Classifying numerical semigroups using polyhedral geometry

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Slides available: `https://cdoneill.sdsu.edu/`

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Example:

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

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Multiplicity: $\mathrm{m}(S) =$ smallest nonzero element

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Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

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The Apéry set is a "one stop shop" for computation.

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Theorem

If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup S with $\mathrm{Ap}(S) = A$ if and only if

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Big idea: the inequalities "$a_i + a_j \geq a_{i+j}$" to define a **cone** C_m.

Kunz cone

Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

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$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

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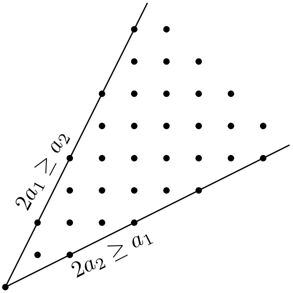

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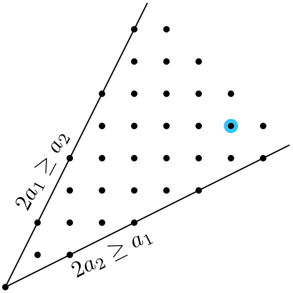

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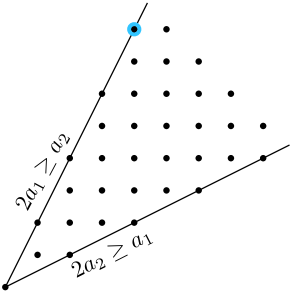

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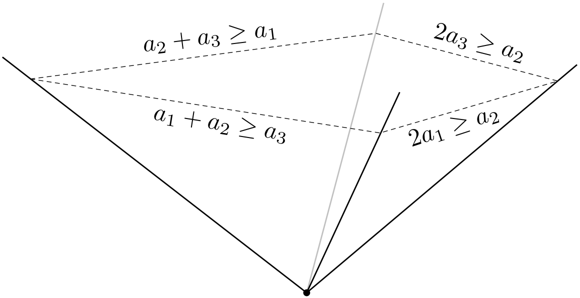

Faces of the Kunz cone

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When are numerical semigroups in (the relative interior of) the same face?

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Big picture: "moduli space" approach for studying XYZ's

- Define a space with XYZ's as points
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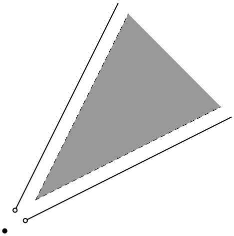
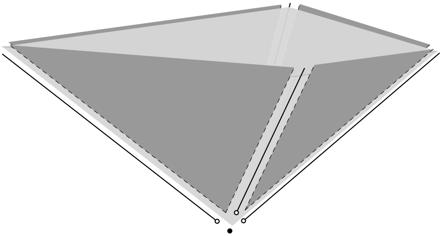
More interesting example: C_m





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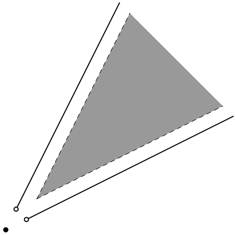

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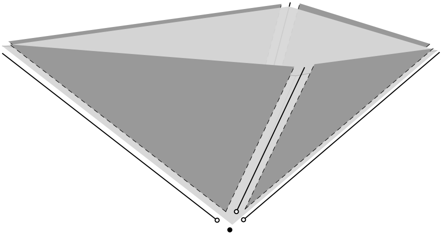

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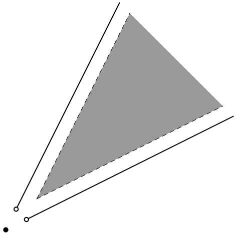

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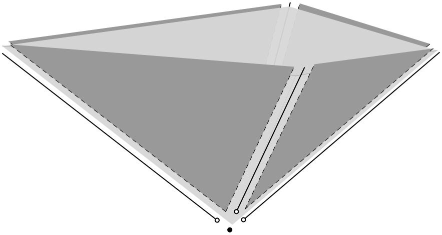

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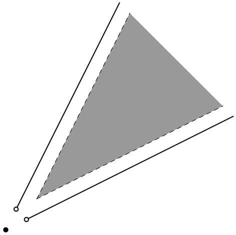

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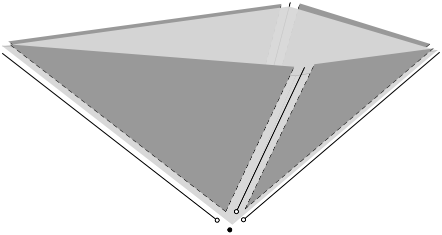

$C_5 \subseteq \mathbb{R}^4$? Cross section:

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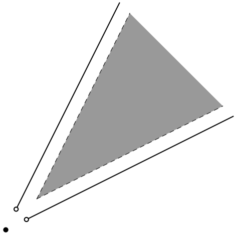

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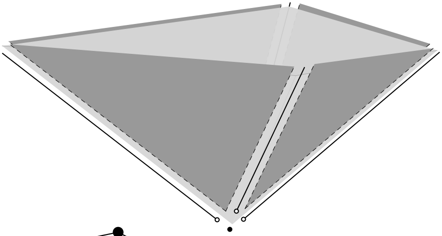
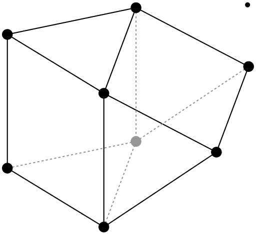


$C_5 \subseteq \mathbb{R}^4$? Cross section:



Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

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$$n_i \not\equiv n_j \bmod n_1 \quad \Longrightarrow \quad k \le \mathrm{m}(S)$$

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$$S = \langle m, a_1, \ldots, a_{m-1} \rangle \quad \text{where} \quad \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\}$$

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What about the other faces?

Faces of the Kunz cone

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Faces of the Kunz cone

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When are numerical semigroups in (the relative interior of) the same face?

Example: $S = \langle 4, 10, 11, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 11\}$$
$$a_1 = 13, \quad a_2 = 10, \quad a_3 = 11$$

$$2a_1 > a_2 \qquad a_1 + a_2 > a_3$$
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Example: $S = \langle 4, 10, 13 \rangle$

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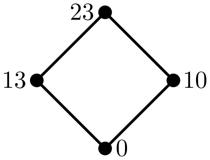
Definition

The *Apéry poset* of S: define $a \preceq a'$ whenever $a' - a \in S$.

$\mathrm{Ap}(S) = \{0, 13, 10, 23\}$



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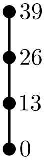

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Faces of the Kunz cone

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When are numerical semigroups in (the relative interior of) the same face?

$$S = \langle 6, 9, 20 \rangle \qquad S' = \langle 6, 26, 27 \rangle$$

$$\mathrm{Ap}(S) = \{0, 49, 20, \ 9, 40, 29\} \qquad \mathrm{Ap}(S') = \{0, 79, 26, 27, 52, 53\}$$

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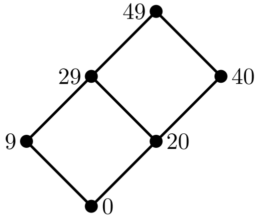

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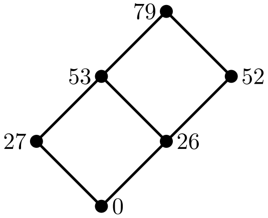

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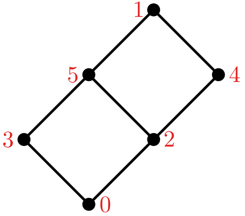

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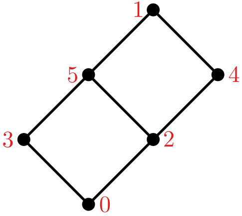

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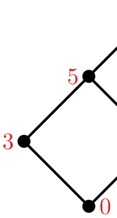

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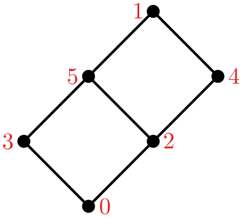

The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

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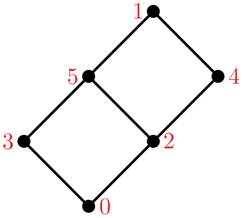

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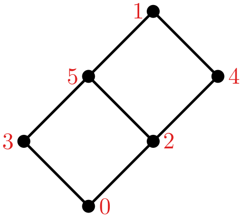

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Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

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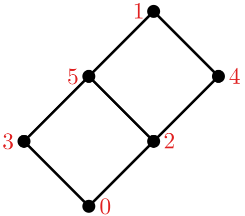

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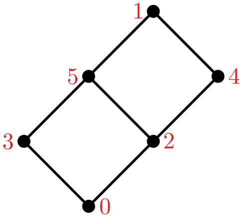

Defining facet equations:

$$2a_2 = a_4$$
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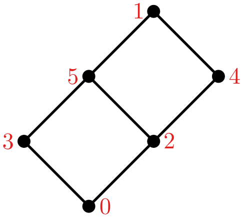

Defining facet equations:

$2a_2 = a_4$	$2 \preceq 4$
$a_2 + a_3 = a_5$	$2 \preceq 5$
	$3 \preceq 5$
$a_2 + a_5 = a_1$	$2 \preceq 1$
	$5 \preceq 1$
$a_3 + a_4 = a_1$	$3 \preceq 1$
	$4 \preceq 1$

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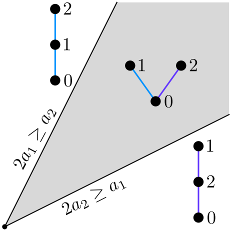
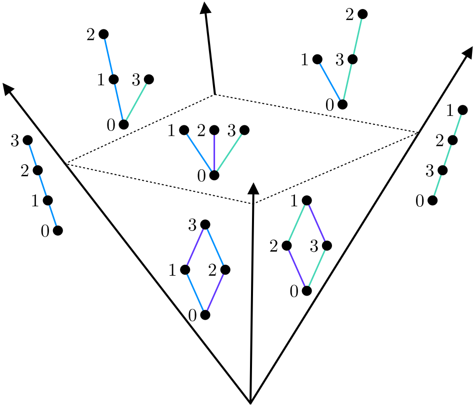
C_3 and C_4



2
1
0
1
2
0
$2a_1 \geq a_2$
$2a_2 \geq a_1$
1
2
0



2
1
3
0
2
1
3
0
3
2
1
0
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1
2
3
0

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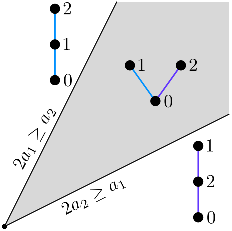

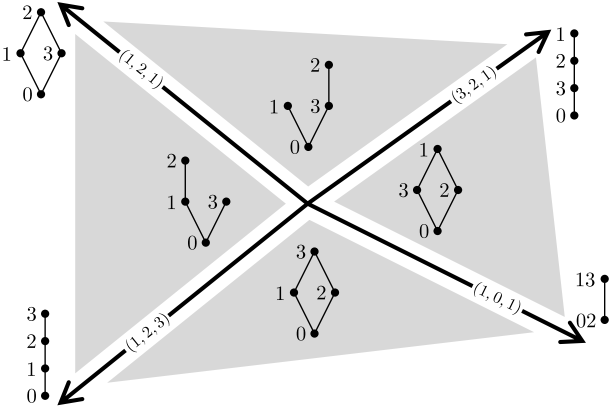

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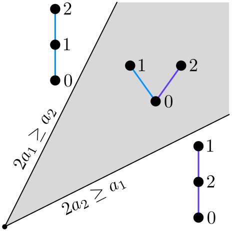

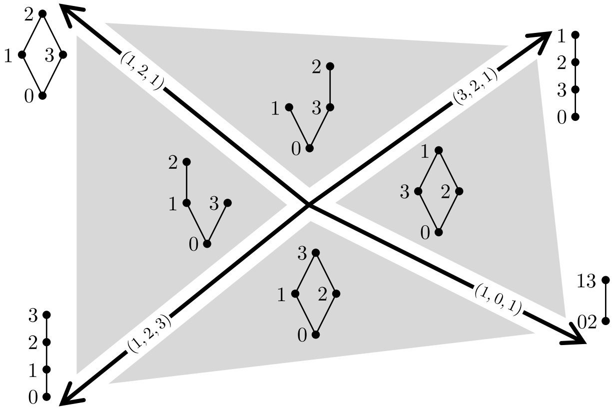

Theorem (Kaplan–O.)

There is a natural labeling of the faces of C_m by finite posets.

Shared properties within a face

What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

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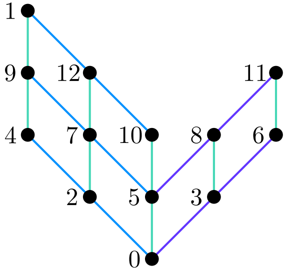

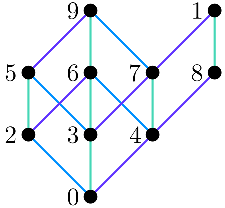

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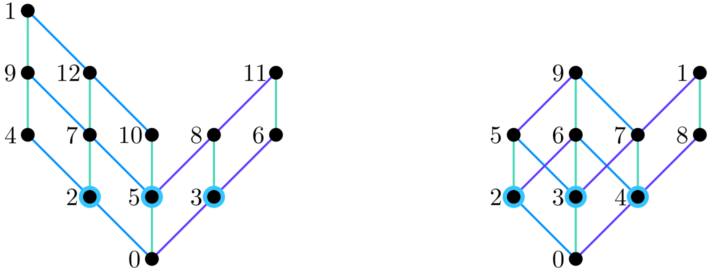

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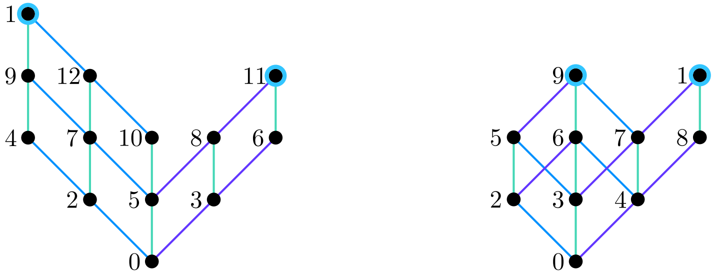

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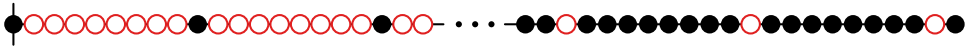
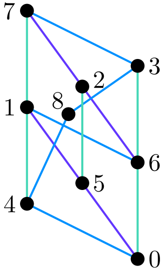
$S = \langle 4, 7 \rangle$

$S = \langle 9, 40, 50, 60 \rangle$





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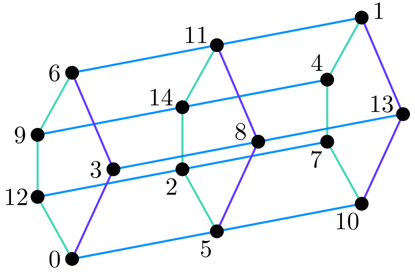

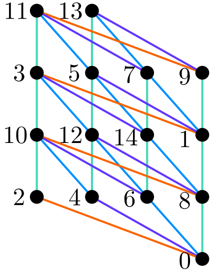

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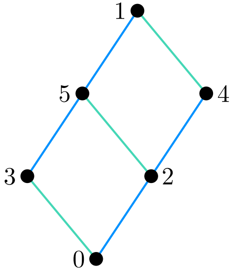
$$I_S = \ker\left(\Bbbk[\overline{x}] \to \Bbbk[t]\right) \quad x_i \mapsto t^{n_i}$$

$$S = \langle 6, 9, 20 \rangle$$

$$I_S = \langle x^3 - y^2, x^4y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$$



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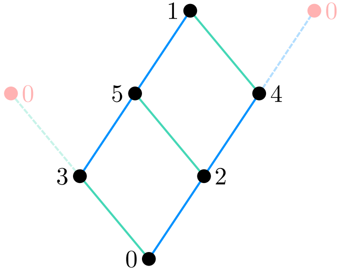

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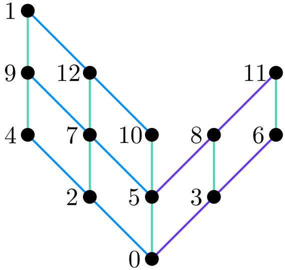

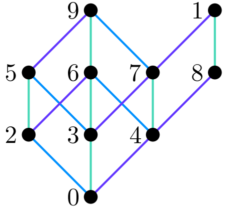

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What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

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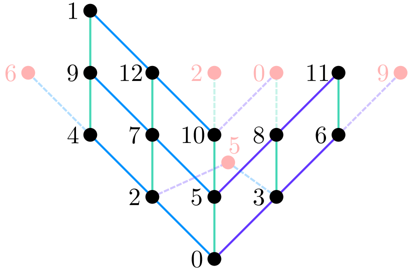

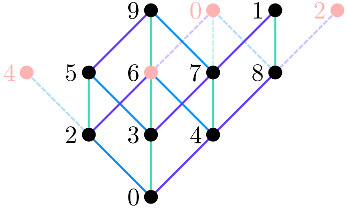

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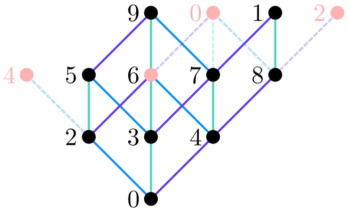
$S = \langle 10, a_2, a_3, a_4 \rangle$

$$I_S = \langle\, x_2^2 - y^* x_4,\ x_2x_4 - x_3^2,\ x_3^2x_4 - y^*,\ x_4^3 - y^* x_2 \,\rangle \subseteq \Bbbk[y, x_2, x_3, x_4]$$



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	1,12	1,13	2,12	2,23	3,13	3,23	2,13	3,12
1,1	$-x_2$	$-x_3$					y^*	y^*
2,2	$-y^*$		x_1	$-x_3$		y^*		
3,3					x_1	x_2	$-y^*$	
2,1	x_1		$-x_2$	y^*	y^*		$-x_3$	
3,1	y^*	x_1			$-x_3$	$-y^*$		$-x_2$
3,2		$-y^*$	$-y^*$	x_2		$-x_3$	x_1	x_1

	1,[3]	2,[3]	3,[3]
1,12	x_3	$-y^*$	
1,13	$-x_2$		y^*
2,12		x_3	$-y^*$
2,23	$-y^*$	x_1	
3,13	y^*		$-x_2$
3,23		$-y^*$	x_1
2,13	x_1	$-x_2$	
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$$0 \leftarrow R \xleftarrow{\;\begin{array}{c|cccc} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \emptyset & y & x_1 & x_2 & x_3 \end{array}\;} R^4 \xleftarrow{\;\begin{array}{c|cccccccccccc} & \mathbf{01} & \mathbf{02} & \mathbf{03} & \mathbf{11} & \mathbf{12} & \mathbf{13} & \mathbf{21} & \mathbf{22} & \mathbf{23} & \mathbf{31} & \mathbf{32} & \mathbf{33} \\ \mathbf{0} & x_1 & x_2 & x_3 & & & -y^* & & -y^* & & -y^* & & \\ \mathbf{1} & -y & & & x_1 & x_2 & x_3 & & & -y^* & & -y^* & \\ \mathbf{2} & & -y & & -y^* & & & x_1 & x_2 & x_3 & & & -y^* \\ \mathbf{3} & & & -y & & -y^* & & -y^* & & & x_1 & x_2 & x_3 \end{array}\;} R^{12}$$

$$\longleftarrow R^{36} \longleftarrow R^{108} \longleftarrow R^{324} \longleftarrow R^{972} \longleftarrow R^{2916} \longleftarrow \cdots$$

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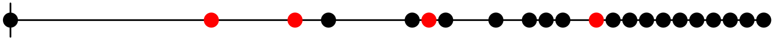

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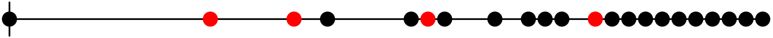
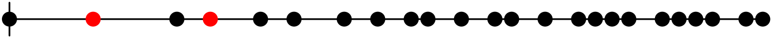


Equality holds when:

- $S = \langle a, b \rangle$



- $S = \langle m, m+1, \ldots, 2m-1 \rangle$

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If S corresponds to $x = (a_1, \ldots, a_{m-1}) \in C_m$,

$$\mathrm{g}(S) = \|x\|_1 - \tfrac{1}{2}m(m-1), \qquad \mathrm{F}(S) = \|x\|_\infty - m,$$

and # generators k is determined by the face $F \subseteq C_m$ containing x.

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\langle 2, 7 \rangle &= \{0, \quad 2, \quad 4, \quad 6, 7, 8, \ldots\} \\
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Not true for $n'_f = \#$ of numerical semigroups with Frobenius number f

$$n'_{11} = 51 \qquad n'_{12} = 40 \qquad n'_{13} = 106$$

References

W. Bruns, P. García-Sánchez, C. O'Neill, D. Wilburne (2020)
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