Classifying numerical semigroups using polyhedral geometry

Christopher O'Neill

San Diego State University

cdoneill@sdsu.edu

Joint with (i) Winfred Bruns, Pedro García-Sánchez, Dane Wilbourne; (ii) Nathan Kaplan;
(iii) J. Autry, *A. Ezell, *T. Gomes, *C. Preuss, *T. Saluja, *E. Torres Dávila
(iv) B. Braun, T. Gomes, E. Miller, C. O'Neill, and A. Sobieska

* = undergraduate student

Slides available: `https://cdoneill.sdsu.edu/`

May 8, 2024

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$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{c} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

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Multiplicity: $\mathrm{m}(S) =$ smallest nonzero element

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Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

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The Apéry set is a "one stop shop" for computation.

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Theorem

If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup S with $\mathrm{Ap}(S) = A$ if and only if

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Big idea: the inequalities "$a_i + a_j \geq a_{i+j}$" to define a **cone** C_m.

Kunz cone

Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

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$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

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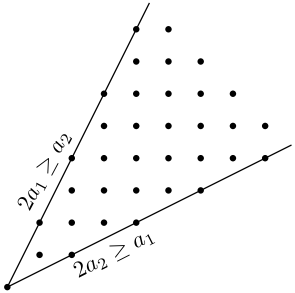

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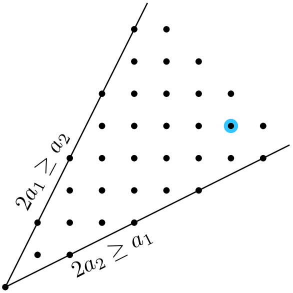

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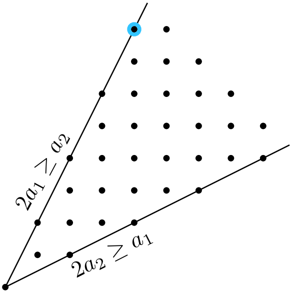

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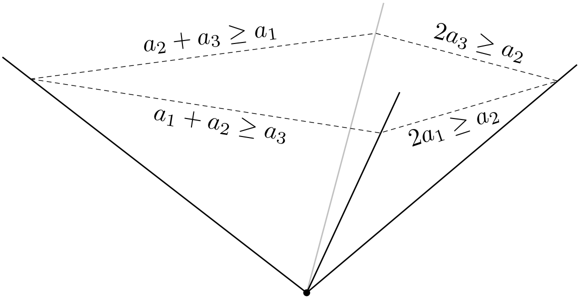

Faces of the Kunz cone

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When are numerical semigroups in (the relative interior of) the same face?

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Big picture: "moduli space" approach for studying XYZ's

- Define a space with XYZ's as points
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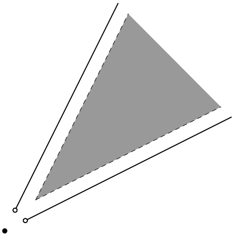
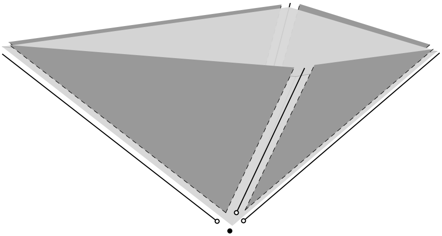
More interesting example: C_m





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$$S = \langle m, a_1, \ldots, a_{m-1} \rangle \quad \text{where} \quad \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\}$$

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Example: $S = \langle 4, 10, 11, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 11\}$$
$$a_1 = 13, \quad a_2 = 10, \quad a_3 = 11$$

$$2a_1 > a_2 \qquad a_1 + a_2 > a_3$$
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Faces of the Kunz cone

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Definition

The *Apéry poset* of S: define $a \preceq a'$ whenever $a' - a \in S$.

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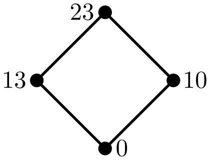

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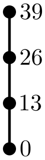

Faces of the Kunz polyhedron

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Faces of the Kunz polyhedron

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When are numerical semigroups in (the relative interior of) the same face?

$$S = \langle 6, 9, 20 \rangle$$
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$$S' = \langle 6, 26, 27 \rangle$$
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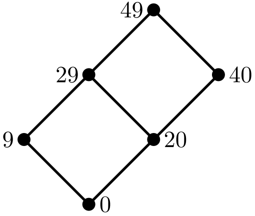

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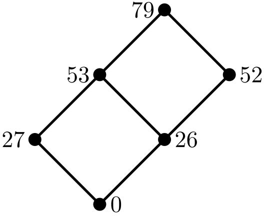

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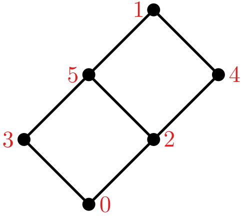

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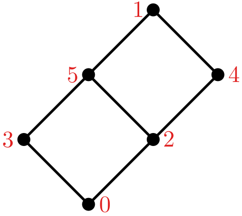

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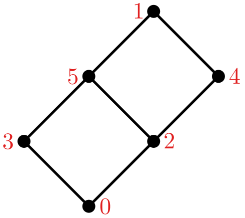

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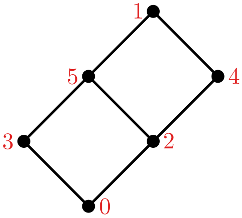

The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

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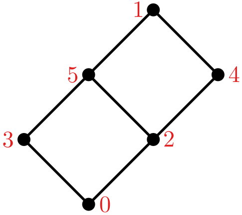

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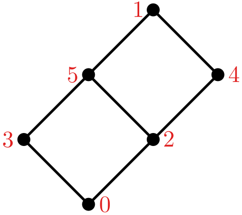

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Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

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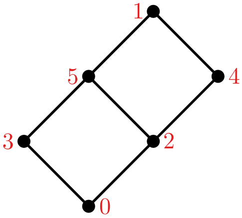

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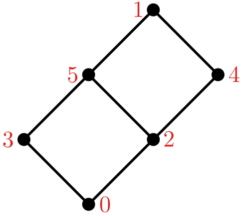

Defining facet equations:

$$2a_2 = a_4$$
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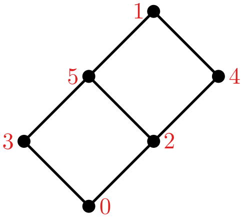

Defining facet equations:

$2a_2 = a_4$	$2 \preceq 4$
$a_2 + a_3 = a_5$	$2 \preceq 5$
	$3 \preceq 5$
$a_2 + a_5 = a_1$	$2 \preceq 1$
	$5 \preceq 1$
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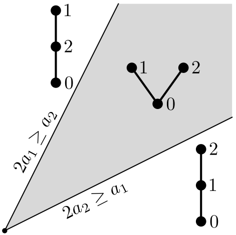
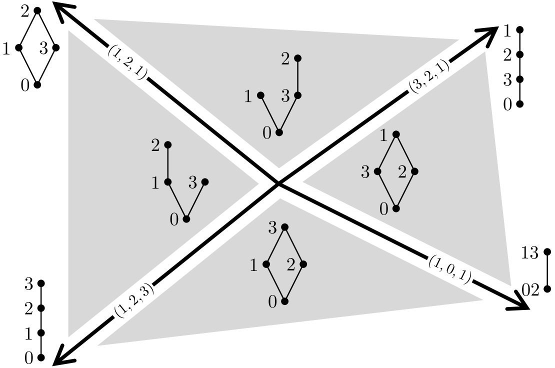
C_3 and C_4



1
2
0
1
2
0
$2a_1 \geq a_2$
$2a_2 \geq a_1$
2
1
0



2
1
3
0
(1, 2, 1)
2
1
3
0
(3, 2, 1)
1
2
3
0
2
1
3
0
1
3
2
0
3
1
2
0
(1, 0, 1)
13
02
3
2
1
0
(1, 2, 3)

A couple of long-standing (**hard**) conjectures

Genus $g = \mathrm{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$: number of "gaps" of S.

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Example: $n_3 = 4$

$$
\begin{aligned}
\langle 2, 7 \rangle &= \{0, \quad 2, \quad 4, \quad 6, 7, 8, \ldots\} \\
\langle 3, 4 \rangle &= \{0, \qquad 3, 4, \quad 6, 7, 8, \ldots\} \\
\langle 3, 5, 7 \rangle &= \{0, \qquad 3, \quad 5, 6, 7, 8, \ldots\} \\
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Suspected: $n_g \geq n_{g-1} + n_{g-2}$ for all g (verified for $g \leq 70$)

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Not true for $n'_f = \#$ of numerical semigroups with Frobenius number f
$n'_{11} = 51 \qquad n'_{12} = 40 \qquad n'_{13} = 106$

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Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathsf{F}(S) + 1 \le k(\mathsf{F}(S) + 1 - \mathsf{g}(S))$.

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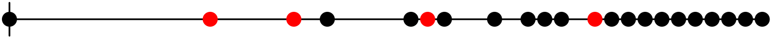

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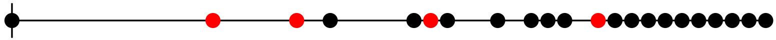

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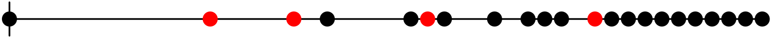
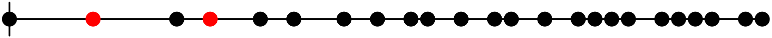


Equality holds when:

- $S = \langle a, b \rangle$



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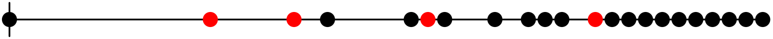

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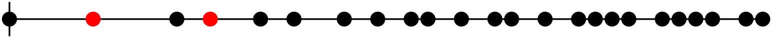

- $S = \langle m, m + 1, \ldots, 2m - 1 \rangle$

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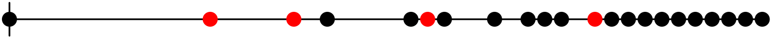

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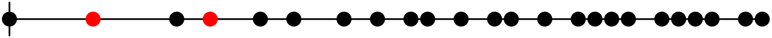

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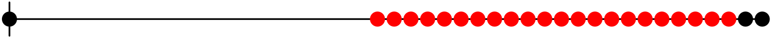

Proved in many special cases, including $\mathrm{g}(S) \le 60$.

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Direct ties to geometry: if S corresponds to $x = (a_1, \ldots, a_{m-1}) \in C_m$,

$$g(S) = \|x\|_1 - \tfrac{1}{2}m(m-1), \qquad F(S) = \|x\|_\infty - m,$$

and $\#$ generators k is determined by the face $F \subseteq \mathcal{C}_m$ containing x.

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For all g, we have $n_g \ge n_{g-1}$.

Direct ties to geometry: if S corresponds to $x = (a_1, \ldots, a_{m-1}) \in C_m$,

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A couple of long-standing (**hard**) conjectures

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Conjecture (Kaplan)

For fixed m, the number of numerical semigroups g gaps is non-decreasing.

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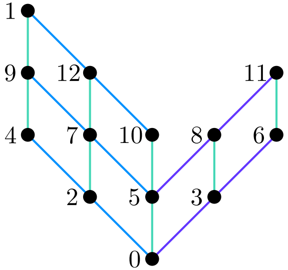

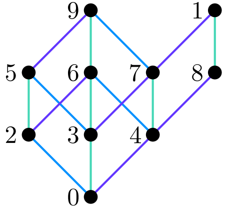

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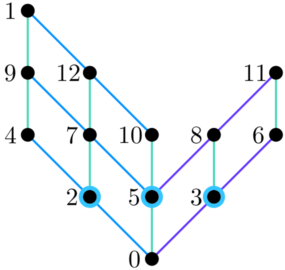

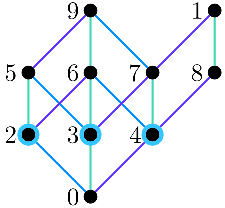

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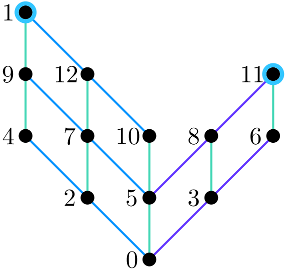

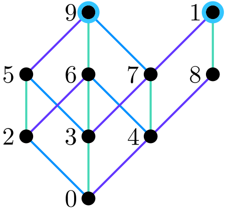

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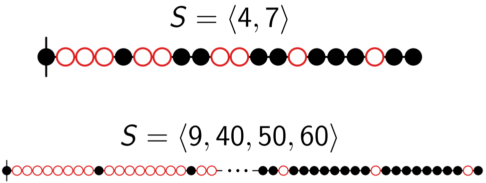

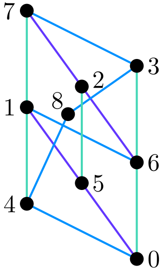

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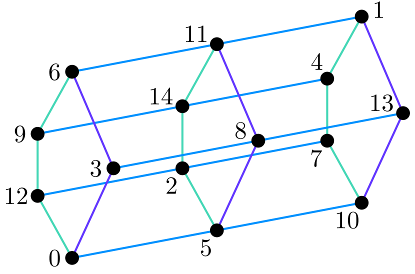

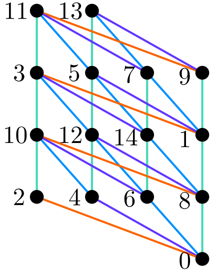

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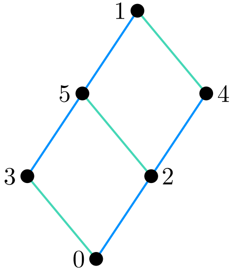

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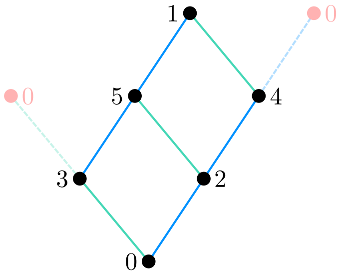

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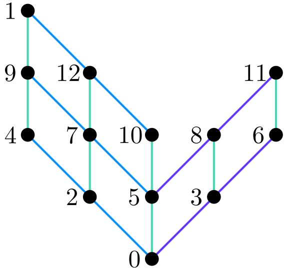

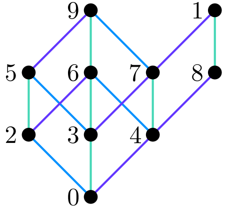

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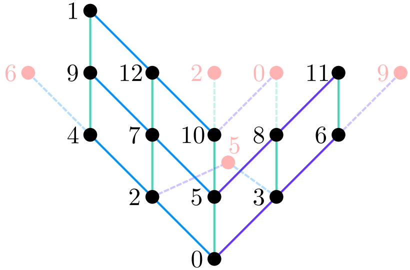

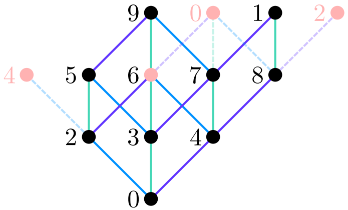

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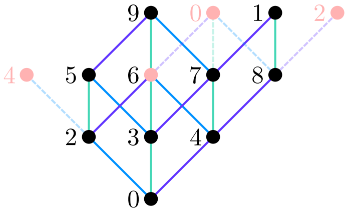
$S = \langle 10, a_2, a_3, a_4 \rangle$

$$\begin{aligned} I_S = \langle\, & x_2^2 - y^* x_4,\ \ x_2 x_4 - x_3^2, \\ & x_3^2 x_4 - y^*, \quad x_4^3 - y^* x_2 \,\rangle \\ \subseteq\ & \Bbbk[y, x_2, x_3, x_4] \end{aligned}$$



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	1,12	1,13	2,12	2,23	3,13	3,23	2,13	3,12
1,1	$-x_2$	$-x_3$					y^*	y^*
2,2	$-y^*$		x_1	$-x_3$		y^*		
3,3					x_1	x_2	$-y^*$	
2,1	x_1		$-x_2$	y^*	y^*		$-x_3$	
3,1	y^*	x_1			$-x_3$	$-y^*$		$-x_2$
3,2		$-y^*$	$-y^*$	x_2		$-x_3$	x_1	x_1

	1,[3]	2,[3]	3,[3]
1,12	x_3	$-y^*$	
1,13	$-x_2$		y^*
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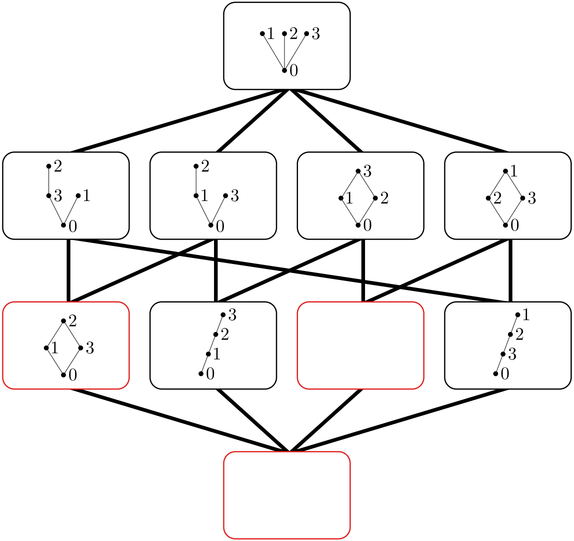
$$0 \leftarrow R \xleftarrow{\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \emptyset & y & x_1 & x_2 & x_3 \end{array}} R^4 \xleftarrow{\begin{array}{c|cccccccccccc} & 01 & 02 & 03 & 11 & 12 & 13 & 21 & 22 & 23 & 31 & 32 & 33 \\ \hline 0 & x_1 & x_2 & x_3 & & & -y^* & & -y^* & & -y^* & & \\ 1 & -y & & & x_1 & x_2 & x_3 & & & -y^* & & -y^* & \\ 2 & & -y & & -y^* & & & x_1 & x_2 & x_3 & & & -y^* \\ 3 & & & -y & & -y^* & & -y^* & & & x_1 & x_2 & x_3 \end{array}} R^{12}$$

$$\longleftarrow R^{36} \longleftarrow R^{108} \longleftarrow R^{324} \longleftarrow R^{972} \longleftarrow R^{2916} \longleftarrow \cdots$$

Posets for the poset-less faces



Face lattice of C_4

Posets for the poset-less faces

Theorem

If $d \mid m$, then there exists a map $C_d \hookrightarrow C_m$ that induces a dimension-preserving injection on face lattices.
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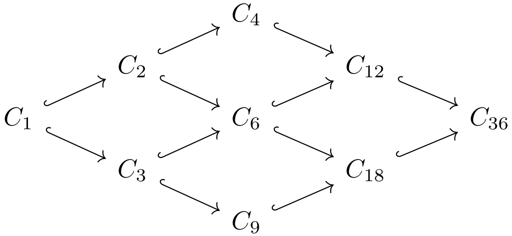

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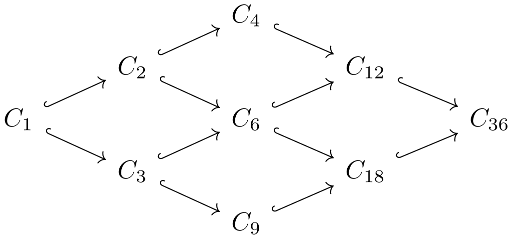
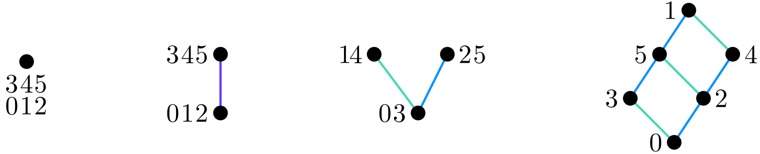


Takeaway: each $F \subseteq C_m$ has a Kunz poset; elements are *cosets* of $H \subseteq \mathbb{Z}_m$



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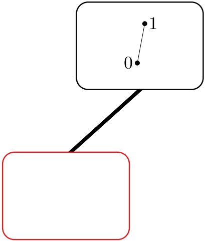
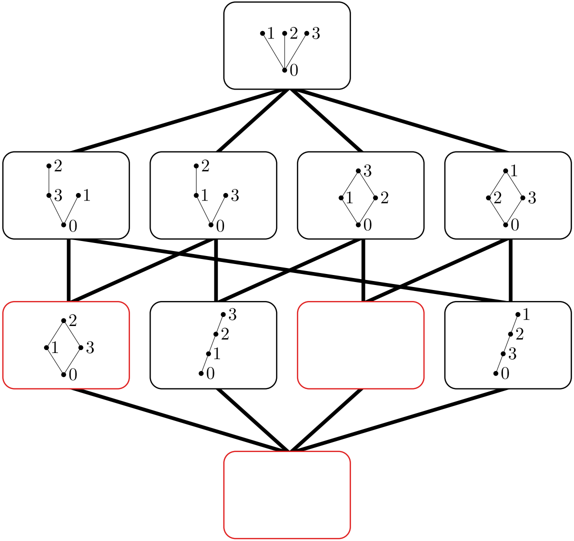


Face lattice of C_2



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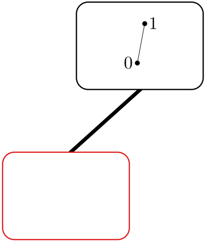

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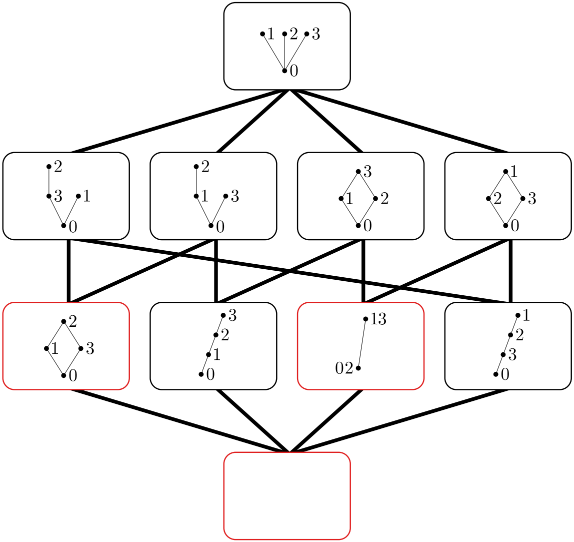

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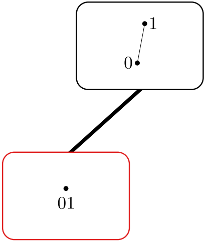

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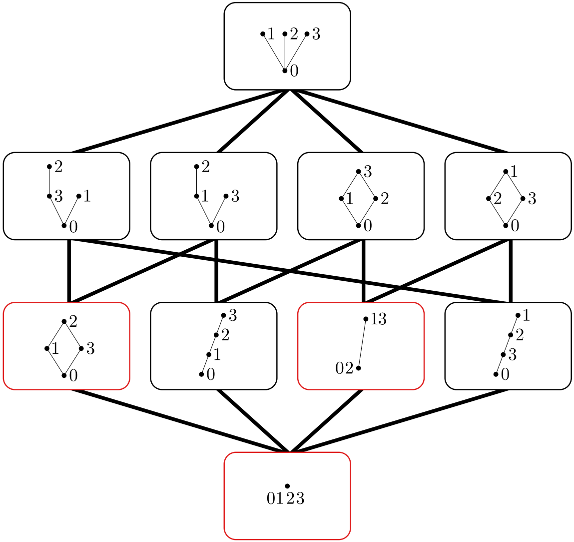

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Gluings and complete intersections

The *gluing* of $S = \langle n_1, \ldots, n_k \rangle$, $S' = \langle n_1', \ldots, n_\ell' \rangle$ by $a, b \in \mathbb{Z}_{\geq 0}$:

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$$\begin{aligned} \langle 70, 105, 112, 150, 200 \rangle &= 7\langle 10, 15, 16 \rangle + 50\langle 3, 4 \rangle \\ &= 7\Big(5\langle 2, 3 \rangle + \langle 16 \rangle\Big) + 50\langle 3, 4 \rangle \end{aligned}$$

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The *gluing* of $S = \langle n_1, \ldots, n_k \rangle$, $S' = \langle n'_1, \ldots, n'_\ell \rangle$ by $a, b \in \mathbb{Z}_{\geq 0}$:

$$T = aS + bS' = \langle an_1, \ldots, an_k, bn'_1, \ldots, bn'_\ell \rangle$$

Requirements: $a \in S', b \in S$ non-generators with $\gcd(a, b) = 1$.

$$\langle 55, 66, 77, 100, 150 \rangle = 11\langle 5, 6, 7 \rangle + 50\langle 2, 3 \rangle$$

Monoscopic gluings: $S' = \langle 1 \rangle$

$$\langle 10, 12, 14, 15 \rangle = 2\langle 5, 6, 7 \rangle + \langle 15 \rangle = 2\langle 5, 6, 7 \rangle + 15\langle 1 \rangle$$

Complete intersections: gluing from the ground up

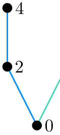
$$\begin{aligned} \langle 70, 105, 112, 150, 200 \rangle &= 7\langle 10, 15, 16 \rangle + 50\langle 3, 4 \rangle \\ &= 7\Big(5\langle 2, 3 \rangle + \langle 16 \rangle\Big) + 50\langle 3, 4 \rangle \\ &= 7\Big(5\Big(\langle 2 \rangle + \langle 3 \rangle\Big) + \langle 16 \rangle\Big) + 50\Big(\langle 3 \rangle + \langle 4 \rangle\Big) \end{aligned}$$

Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$ $\qquad$ $T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$

Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$



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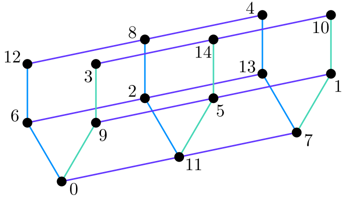

Gluings and complete intersections

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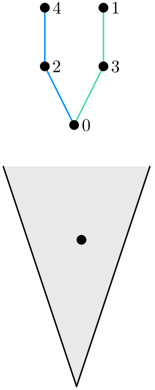

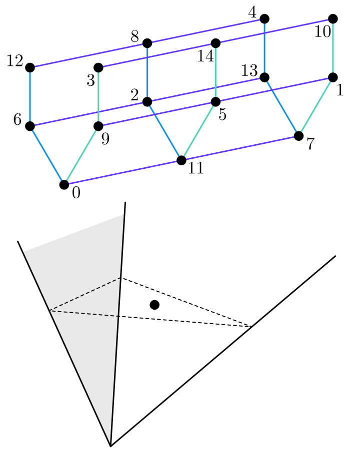

Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$

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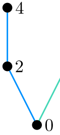

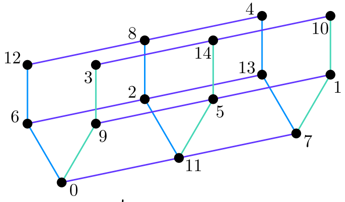

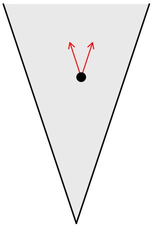

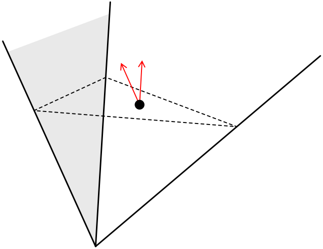

Gluings and complete intersections

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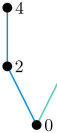
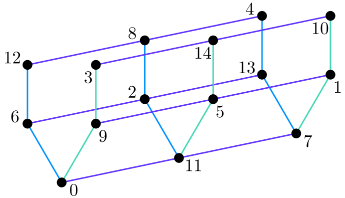
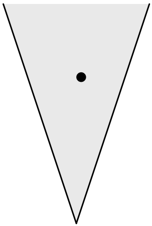
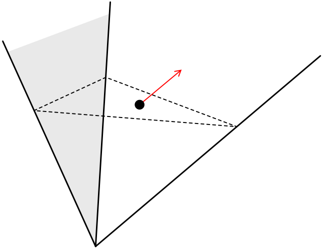


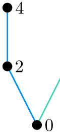


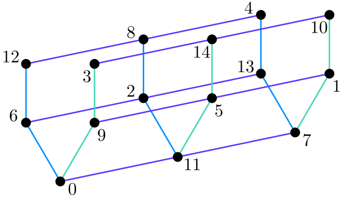


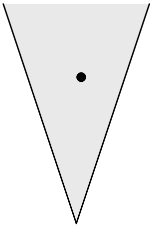


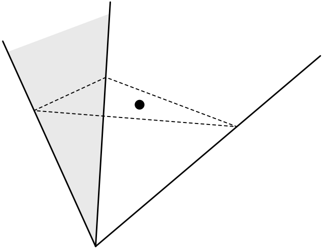
Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$ $T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$



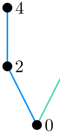


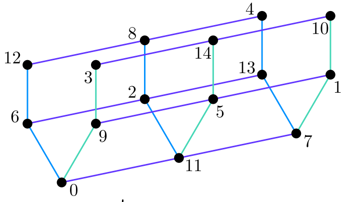


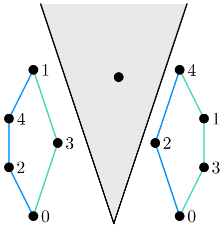


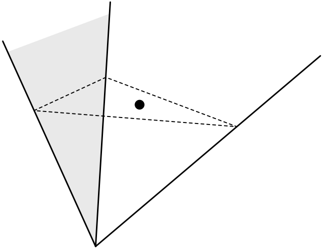
Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$ $\qquad$ $T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$



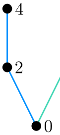


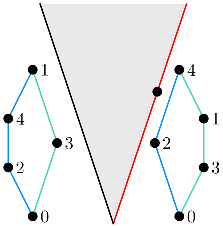




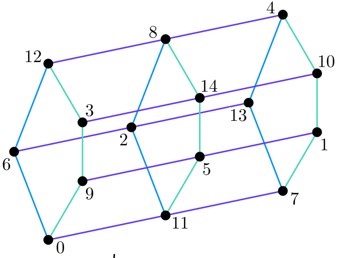
Gluings and complete intersections

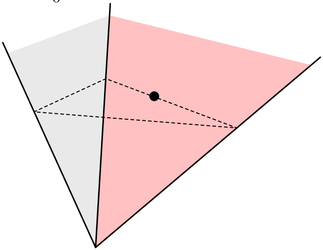
$S = \langle 5, 12, {\color{red}8} \rangle$





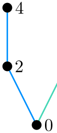
$T = 3{\color{red}S} + \langle 41 \rangle = \langle 15, 36, {\color{red}24}, 41 \rangle$

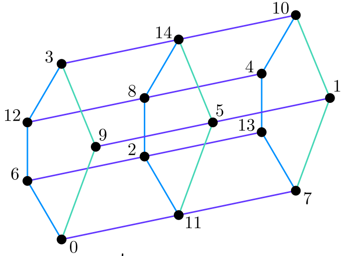


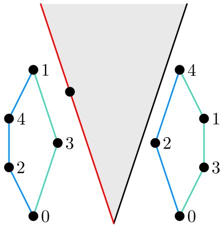


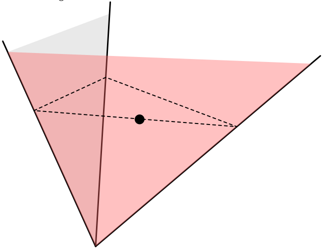
Gluings and complete intersections

$S = \langle 5, 12, {\color{red}18} \rangle$ $\qquad$ $T = 3{\color{red}S} + \langle 41 \rangle = \langle 15, 36, {\color{red}54}, 41 \rangle$



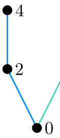


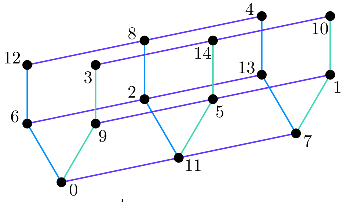


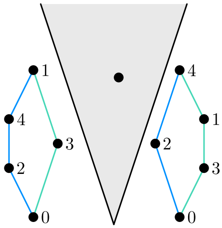


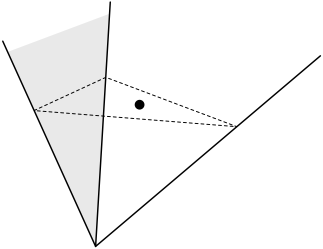
Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$ $\qquad$ $T = 3S + \langle 41 \rangle = \langle 15, 36, 39, 41 \rangle$



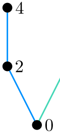


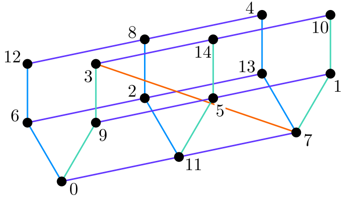


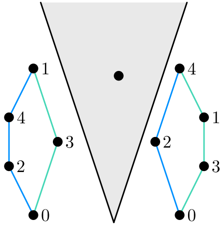


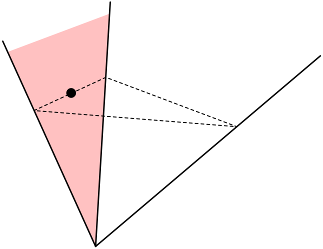
Gluings and complete intersections

$S = \langle 5, 12, 13 \rangle$ $\qquad$ $T = 3S + \langle 26 \rangle = \langle 15, 36, 39, 26 \rangle$





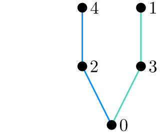


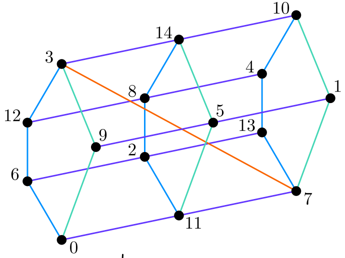


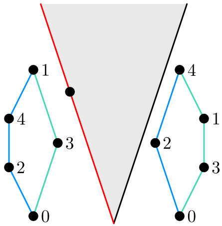
Gluings and complete intersections

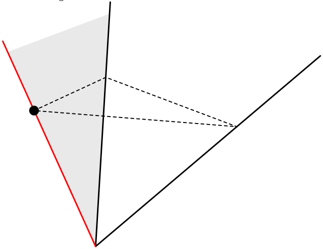
$S = \langle 5, 12, 13 \rangle$

$T = 3S + \langle \textcolor{red}{26} \rangle = \langle 15, 36, 39, \textcolor{red}{26} \rangle$





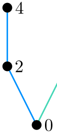


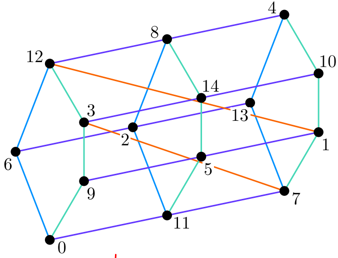


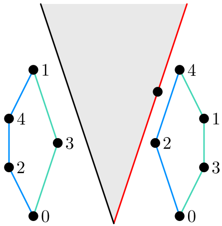
Gluings and complete intersections

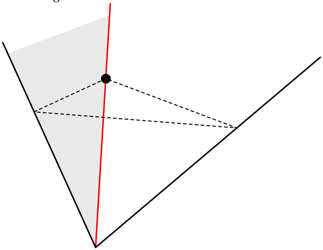
$S = \langle 5, 12, 13 \rangle$

$T = 3S + \langle 26 \rangle = \langle 15, 36, 39, 26 \rangle$









Gluings and complete intersections

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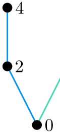

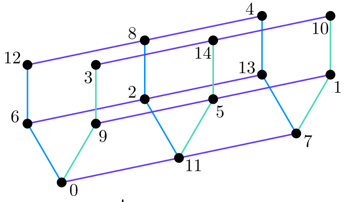

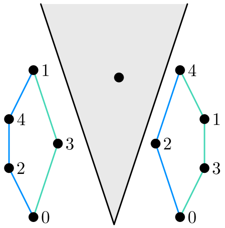

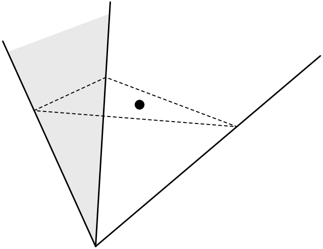

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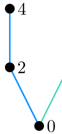

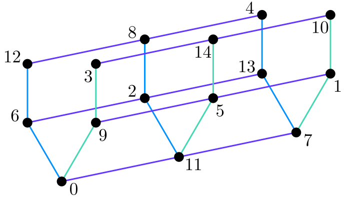

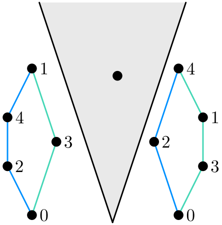

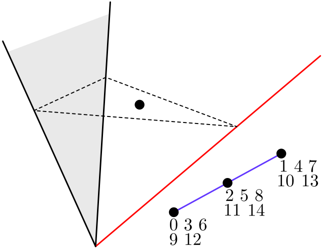

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Thanks!