Classifying numerical semigroups using polyhedral geometry

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Slides available: `https://cdoneill.sdsu.edu/`

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$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

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Multiplicity: $\mathrm{m}(S) =$ smallest nonzero element

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Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

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- $|\mathrm{Ap}(S)| = m$

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The Apéry set is a "one stop shop" for computation.

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Theorem

If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup S with $\mathrm{Ap}(S) = A$ if and only if

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Big idea: the inequalities "$a_i + a_j \geq a_{i+j}$" to define a **cone** C_m.

Kunz cone

Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

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$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\ge 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

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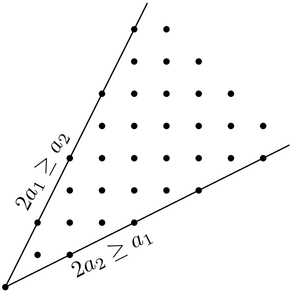

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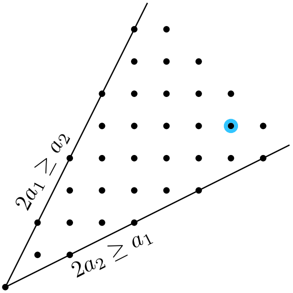

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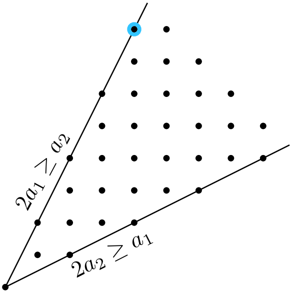

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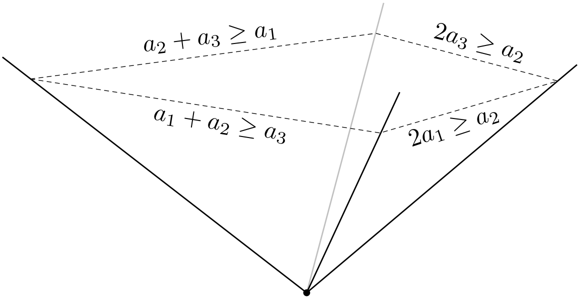

Faces of the Kunz cone

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When are numerical semigroups in (the relative interior of) the same face?

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Big picture: "moduli space" approach for studying XYZ's

- Define a space with XYZ's as points
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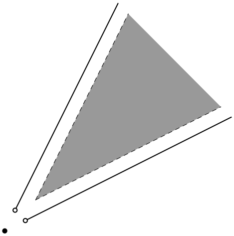
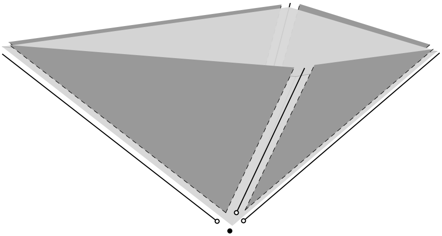
More interesting example: C_m





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Example: $S = \langle 4, 10, 11, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 11\}$$
$$a_1 = 13, \quad a_2 = 10, \quad a_3 = 11$$

$$2a_1 > a_2 \qquad a_1 + a_2 > a_3$$
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Faces of the Kunz cone

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Definition

The *Apéry poset* of S: define $a \preceq a'$ whenever $a' - a \in S$.

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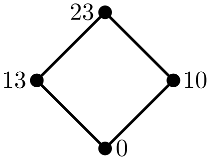

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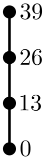

Faces of the Kunz polyhedron

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$$S = \langle 6, 9, 20 \rangle \qquad S' = \langle 6, 26, 27 \rangle$$
$$\mathrm{Ap}(S) = \{0, 49, 20, \; 9, 40, 29\} \qquad \mathrm{Ap}(S') = \{0, 79, 26, 27, 52, 53\}$$

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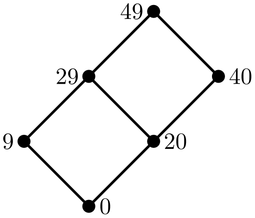

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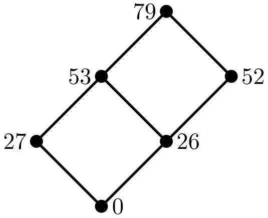

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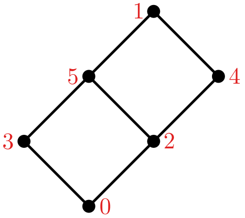

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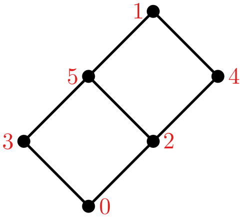

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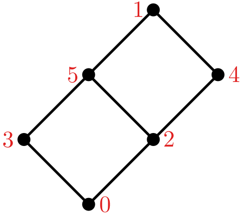

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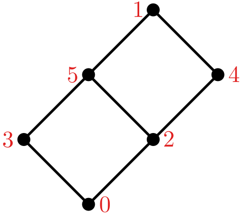

The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

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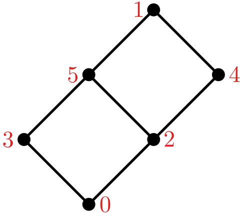

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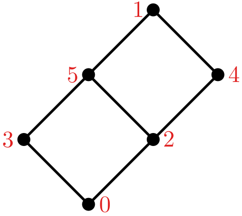

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Theorem (Bruns–García-Sánchez–O.–Wilburne)

Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

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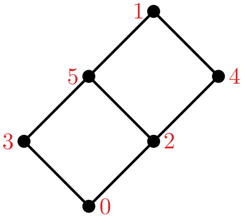

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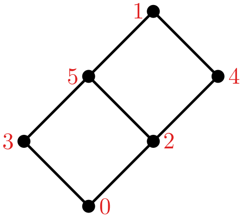

Defining facet equations:

$$2a_2 = a_4$$
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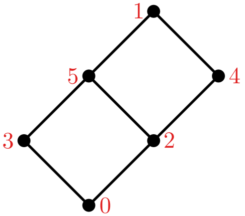

Defining facet equations:

$$\begin{array}{rcl}
2a_2 = a_4 & & 2 \preceq 4 \\
a_2 + a_3 = a_5 & & 2 \preceq 5 \\
 & & 3 \preceq 5 \\
a_2 + a_5 = a_1 & & 2 \preceq 1 \\
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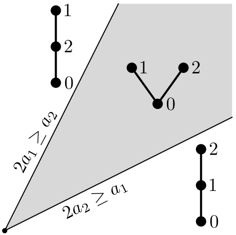
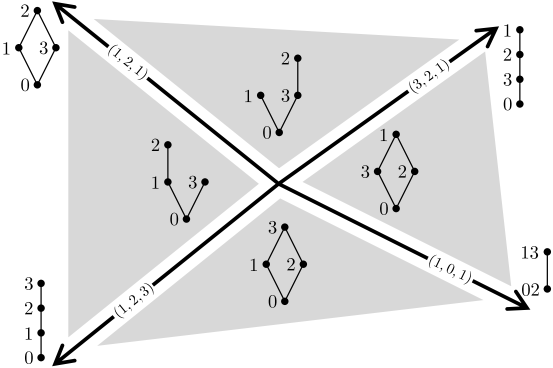
C_3 and C_4



1
2
0
1
2
0
$2a_1 \geq a_2$
$2a_2 \geq a_1$
2
1
0



2
1
3
0
(1, 2, 1)
(3, 2, 1)
(1, 0, 1)
(1, 2, 3)
1
2
3
0
13
02
3
2
1
0

A couple of long-standing (**hard**) conjectures

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Example: $n_3 = 4$

$$\begin{aligned}
\langle 2,7\rangle &= \{0, \quad 2, \quad 4, \quad 6,7,8,\ldots\} \\
\langle 3,4\rangle &= \{0, \qquad 3,4, \quad 6,7,8,\ldots\} \\
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Not true for $n'_f = \#$ of numerical semigroups with Frobenius number f

$$n'_{11} = 51 \qquad n'_{12} = 40 \qquad n'_{13} = 106$$

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For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \leq k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

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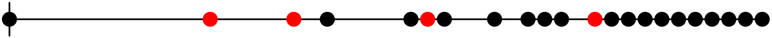

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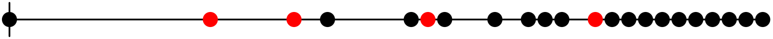

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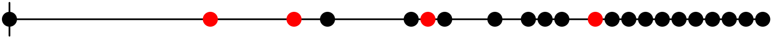

Equality holds when:

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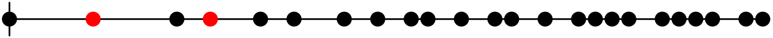

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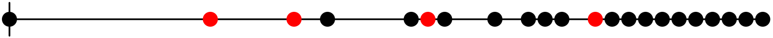

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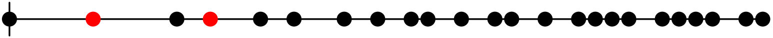

- $S = \langle m, m+1, \ldots, 2m-1 \rangle$

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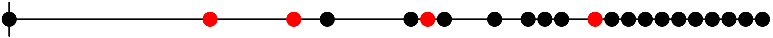

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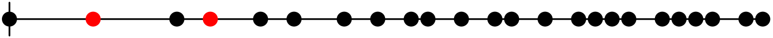

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Proved in many special cases, including $\mathrm{g}(S) \le 60$.

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Direct ties to geometry: if S corresponds to $x = (a_1, \ldots, a_{m-1}) \in C_m$,

$$g(S) = \|x\|_1 - \tfrac{1}{2}m(m-1), \qquad F(S) = \|x\|_\infty - m,$$

and $\#$ generators k is determined by the face $F \subseteq \mathcal{C}_m$ containing x.

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Theorem (Bruns-García-Sánchez-O.-Wilburne, 2020)

Wilf's conjecture holds for all numerical semigroups S with $m \le 18$.

A couple of long-standing (**hard**) conjectures

Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \le k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

Bras-Amoros Conjecture

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Conjecture (Kaplan)

For fixed m, the number of numerical semigroups g gaps is non-decreasing.

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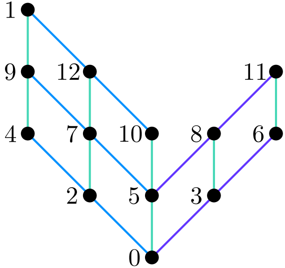

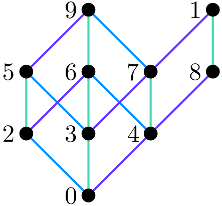

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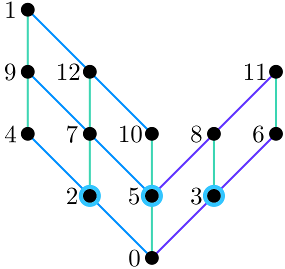

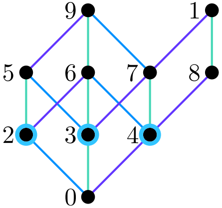

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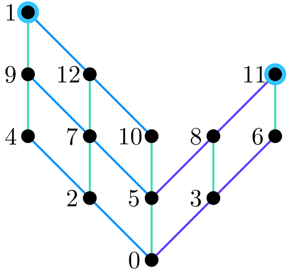

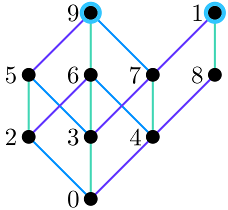

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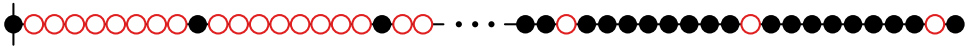
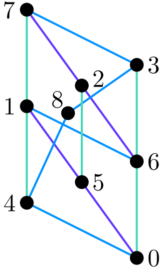
$S = \langle 9, 40, 50, 60 \rangle$





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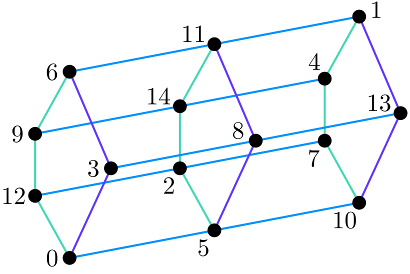

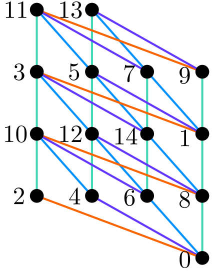

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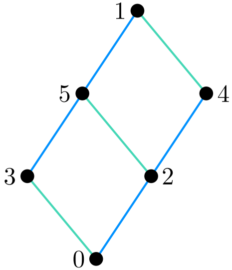

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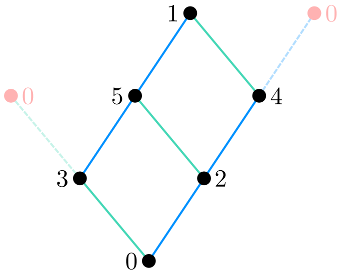

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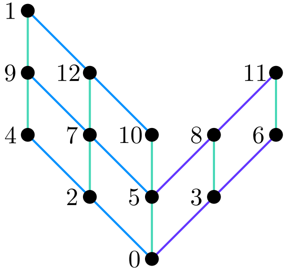

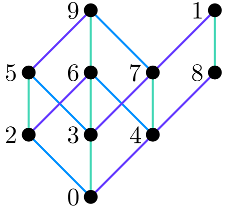

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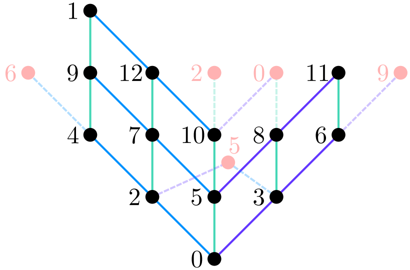

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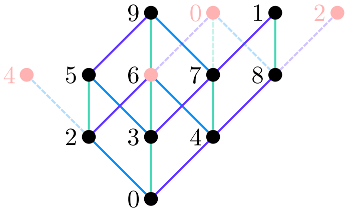
$S = \langle 10, a_2, a_3, a_4 \rangle$

$$I_S = \langle\, x_2^2 - y^* x_4,\; x_2 x_4 - x_3^2,\; x_3^2 x_4 - y^*,\; x_4^3 - y^* x_2 \,\rangle \subseteq \Bbbk[y, x_2, x_3, x_4]$$



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$$
\begin{array}{c}
\begin{array}{c|cccccccc}
 & \mathbf{1,12} & \mathbf{1,13} & \mathbf{2,12} & \mathbf{2,23} & \mathbf{3,13} & \mathbf{3,23} & \mathbf{2,13} & \mathbf{3,12} \\
\hline
\mathbf{1,1} & -x_2 & -x_3 & & & & & y^* & y^* \\
\mathbf{2,2} & -y^* & & x_1 & -x_3 & & y^* & & \\
\mathbf{3,3} & & & & & x_1 & x_2 & -y^* & \\
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0 \leftarrow R \leftarrow R^6 \longleftarrow R^8 \longleftarrow R^3 \leftarrow 0
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$$
0 \leftarrow R \xleftarrow{\begin{array}{cc} & \begin{array}{cccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \emptyset & \left[\begin{array}{cccc} y & x_1 & x_2 & x_3 \end{array}\right] \end{array}} R^4 \xleftarrow{\begin{array}{cc} & \begin{array}{cccccccccccc} \mathbf{01} & \mathbf{02} & \mathbf{03} & \mathbf{11} & \mathbf{12} & \mathbf{13} & \mathbf{21} & \mathbf{22} & \mathbf{23} & \mathbf{31} & \mathbf{32} & \mathbf{33} \end{array} \\ \begin{array}{c} \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} & \left[\begin{array}{cccccccccccc} x_1 & x_2 & x_3 & & & -y^* & & -y^* & & -y^* & & \\ -y & & & x_1 & x_2 & x_3 & & & -y^* & & -y^* & \\ & -y & & -y^* & & & x_1 & x_2 & x_3 & & & -y^* \\ & & -y & & -y^* & & -y^* & & & x_1 & x_2 & x_3 \end{array}\right] \end{array}} R^{12}
$$

$$
\longleftarrow R^{36} \longleftarrow R^{108} \longleftarrow R^{324} \longleftarrow R^{972} \longleftarrow R^{2916} \longleftarrow \cdots
$$

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Fix a numerical semigroup S with

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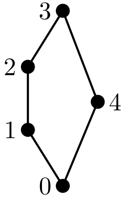

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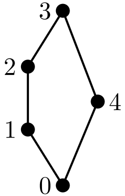

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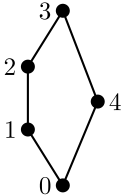

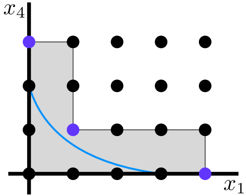

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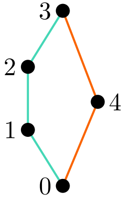

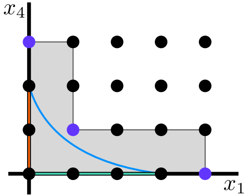

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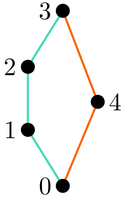

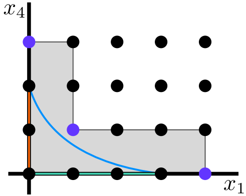

$$I_S + \langle y \rangle = \langle x_1^4, x_1x_4, x_4^3, y, \quad x_1^3 - x_4^2, \quad x_1^2 - x_2, x_1^3 - x_3 \rangle$$

$$= (\text{Artinian monomial ideal}) + (\text{binomials under staircase})$$

A commutative algebra view of Kunz posets

Fix a numerical semigroup S with

$$\mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\}, \qquad a_i \equiv i \bmod m$$

Defining toric ideal: $I_S = \ker(\Bbbk[y, x_1, \ldots, x_{m-1}] \to \Bbbk[t])$

$$y \mapsto t^m \qquad x_i \mapsto t^{a_i}$$

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The *Apéry resolution* for I_S, minimal if and only if S is MED:

$$0 \leftarrow R \xleftarrow{\begin{array}{c|cccccc} & \mathbf{1,1} & \mathbf{2,2} & \mathbf{3,3} & \mathbf{2,1} & \mathbf{3,1} & \mathbf{3,2} \\ \varnothing & x_1^2 - x_2 y^* & x_2^2 - y^* & x_3^2 - x_2 y^* & x_1x_2 - x_3 y^* & x_1x_3 - y^* & x_2x_3 - x_1 y^* \end{array}} \cdots$$

$$R^6 \xleftarrow{\begin{array}{c|cccccccc} & \mathbf{1,12} & \mathbf{1,13} & \mathbf{2,12} & \mathbf{2,23} & \mathbf{3,13} & \mathbf{3,23} & \mathbf{2,13} & \mathbf{3,12} \\ \hline \mathbf{1,1} & -x_2 & -x_3 & & & & & y^* & y^* \\ \mathbf{2,2} & -y^* & & x_1 & -x_3 & & y^* & & \\ \mathbf{3,3} & & & & & x_1 & x_2 & -y^* & \\ \mathbf{2,1} & x_1 & & -x_2 & y^* & y^* & & -x_3 & \\ \mathbf{3,1} & y^* & x_1 & & & -x_3 & -y^* & & -x_2 \\ \mathbf{3,2} & & -y^* & -y^* & x_2 & & -x_3 & x_1 & x_1 \end{array}} R^8 \xleftarrow{\begin{array}{c|ccc} & \mathbf{1,[3]} & \mathbf{2,[3]} & \mathbf{3,[3]} \\ \hline \mathbf{1,12} & x_3 & -y^* & \\ \mathbf{1,13} & -x_2 & & y^* \\ \mathbf{2,12} & & x_3 & -y^* \\ \mathbf{2,23} & -y^* & x_1 & \\ \mathbf{3,13} & y^* & & -x_2 \\ \mathbf{3,23} & & -y^* & x_1 \\ \mathbf{2,13} & x_1 & -x_2 & \\ \mathbf{3,12} & -x_1 & & x_3 \end{array}} R^3 \leftarrow 0$$

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$$R^6 \xleftarrow{\begin{array}{c|cccccccc} & \mathbf{1,12} & \mathbf{1,13} & \mathbf{2,12} & \mathbf{2,23} & \mathbf{3,13} & \mathbf{3,23} & \mathbf{2,13} & \mathbf{3,12} \\ \hline \mathbf{1,1} & -x_2 & -x_3 & & & & & y^* & y^* \\ \mathbf{2,2} & -y^* & & x_1 & -x_3 & & y^* & & \\ \mathbf{3,3} & & & & & x_1 & x_2 & -y^* & \\ \mathbf{2,1} & x_1 & & -x_2 & y^* & y^* & & -x_3 & \\ \mathbf{3,1} & y^* & x_1 & & & -x_3 & -y^* & & -x_2 \\ \mathbf{3,2} & & -y^* & -y^* & x_2 & & -x_3 & x_1 & x_1 \end{array}} R^8 \xleftarrow{\begin{array}{c|ccc} & \mathbf{1,[3]} & \mathbf{2,[3]} & \mathbf{3,[3]} \\ \hline \mathbf{1,12} & x_3 & -y^* & \\ \mathbf{1,13} & -x_2 & & y^* \\ \mathbf{2,12} & & x_3 & -y^* \\ \mathbf{2,23} & -y^* & x_1 & \\ \mathbf{3,13} & y^* & & -x_2 \\ \mathbf{3,23} & & -y^* & x_1 \\ \mathbf{2,13} & x_1 & -x_2 & \\ \mathbf{3,12} & -x_1 & & x_3 \end{array}} R^3 \leftarrow 0$$

Can "specialize" to a minimal resolution, consistent across the face of C_m

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- "the same resolution" vs. "structure consistent across the face"

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Words of caution:

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- "up to isomorphism" isn't good enough

  $S = \langle 4, 9, 10, 11 \rangle$:

  $$\begin{aligned} I_S &= \langle x_1^2 - x_2 y^2, \underline{x_2^2 - y^5}, x_3^2 - x_2 y^3, x_1x_2 - x_3 y^2, \underline{x_1x_3 - y^5}, x_2x_3 - x_1 y^3 \rangle \\ &= \langle x_1^2 - x_2 y^2, x_2^2 - y^5, x_3^2 - x_2 y^3, x_1x_2 - x_3 y^2, x_1x_3 - x_2^2, x_2x_3 - x_1 y^3 \rangle \end{aligned}$$

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  $S = \langle 4, 9, 10, 11 \rangle$:
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- watch the number of variables

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- watch the number of variables

  $S = \langle 4, 5, 7 \rangle$:

  $$\begin{aligned} I_S &= \langle x_1^3 - x_3y^2, x_1x_3 - y^3, x_3^2 - x_1^2y, x_1^2 - x_2 \rangle \subseteq \Bbbk[y, x_1, x_2, x_3] \\ J_S &= \langle x_1^3 - x_3y^2, x_1x_3 - y^3, x_3^2 - x_1^2y \rangle \subseteq \Bbbk[y, x_1, x_3] \end{aligned}$$

A commutative algebra view of Kunz posets

Fix a numerical semigroup $S = \langle m, n_1, \ldots, n_k \rangle$ with

$$\mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\}, \qquad a_i \equiv i \bmod m$$

Defining toric ideal: $I_S = \ker(\mathbb{k}[y, x_1, \ldots, x_{m-1}] \to \mathbb{k}[t])$

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A commutative algebra view of Kunz posets

Fix a numerical semigroup $S = \langle m, n_1, \ldots, n_k \rangle$ with

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Defining toric ideal: $I_S = \ker(\Bbbk[y, x_1, \ldots, x_{m-1}] \to \Bbbk[t])$

$$y \mapsto t^m \qquad x_i \mapsto t^{a_i}$$

The *infinite Apéry resolution* of $\Bbbk$ over $R = \Bbbk[S]$:

$$0 \leftarrow R \xleftarrow{\begin{array}{c|cccc} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \hline \emptyset & y & x_1 & x_2 & x_3 \end{array}} R^4 \leftarrow$$

$$\begin{array}{c|cccccccccccc} & \mathbf{01} & \mathbf{02} & \mathbf{03} & \mathbf{11} & \mathbf{12} & \mathbf{13} & \mathbf{21} & \mathbf{22} & \mathbf{23} & \mathbf{31} & \mathbf{32} & \mathbf{33} \\ \hline \mathbf{0} & x_1 & x_2 & x_3 & & & -y^{b_{13}} & & -y^{b_{22}} & & -y^{b_{13}} & & \\ \mathbf{1} & -y & & & x_1 & x_2 & x_3 & & & -y^{b_{23}} & & -y^{b_{23}} & \\ \mathbf{2} & & -y & & -y^{b_{11}} & & & x_1 & x_2 & x_3 & & & -y^{b_{33}} \\ \mathbf{3} & & & -y & & -y^{b_{12}} & & -y^{b_{12}} & & & x_1 & x_2 & x_3 \end{array}$$

$$R^{12} \leftarrow$$

$$\begin{array}{c|cccccccccccc} & \mathbf{01*} & \mathbf{02*} & \mathbf{03*} & \mathbf{11*} & \mathbf{12*} & \mathbf{13*} & \mathbf{21*} & \mathbf{22*} & \mathbf{23*} & \mathbf{31*} & \mathbf{32*} & \mathbf{33*} \\ \hline \mathbf{0*} & & & & & & y^{b_{13}} I & & y^{b_{22}} I & & y^{b_{13}} I & & \\ \mathbf{1*} & yI & & & & & & & & y^{b_{23}} I & & y^{b_{23}} I & \\ \mathbf{2*} & & yI & & y^{b_{11}} I & & & & & & & & y^{b_{33}} I \\ \mathbf{3*} & & & yI & & y^{b_{12}} I & & y^{b_{12}} I & & & & & \end{array}$$

$$R^{36} \leftarrow \cdots$$

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Thanks!