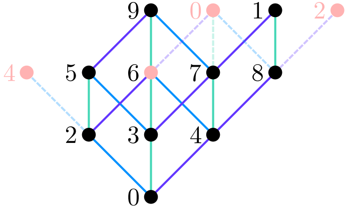

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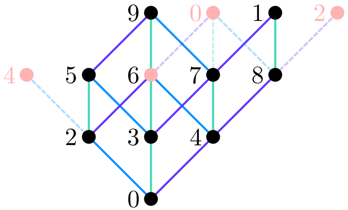
$$I_S = \langle\, x_2^2 - y^* x_4,\ \ x_2 x_4 - x_3^2,\ \ x_3^2 x_4 - y^*,\ \ x_4^3 - y^* x_2 \,\rangle \subseteq \Bbbk[y, x_2, x_3, x_4]$$



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- Betti numbers of I_S over $\Bbbk[\overline{x}]$

$$0 \leftarrow R \leftarrow R^6 \longleftarrow R^8 \longleftarrow R^3 \leftarrow 0$$

	1,12	1,13	2,12	2,23	3,13	3,23	2,13	3,12
1,1	$-x_2$	$-x_3$					y^*	y^*
2,2	$-y^*$		x_1	$-x_3$		y^*		
3,3					x_1	x_2	$-y^*$	
2,1	x_1		$-x_2$	y^*	y^*		$-x_3$	
3,1	y^*	x_1			$-x_3$	$-y^*$		$-x_2$
3,2		$-y^*$	$-y^*$	x_2		$-x_3$	x_1	x_1

	1,[3]	2,[3]	3,[3]
1,12	x_3	$-y^*$	
1,13	$-x_2$		y^*
2,12		x_3	$-y^*$
2,23	$-y^*$	x_1	
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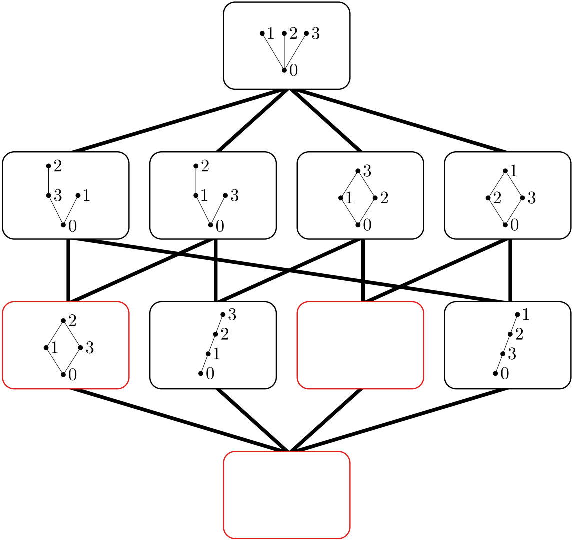
- Betti numbers of $\Bbbk$ over $\Bbbk[\overline{x}]/I_S$

$$0 \leftarrow R \xleftarrow{\begin{array}{c} \begin{array}{cccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \emptyset \; [\, y \;\; x_1 \;\; x_2 \;\; x_3 \,] \end{array}} R^4 \xleftarrow{\begin{array}{c|cccccccccccc} & \mathbf{01} & \mathbf{02} & \mathbf{03} & \mathbf{11} & \mathbf{12} & \mathbf{13} & \mathbf{21} & \mathbf{22} & \mathbf{23} & \mathbf{31} & \mathbf{32} & \mathbf{33} \\ \hline \mathbf{0} & x_1 & x_2 & x_3 & & & -y^* & & -y^* & & -y^* & & \\ \mathbf{1} & -y & & & x_1 & x_2 & x_3 & & & -y^* & & -y^* & \\ \mathbf{2} & & -y & & -y^* & & & x_1 & x_2 & x_3 & & & -y^* \\ \mathbf{3} & & & -y & & -y^* & & -y^* & & & x_1 & x_2 & x_3 \end{array}} R^{12}$$

$$\longleftarrow R^{36} \longleftarrow R^{108} \longleftarrow R^{324} \longleftarrow R^{972} \longleftarrow R^{2916} \longleftarrow \cdots$$

Posets for the poset-less faces



Face lattice of C_4

Posets for the poset-less faces

Theorem

If $d \mid m$, then there exists a map $C_d \hookrightarrow C_m$ that induces a dimension-preserving injection on face lattices.

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For $x \in C_m$, the coordinates of 0's form a subgroup of $\mathbb{Z}_m$

Example: $\mathbb{Z}_{10}$

$(*, *, *, *, 0, *, *, *, *)$ $\quad$ $(*, 0, *, 0, *, 0, *, 0, *)$

$(0, 0, 0, 0, 0, 0, 0, 0, 0)$ $\quad$ $(*, *, *, *, *, *, *, *, *)$

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Maps are induced by $\mathbb{Z}_m / \langle d \rangle \cong \mathbb{Z}_d$

$C_5 \hookrightarrow C_{10}$: $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, x_4, 0, x_1, x_2, x_3, x_4)$

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Rays of $C_2 \subseteq \mathbb{R}^1$:
(1)

Rays of $C_4 \subseteq \mathbb{R}^3$:
(1, 0, 1)
(1, 2, 1)
(1, 2, 3)
(3, 2, 1)

Rays of $C_3 \subseteq \mathbb{R}^2$:
(1, 2)
(2, 1)

Rays of $C_{12} \subseteq \mathbb{R}^{11}$:
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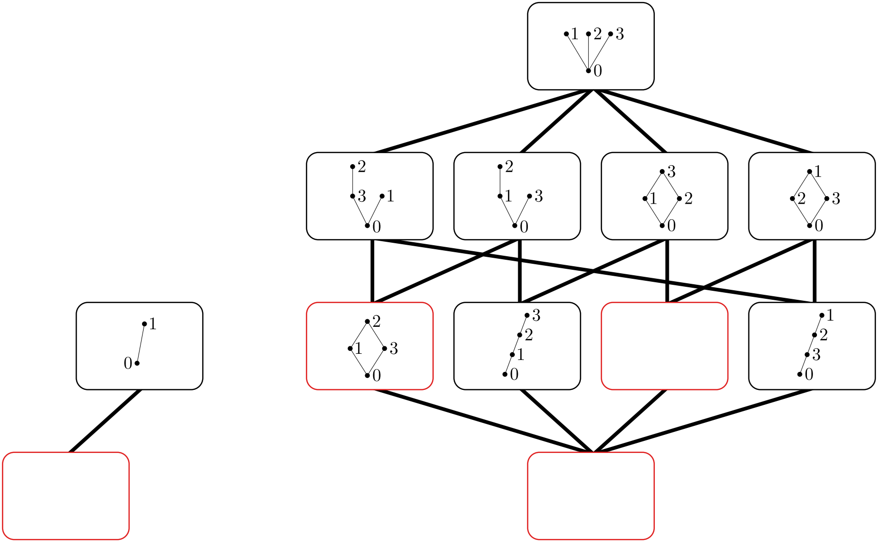
(1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1)
(1, 2, 3, 0, 1, 2, 3, 0, 1, 2, 3)
(3, 2, 1, 0, 3, 2, 1, 0, 3, 2, 1)
⋮
⋮
(1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2)
(2, 1, 0, 2, 1, 0, 2, 1, 0, 2, 1)
⋮

Posets for the poset-less faces



Face lattice of C_2

Face lattice of C_4

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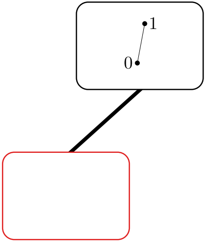

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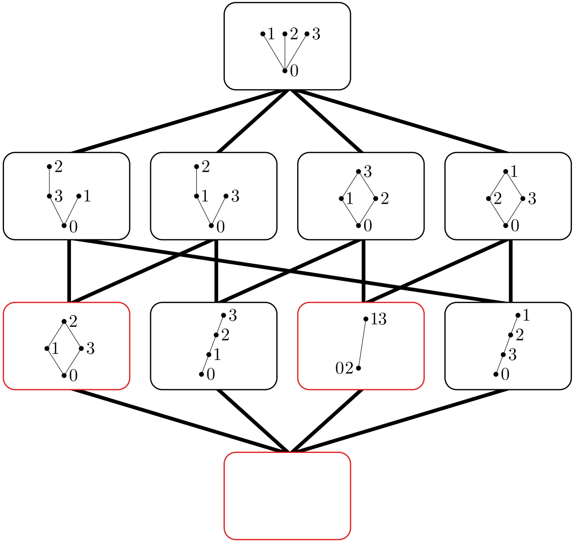

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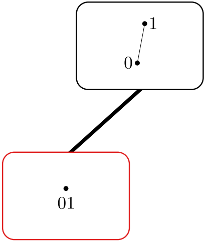

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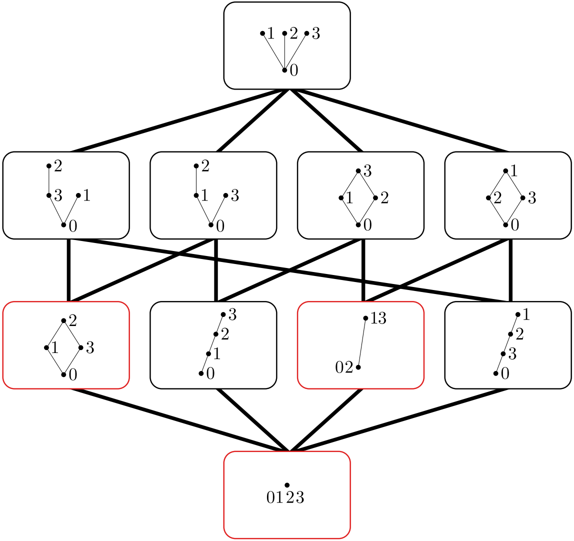

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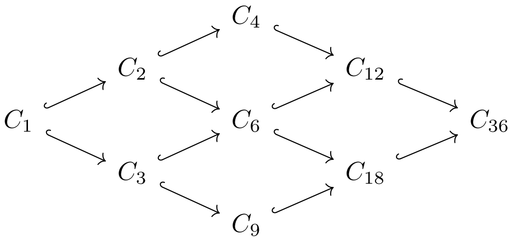

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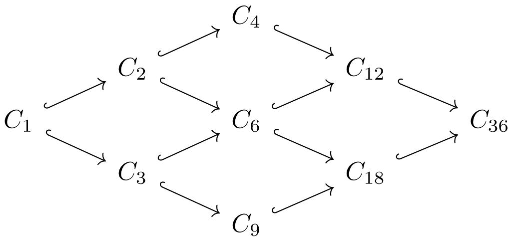

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- Define $C_1 = \mathbb{R}^0 = \{\bullet\}$, and for $m \geq 1$, define $C_1 \hookrightarrow C_m$ with $\bullet \mapsto 0$

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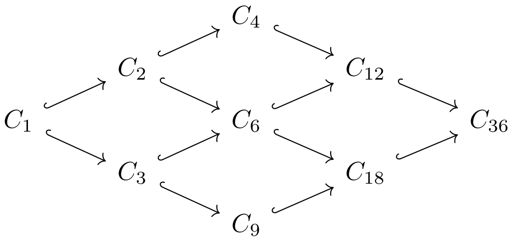

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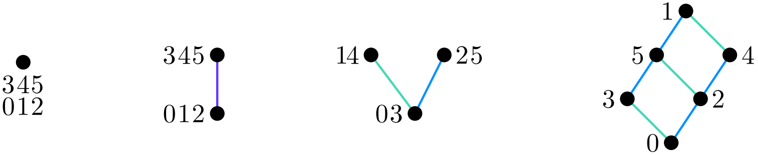

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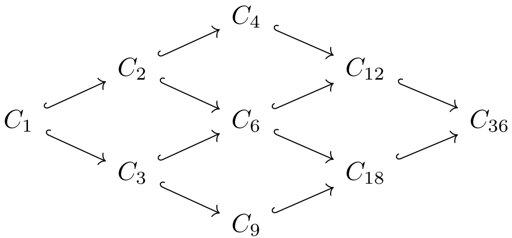

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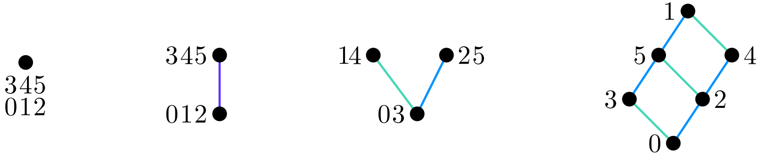

- Categorical limit: polyhedral complex C_∞ with C_m as subcomplexes

Gluings and complete intersections

The *gluing* of $S = \langle n_1, \ldots, n_k \rangle$, $S' = \langle n'_1, \ldots, n'_\ell \rangle$ by $a, b \in \mathbb{Z}_{\geq 0}$:

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Requirements: $a \in S', b \in S$ non-generators with $\gcd(a, b) = 1$.

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$$\langle 10, 12, 14, 15 \rangle = 2\langle 5, 6, 7 \rangle + \langle 15 \rangle = 2\langle 5, 6, 7 \rangle + 15\langle 1 \rangle$$

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Monoscopic gluings: $S' = \langle 1 \rangle$

$$\langle 10, 12, 14, 15 \rangle = 2\langle 5, 6, 7 \rangle + \langle 15 \rangle = 2\langle 5, 6, 7 \rangle + 15\langle 1 \rangle$$

Complete intersections: gluing from the ground up

Gluings and complete intersections

The *gluing* of $S = \langle n_1, \ldots, n_k \rangle$, $S' = \langle n'_1, \ldots, n'_\ell \rangle$ by $a, b \in \mathbb{Z}_{\geq 0}$:

$$T = aS + bS' = \langle an_1, \ldots, an_k, bn'_1, \ldots, bn'_\ell \rangle$$

Requirements: $a \in S', b \in S$ non-generators with $\gcd(a, b) = 1$.

$$\langle 55, 66, 77, 100, 150 \rangle = 11\langle 5, 6, 7 \rangle + 50\langle 2, 3 \rangle$$

Monoscopic gluings: $S' = \langle 1 \rangle$

$$\langle 10, 12, 14, 15 \rangle = 2\langle 5, 6, 7 \rangle + \langle 15 \rangle = 2\langle 5, 6, 7 \rangle + 15\langle 1 \rangle$$

Complete intersections: gluing from the ground up

$\langle 70, 105, 112, 150, 200 \rangle$

Gluings and complete intersections

The *gluing* of $S = \langle n_1, \ldots, n_k \rangle$, $S' = \langle n'_1, \ldots, n'_\ell \rangle$ by $a, b \in \mathbb{Z}_{\geq 0}$:

$$T = aS + bS' = \langle an_1, \ldots, an_k, bn'_1, \ldots, bn'_\ell \rangle$$

Requirements: $a \in S', b \in S$ non-generators with $\gcd(a, b) = 1$.

$$\langle 55, 66, 77, 100, 150 \rangle = 11\langle 5, 6, 7 \rangle + 50\langle 2, 3 \rangle$$

Monoscopic gluings: $S' = \langle 1 \rangle$

$$\langle 10, 12, 14, 15 \rangle = 2\langle 5, 6, 7 \rangle + \langle 15 \rangle = 2\langle 5, 6, 7 \rangle + 15\langle 1 \rangle$$

Complete intersections: gluing from the ground up

$$\langle 70, 105, 112, 150, 200 \rangle = 7\langle 10, 15, 16 \rangle + 50\langle 3, 4 \rangle$$

Gluings and complete intersections

The *gluing* of $S = \langle n_1, \ldots, n_k \rangle$, $S' = \langle n'_1, \ldots, n'_\ell \rangle$ by $a, b \in \mathbb{Z}_{\geq 0}$:

$$T = aS + bS' = \langle an_1, \ldots, an_k, bn'_1, \ldots, bn'_\ell \rangle$$

Requirements: $a \in S', b \in S$ non-generators with $\gcd(a, b) = 1$.

$$\langle 55, 66, 77, 100, 150 \rangle = 11\langle 5, 6, 7 \rangle + 50\langle 2, 3 \rangle$$

Monoscopic gluings: $S' = \langle 1 \rangle$

$$\langle 10, 12, 14, 15 \rangle = 2\langle 5, 6, 7 \rangle + \langle 15 \rangle = 2\langle 5, 6, 7 \rangle + 15\langle 1 \rangle$$

Complete intersections: gluing from the ground up

$$\begin{aligned} \langle 70, 105, 112, 150, 200 \rangle &= 7\langle 10, 15, 16 \rangle + 50\langle 3, 4 \rangle \\ &= 7\Big(5\langle 2, 3 \rangle + \langle 16 \rangle\Big) + 50\langle 3, 4 \rangle \end{aligned}$$

Gluings and complete intersections

The *gluing* of $S = \langle n_1, \ldots, n_k \rangle$, $S' = \langle n'_1, \ldots, n'_\ell \rangle$ by $a, b \in \mathbb{Z}_{\geq 0}$:

$$T = aS + bS' = \langle an_1, \ldots, an_k, bn'_1, \ldots, bn'_\ell \rangle$$

Requirements: $a \in S', b \in S$ non-generators with $\gcd(a, b) = 1$.

$$\langle 55, 66, 77, 100, 150 \rangle = 11\langle 5, 6, 7 \rangle + 50\langle 2, 3 \rangle$$

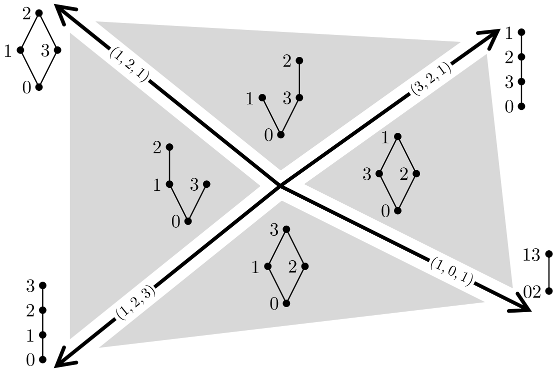
Monoscopic gluings: $S' = \langle 1 \rangle$

$$\langle 10, 12, 14, 15 \rangle = 2\langle 5, 6, 7 \rangle + \langle 15 \rangle = 2\langle 5, 6, 7 \rangle + 15\langle 1 \rangle$$

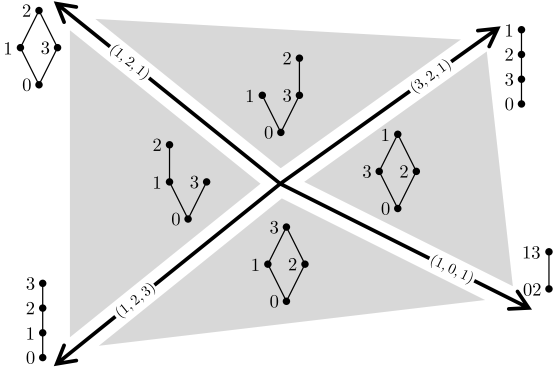
Complete intersections: gluing from the ground up

$$\begin{aligned} \langle 70, 105, 112, 150, 200 \rangle &= 7\langle 10, 15, 16 \rangle + 50\langle 3, 4 \rangle \\ &= 7\Big(5\langle 2, 3 \rangle + \langle 16 \rangle\Big) + 50\langle 3, 4 \rangle \\ &= 7\Big(5\Big(\langle 2 \rangle + \langle 3 \rangle\Big) + \langle 16 \rangle\Big) + 50\Big(\langle 3 \rangle + \langle 4 \rangle\Big) \end{aligned}$$

Gluings and complete intersections



Gluings and complete intersections



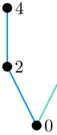
$S = \langle 4, 10, 21 \rangle$
$= 2\langle 2, 5 \rangle + \langle 21 \rangle$

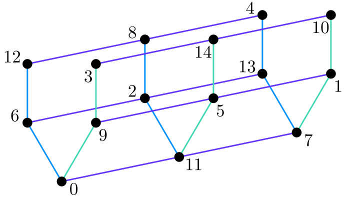
$S = \langle 4, 10, 23 \rangle$
$= 2\langle 2, 5 \rangle + \langle 23 \rangle$

Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$

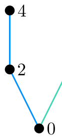
$T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$



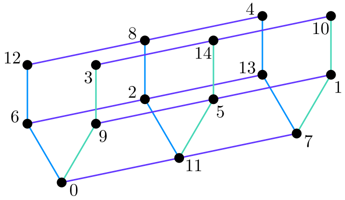


Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$

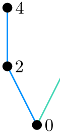


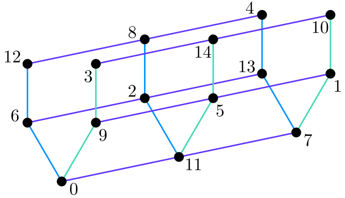
$T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$

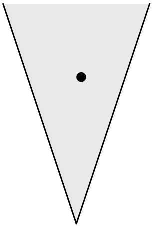


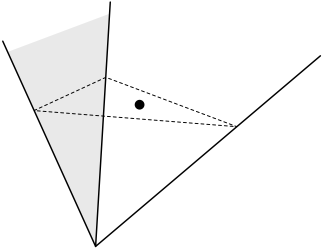
Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$ $\qquad$ $T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$

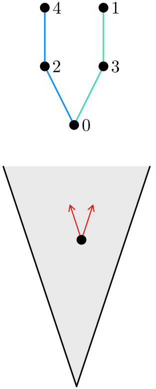


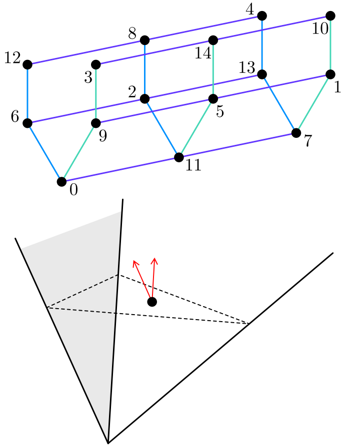






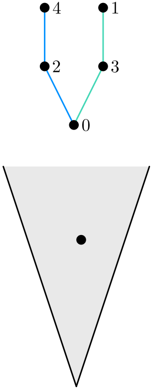
$S = \langle 5, 12, 13 \rangle$ $\qquad$ $T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$

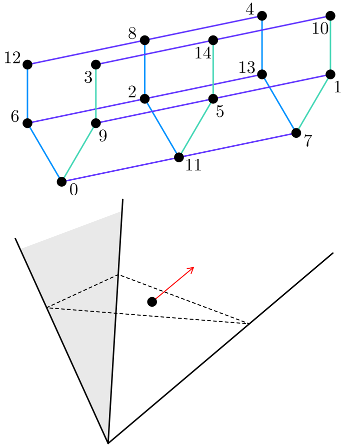




Gluings and complete intersections

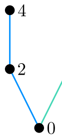
$S = \langle 5, 12, 13 \rangle$ $\qquad$ $T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$

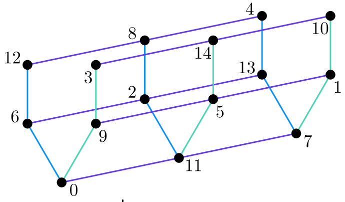


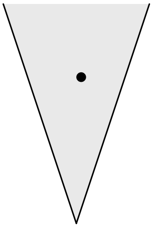


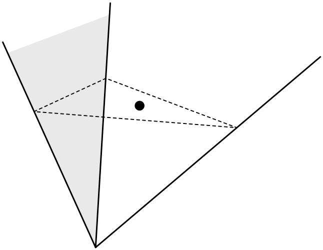
Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$ $\qquad$ $T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$



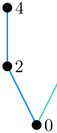


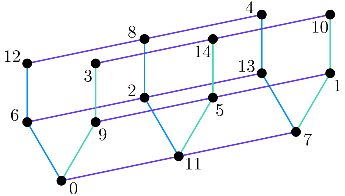


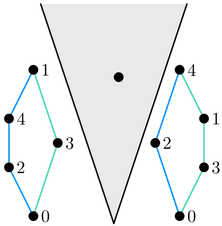


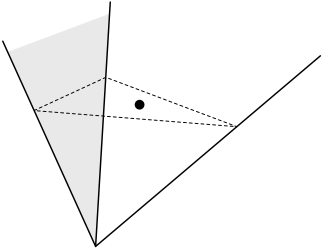
Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$ $\qquad$ $T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$





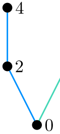


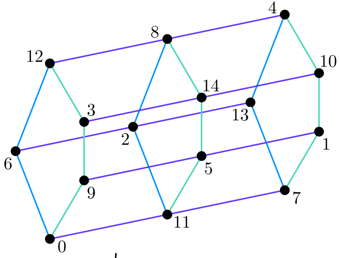


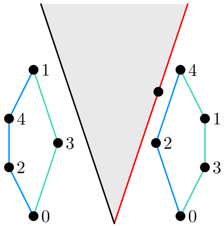
Gluings and complete intersections

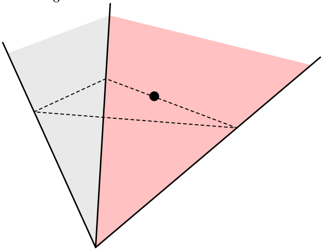
$S = \langle 5, 12, {\color{red}8} \rangle$

$T = 3{\color{red}S} + \langle 41 \rangle = \langle 15, 36, {\color{red}24}, 41 \rangle$



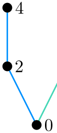


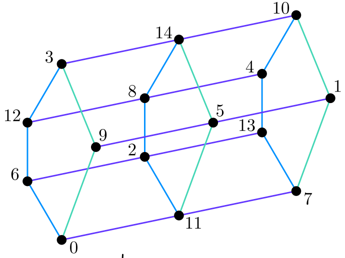


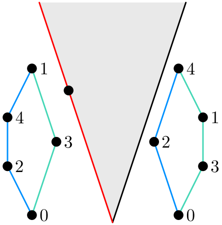


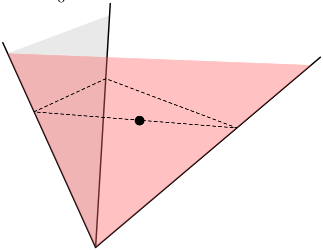
Gluings and complete intersections

$S = \langle 5, 12, {\color{red}18} \rangle$ $\qquad$ $T = 3{\color{red}S} + \langle 41 \rangle = \langle 15, 36, {\color{red}54}, 41 \rangle$





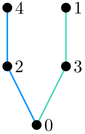


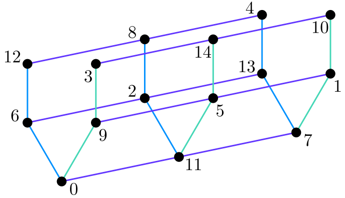


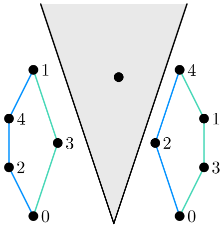
Gluings and complete intersections

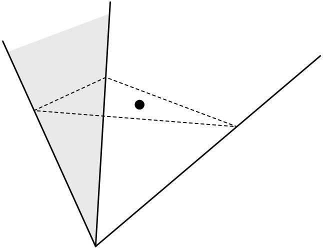
$S = \langle 5, 12, 13 \rangle$

$T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$



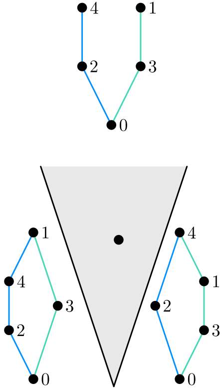


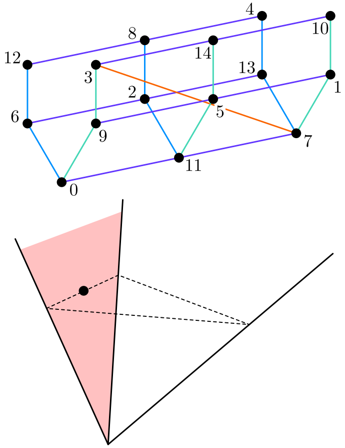




Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$ $\qquad$ $T = 3S + \langle 26 \rangle = \langle 15, 36, 39, 26 \rangle$

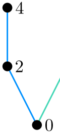


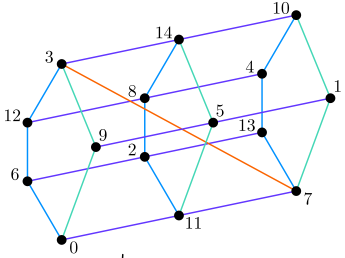


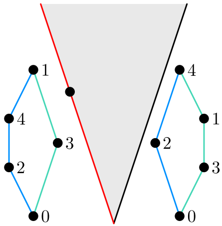
Gluings and complete intersections

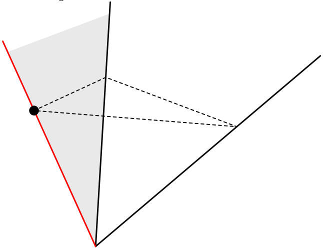
$S = \langle 5, 12, 13 \rangle$

$T = 3S + \langle 26 \rangle = \langle 15, 36, 39, 26 \rangle$



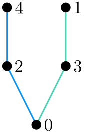


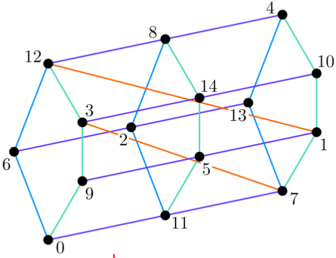


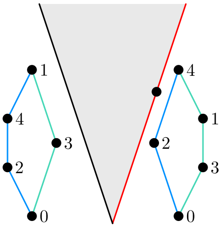


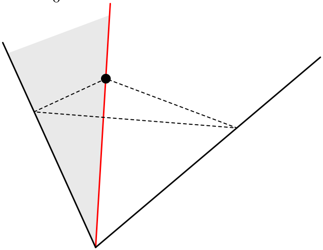
Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$ $T = 3S + \langle 26 \rangle = \langle 15, 36, 39, 26 \rangle$



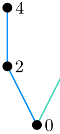


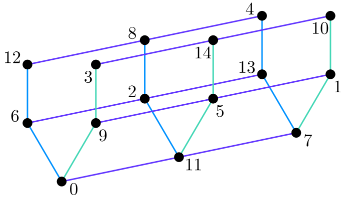


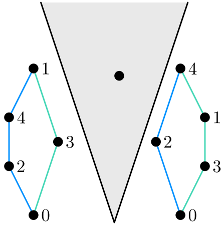


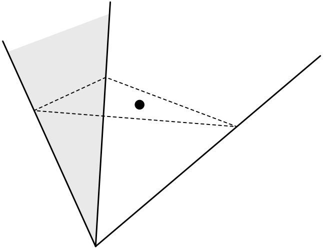
Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$ $\qquad$ $T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$

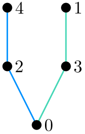


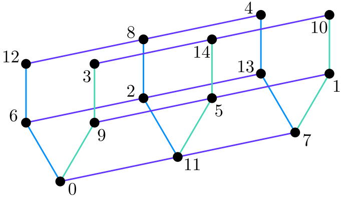


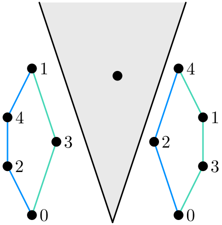


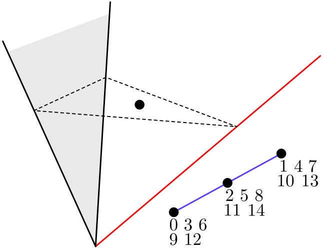


$S = \langle 5, 12, 13 \rangle$ $\quad\quad$ $T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$









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