# Classifying numerical semigroups using polyhedral geometry

Christopher O'Neill

San Diego State University

*cdoneill@sdsu.edu*

Joint with (i) Winfred Bruns, Pedro García-Sánchez, Dane Wilbourne; (ii) Nathan Kaplan;
(iii) J. Autry, *A. Ezell, *T. Gomes, *C. Preuss, *T. Saluja, *E. Torres Dávila
(iv) B. Braun, T. Gomes, E. Miller, C. O'Neill, and A. Sobieska

* = undergraduate student

Slides available: `https://cdoneill.sdsu.edu/`

November 8, 2024

# Numerical semigroups

**Definition**

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

# Numerical semigroups

**Definition**

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example:

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

# Numerical semigroups

**Definition**

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example: "McNugget Semigroup"

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

# Numerical semigroups

## Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example: "McNugget Semigroup"

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Example: $S = \langle 6, 9, 18, 20, 32 \rangle$

# Numerical semigroups

## Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example: "McNugget Semigroup"

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Example: $S = \langle 6, 9, \cancel{18}, 20, \cancel{32} \rangle$

# Numerical semigroups

## Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example: “McNugget Semigroup”

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Example: $S = \langle 6, 9, \cancel{18}, 20, \cancel{32} \rangle = McN$

# Numerical semigroups

## Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example: "McNugget Semigroup"

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Example: $S = \langle 6, 9, \cancel{18}, 20, \cancel{32} \rangle = McN$

## Fact

Every numerical semigroup has a unique minimal generating set.

# Numerical semigroups

## Definition

A *numerical semigroup* $S \subseteq \mathbb{Z}_{\geq 0}$: closed under **addition**, $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Example: "McNugget Semigroup"

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

Example: $S = \langle 6, 9, \cancel{18}, 20, \cancel{32} \rangle = McN$

## Fact

Every numerical semigroup has a unique minimal generating set.

*Multiplicity*: $\mathrm{m}(S) =$ smallest nonzero element

# Apéry sets

Fix a numerical semigroup $S$ with $\mathrm{m}(S) = m$.

# Apéry sets

Fix a numerical semigroup $S$ with $\mathrm{m}(S) = m$.

## Definition

The *Apéry set* of $S$ is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

# Apéry sets

Fix a numerical semigroup $S$ with $\mathrm{m}(S) = m$.

## Definition

The *Apéry set* of $S$ is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

# Apéry sets

Fix a numerical semigroup $S$ with $\mathrm{m}(S) = m$.

**Definition**

The *Apéry set* of $S$ is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

# Apéry sets

Fix a numerical semigroup $S$ with $\mathrm{m}(S) = m$.

## Definition

The *Apéry set* of $S$ is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

# Apéry sets

Fix a numerical semigroup $S$ with $\mathrm{m}(S) = m$.

**Definition**

The *Apéry set* of $S$ is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

# Apéry sets

Fix a numerical semigroup $S$ with $\mathrm{m}(S) = m$.

**Definition**

The *Apéry set* of $S$ is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

Observations:

# Apéry sets

Fix a numerical semigroup $S$ with $\mathrm{m}(S) = m$.

**Definition**

The *Apéry set* of $S$ is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

Observations:

- The elements of $\mathrm{Ap}(S)$ are distinct modulo $m$

# Apéry sets

Fix a numerical semigroup $S$ with $\mathrm{m}(S) = m$.

**Definition**

The *Apéry set* of $S$ is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

If $S = \langle 6, 9, 20 \rangle$, then

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

Observations:

- The elements of $\mathrm{Ap}(S)$ are distinct modulo $m$
- $|\,\mathrm{Ap}(S)| = m$

# Apéry sets

Fix a numerical semigroup $S$ with $\mathrm{m}(S) = m$.

## Definition

The *Apéry set* of $S$ is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

# Apéry sets

Fix a numerical semigroup $S$ with $\mathrm{m}(S) = m$.

**Definition**

The *Apéry set* of $S$ is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

Many things can be easily recovered from the Apéry set.

# Apéry sets

Fix a numerical semigroup $S$ with $\mathrm{m}(S) = m$.

**Definition**

The *Apéry set* of $S$ is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

Many things can be easily recovered from the Apéry set.

- Fast membership test:

$$n \in S \text{ if } n \geq a \text{ for } a \in \mathrm{Ap}(S) \text{ with } a \equiv n \bmod m$$

# Apéry sets

Fix a numerical semigroup $S$ with $\mathrm{m}(S) = m$.

**Definition**

The *Apéry set* of $S$ is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

Many things can be easily recovered from the Apéry set.

- Fast membership test:

$$n \in S \text{ if } n \geq a \text{ for } a \in \mathrm{Ap}(S) \text{ with } a \equiv n \bmod m$$

- Frobenius number: $\mathrm{F}(S) = \max(\mathrm{Ap}(S)) - m$

# Apéry sets

Fix a numerical semigroup $S$ with $\mathrm{m}(S) = m$.

**Definition**

The *Apéry set* of $S$ is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

Many things can be easily recovered from the Apéry set.

- Fast membership test:

$$n \in S \text{ if } n \geq a \text{ for } a \in \mathrm{Ap}(S) \text{ with } a \equiv n \bmod m$$

- Frobenius number: $\mathrm{F}(S) = \max(\mathrm{Ap}(S)) - m$
- Number of gaps (the *genus*):

$$\mathrm{g}(S) = |\mathbb{N} \setminus S| = \sum_{a \in \mathrm{Ap}(S)} \left\lfloor \frac{a}{m} \right\rfloor$$

# Apéry sets

Fix a numerical semigroup $S$ with $\mathrm{m}(S) = m$.

**Definition**

The *Apéry set* of $S$ is

$$\mathrm{Ap}(S) = \{a \in S : a - m \notin S\}$$

Many things can be easily recovered from the Apéry set.

- Fast membership test:

$$n \in S \text{ if } n \geq a \text{ for } a \in \mathrm{Ap}(S) \text{ with } a \equiv n \bmod m$$

- Frobenius number: $\mathrm{F}(S) = \max(\mathrm{Ap}(S)) - m$
- Number of gaps (the *genus*):

$$\mathrm{g}(S) = |\mathbb{N} \setminus S| = \sum_{a \in \mathrm{Ap}(S)} \left\lfloor \frac{a}{m} \right\rfloor$$

The Apéry set is a "one stop shop" for computation.

# Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ the Apéry set of some numerical semigroup?

# Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 5, \quad a_1 = 11,\ a_2 = 7,\ a_3 = 23,\ a_4 = 19$$

# Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 5, \quad a_1 = 11,\ a_2 = 7,\ a_3 = 23,\ a_4 = 19$$

but $a_1 + a_2 \equiv 3 \bmod 5$ and $a_1 + a_2 < a_3$.

# Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 5, \quad a_1 = 11,\ a_2 = 7,\ a_3 = 23,\ a_4 = 19$$

but $a_1 + a_2 \equiv 3 \bmod 5$ and $a_1 + a_2 < a_3$.

Is $\{0, 13, 14, 27, 10, 11\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 6, \quad a_1 = 13,\ a_2 = 14,\ a_3 = 27,\ a_4 = 10,\ a_5 = 11$$

# Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 5, \quad a_1 = 11,\ a_2 = 7,\ a_3 = 23,\ a_4 = 19$$

but $a_1 + a_2 \equiv 3 \bmod 5$ and $a_1 + a_2 < a_3$.

Is $\{0, 13, 14, 27, 10, 11\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 6, \quad a_1 = 13,\ a_2 = 14,\ a_3 = 27,\ a_4 = 10,\ a_5 = 11$$

but $a_4 + a_5 \equiv 3 \bmod 6$ and $a_4 + a_5 < a_3$.

# Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 5, \quad a_1 = 11,\ a_2 = 7,\ a_3 = 23,\ a_4 = 19$$

but $a_1 + a_2 \equiv 3 \bmod 5$ and $a_1 + a_2 < a_3$.

Is $\{0, 13, 14, 27, 10, 11\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 6, \quad a_1 = 13,\ a_2 = 14,\ a_3 = 27,\ a_4 = 10,\ a_5 = 11$$

but $a_4 + a_5 \equiv 3 \bmod 6$ and $a_4 + a_5 < a_3$.

**Theorem**

*If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup $S$ with $\mathrm{Ap}(S) = A$ if and only if*

$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

# Apéry sets

Is $A = \{0, 11, 7, 23, 19\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 5, \quad a_1 = 11,\ a_2 = 7,\ a_3 = 23,\ a_4 = 19$$

but $a_1 + a_2 \equiv 3 \bmod 5$ and $a_1 + a_2 < a_3$.

Is $\{0, 13, 14, 27, 10, 11\}$ the Apéry set of some numerical semigroup?

$$m = |A| = 6, \quad a_1 = 13,\ a_2 = 14,\ a_3 = 27,\ a_4 = 10,\ a_5 = 11$$

but $a_4 + a_5 \equiv 3 \bmod 6$ and $a_4 + a_5 < a_3$.

**Theorem**

*If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup $S$ with $\mathrm{Ap}(S) = A$ if and only if*

$$a_i + a_j \geq a_{i+j} \qquad \textit{whenever} \qquad i + j \neq 0.$$

Big idea: the inequalities "$a_i + a_j \geq a_{i+j}$" to define a **cone** $C_m$.

# Kunz cone

## Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

# Kunz cone

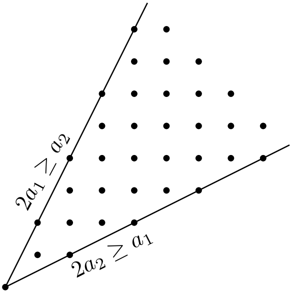
**Definition**

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

Example: $C_3$



# Kunz cone

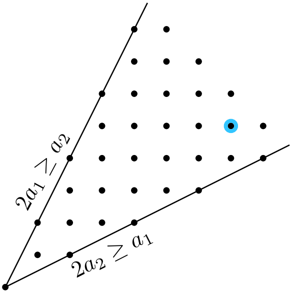
## Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathsf{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

Example: $C_3$




$S = \langle 3, 5, 7 \rangle$
$\mathrm{Ap}(S) = \{0, 7, 5\}$

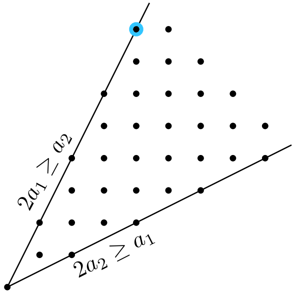
# Kunz cone

## Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities
$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

Example: $C_3$




$S = \langle 3, 5, 7 \rangle$
$\mathrm{Ap}(S) = \{0, 7, 5\}$

$S = \langle 3, 4 \rangle$
$\mathrm{Ap}(S) = \{0, 4, 8\}$

# Kunz cone

## Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

# Kunz cone

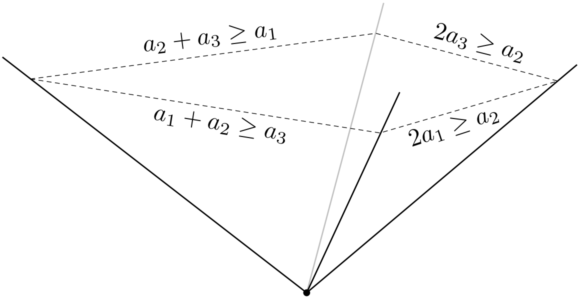
## Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

Example: $C_4$



# Faces of the Kunz cone

**Question**

When are numerical semigroups in (the relative interior of) the same face?

# Faces of the Kunz cone

**Question**

When are numerical semigroups in (the relative interior of) the same face?

First steps: $S \in \text{Int}(C_m)$ if and only if $S$ has *max embedding dimension*

# Faces of the Kunz cone

**Question**

When are numerical semigroups in (the relative interior of) the same face?

First steps: $S \in \mathrm{Int}(C_m)$ if and only if $S$ has *max embedding dimension*

If $S = \langle n_1, \ldots, n_k \rangle$, then

$$n_i \not\equiv n_j \bmod n_1 \quad \Longrightarrow \quad k \leq \mathrm{m}(S)$$

# Faces of the Kunz cone

**Question**

When are numerical semigroups in (the relative interior of) the same face?

First steps: $S \in \mathrm{Int}(C_m)$ if and only if $S$ has *max embedding dimension*

If $S = \langle n_1, \ldots, n_k \rangle$, then

$$n_i \not\equiv n_j \bmod n_1 \quad \Longrightarrow \quad k \le \mathrm{m}(S)$$

If $k = \mathrm{m}(S)$, then $S$ has *max embedding dimension*

# Faces of the Kunz cone

**Question**

When are numerical semigroups in (the relative interior of) the same face?

First steps: $S \in \mathrm{Int}(C_m)$ if and only if $S$ has *max embedding dimension*

If $S = \langle n_1, \ldots, n_k \rangle$, then

$$n_i \not\equiv n_j \bmod n_1 \quad \Longrightarrow \quad k \le \mathrm{m}(S)$$

If $k = \mathrm{m}(S)$, then $S$ has *max embedding dimension*

$$S = \langle m, a_1, \ldots, a_{m-1} \rangle \quad \text{where} \quad \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\}$$

# Faces of the Kunz cone

**Question**

When are numerical semigroups in (the relative interior of) the same face?

First steps: $S \in \mathrm{Int}(C_m)$ if and only if $S$ has *max embedding dimension*

If $S = \langle n_1, \ldots, n_k \rangle$, then

$$n_i \not\equiv n_j \bmod n_1 \quad \Longrightarrow \quad k \leq \mathrm{m}(S)$$

If $k = \mathrm{m}(S)$, then $S$ has *max embedding dimension*

$$S = \langle m, a_1, \ldots, a_{m-1} \rangle \quad \text{where} \quad \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\}$$

Geometrically: “most” numerical semigroups with $\mathrm{m}(S) = m$ are MED

# Faces of the Kunz cone

**Question**

When are numerical semigroups in (the relative interior of) the same face?

First steps: $S \in \mathrm{Int}(C_m)$ if and only if $S$ has *max embedding dimension*

If $S = \langle n_1, \ldots, n_k \rangle$, then

$$n_i \not\equiv n_j \bmod n_1 \quad \Longrightarrow \quad k \le \mathrm{m}(S)$$

If $k = \mathrm{m}(S)$, then $S$ has *max embedding dimension*

$$S = \langle m, a_1, \ldots, a_{m-1} \rangle \quad \text{where} \quad \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\}$$

Geometrically: "most" numerical semigroups with $\mathrm{m}(S) = m$ are MED

What about the other faces?

# Faces of the Kunz cone

**Question**

When are numerical semigroups in (the relative interior of) the same face?

# Faces of the Kunz cone

**Question**

When are numerical semigroups in (the relative interior of) the same face?

Example: $S = \langle 4, 10, 11, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 11\}$$
$$a_1 = 13, \quad a_2 = 10, \quad a_3 = 11$$

$$2a_1 > a_2 \qquad a_1 + a_2 > a_3$$
$$2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

# Faces of the Kunz cone

**Question**

When are numerical semigroups in (the relative interior of) the same face?

Example: $S = \langle 4, 10, 11, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 11\} \qquad 2a_1 > a_2 \qquad a_1 + a_2 > a_3$$
$$a_1 = 13, \quad a_2 = 10, \quad a_3 = 11 \qquad 2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

Example: $S = \langle 4, 10, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 23\} \qquad 2a_1 > a_2 \qquad a_1 + a_2 = a_3$$
$$a_1 = 13, \quad a_2 = 10, \quad a_3 = 23 \qquad 2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

# Faces of the Kunz cone

**Question**

When are numerical semigroups in (the relative interior of) the same face?

Example: $S = \langle 4, 10, 11, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 11\} \qquad 2a_1 > a_2 \qquad a_1 + a_2 > a_3$$
$$a_1 = 13, \quad a_2 = 10, \quad a_3 = 11 \qquad 2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

Example: $S = \langle 4, 10, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 23\} \qquad 2a_1 > a_2 \qquad a_1 + a_2 = a_3$$
$$a_1 = 13, \quad a_2 = 10, \quad a_3 = 23 \qquad 2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

Example: $S = \langle 4, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 26, 39\} \qquad 2a_1 = a_2 \qquad a_1 + a_2 = a_3$$
$$a_1 = 13, \quad a_2 = 26, \quad a_3 = 39 \qquad 2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

# Faces of the Kunz cone

**Question**

When are numerical semigroups in (the relative interior of) the same face?

# Faces of the Kunz cone

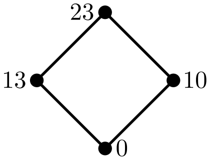
## Question

When are numerical semigroups in (the relative interior of) the same face?

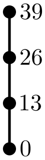
## Definition

The *Apéry poset* of $S$: define $a \preceq a'$ whenever $a' - a \in S$.

$\mathrm{Ap}(S) = \{0, 13, 10, 23\}$




$\mathrm{Ap}(S) = \{0, 13, 26, 39\}$



# Faces of the Kunz cone

**Question**

When are numerical semigroups in (the relative interior of) the same face?

# Faces of the Kunz cone

## Question

When are numerical semigroups in (the relative interior of) the same face?

$$S = \langle 6, 9, 20 \rangle$$
$$\mathrm{Ap}(S) = \{0, 49, 20, \ \ 9, 40, 29\}$$

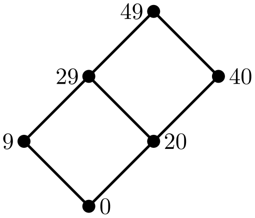
$$S' = \langle 6, 26, 27 \rangle$$
$$\mathrm{Ap}(S') = \{0, 79, 26, 27, 52, 53\}$$

# Faces of the Kunz cone

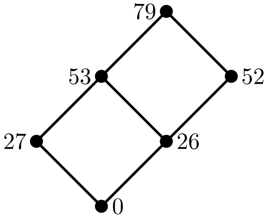
## Question

When are numerical semigroups in (the relative interior of) the same face?

$$S = \langle 6, 9, 20 \rangle$$
$$\mathrm{Ap}(S) = \{0, 49, 20, \ \ 9, 40, 29\}$$




$$S' = \langle 6, 26, 27 \rangle$$
$$\mathrm{Ap}(S') = \{0, 79, 26, 27, 52, 53\}$$



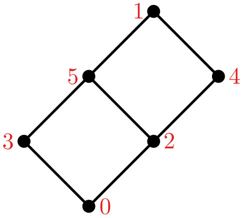
# Faces of the Kunz cone

## Question

When are numerical semigroups in (the relative interior of) the same face?

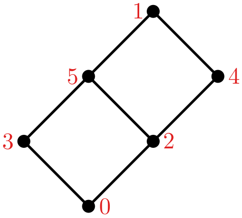
$S = \langle 6, 9, 20 \rangle$

$\mathrm{Ap}(S) = \{0, \underset{1}{49}, \underset{2}{20}, \underset{3}{9}, \underset{4}{40}, \underset{5}{29}\}$




$S' = \langle 6, 26, 27 \rangle$

$\mathrm{Ap}(S') = \{0, \underset{1}{79}, \underset{2}{26}, \underset{3}{27}, \underset{4}{52}, \underset{5}{53}\}$



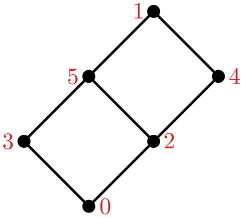
# Faces of the Kunz cone

**Question**

When are numerical semigroups in (the relative interior of) the same face?

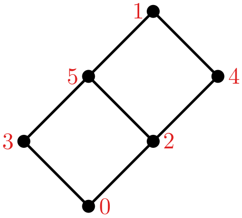
$S = \langle 6, 9, 20 \rangle$

$\mathrm{Ap}(S) = \{0, \underset{1}{49}, \underset{2}{20}, \underset{3}{9}, \underset{4}{40}, \underset{5}{29}\}$




$S' = \langle 6, 26, 27 \rangle$

$\mathrm{Ap}(S') = \{0, \underset{1}{79}, \underset{2}{26}, \underset{3}{27}, \underset{4}{52}, \underset{5}{53}\}$



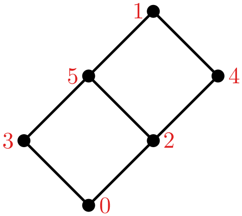

The *Kunz poset* of $S$: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

# Faces of the Kunz cone

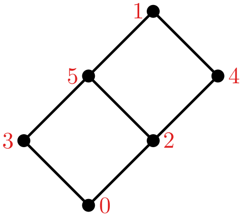
## Question

When are numerical semigroups in (the relative interior of) the same face?

$$S = \langle 6, 9, 20 \rangle$$
$$\mathrm{Ap}(S) = \{0, \underset{1}{49}, \underset{2}{20}, \underset{3}{9}, \underset{4}{40}, \underset{5}{29}\}$$




$$S' = \langle 6, 26, 27 \rangle$$
$$\mathrm{Ap}(S') = \{0, \underset{1}{79}, \underset{2}{26}, \underset{3}{27}, \underset{4}{52}, \underset{5}{53}\}$$




The *Kunz poset* of $S$: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

## Theorem (Bruns–García-Sánchez–O.–Wilburne)

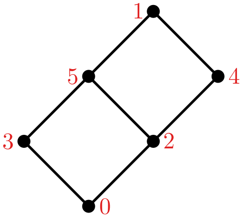
*Numerical semigroups lie in the relative interior of the same face of $C_m$ if and only if their Kunz posets are identical.*

# Faces of the Kunz cone

## Question

When are numerical semigroups in (the relative interior of) the same face?

$$S = \langle 6, 9, 20 \rangle$$
$$\mathrm{Ap}(S) = \{0, \underset{1}{49}, \underset{2}{20}, \underset{3}{9}, \underset{4}{40}, \underset{5}{29}\}$$




The *Kunz poset* of $S$: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

## Theorem (Bruns–García-Sánchez–O.–Wilburne)

*Numerical semigroups lie in the relative interior of the same face of $C_m$ if and only if their Kunz posets are identical.*

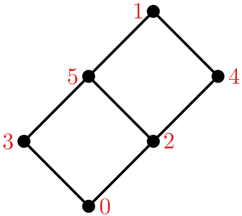
# Faces of the Kunz cone

## Question

When are numerical semigroups in (the relative interior of) the same face?

$S = \langle 6, 9, 20 \rangle$

$\mathrm{Ap}(S) = \{0, \underset{1}{49}, \underset{2}{20}, \underset{3}{9}, \underset{4}{40}, \underset{5}{29}\}$




Defining facet equations:

$$2a_2 = a_4$$
$$a_2 + a_3 = a_5$$
$$a_2 + a_5 = a_1$$
$$a_3 + a_4 = a_1$$

The *Kunz poset* of $S$: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

## Theorem (Bruns–García-Sánchez–O.–Wilburne)

*Numerical semigroups lie in the relative interior of the same face of $C_m$ if and only if their Kunz posets are identical.*

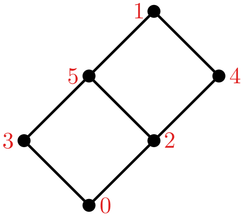
# Faces of the Kunz cone

## Question

When are numerical semigroups in (the relative interior of) the same face?

$$S = \langle 6, 9, 20 \rangle$$

$$\mathrm{Ap}(S) = \{0, \underset{1}{49}, \underset{2}{20}, \underset{3}{9}, \underset{4}{40}, \underset{5}{29}\}$$




Defining facet equations:

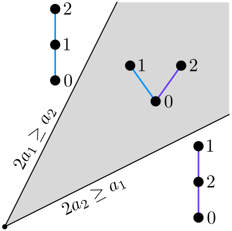
| | |
|---|---|
| $2a_2 = a_4$ | $2 \preceq 4$ |
| $a_2 + a_3 = a_5$ | $2 \preceq 5$ |
| | $3 \preceq 5$ |
| $a_2 + a_5 = a_1$ | $2 \preceq 1$ |
| | $5 \preceq 1$ |
| $a_3 + a_4 = a_1$ | $3 \preceq 1$ |
| | $4 \preceq 1$ |

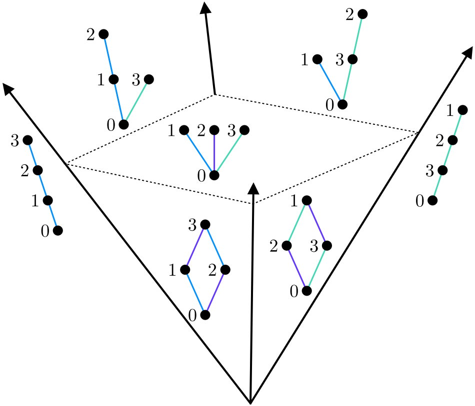
The *Kunz poset* of $S$: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

## Theorem (Bruns–García-Sánchez–O.–Wilburne)

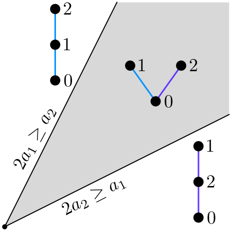
*Numerical semigroups lie in the relative interior of the same face of $C_m$ if and only if their Kunz posets are identical.*

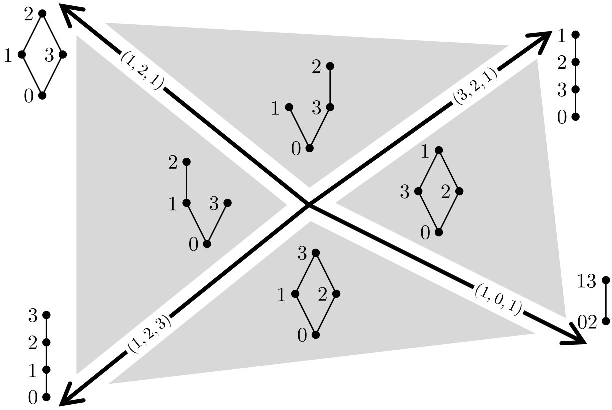
# $C_3$ and $C_4$



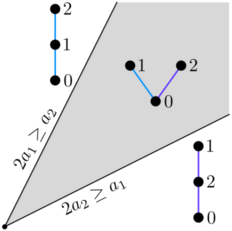



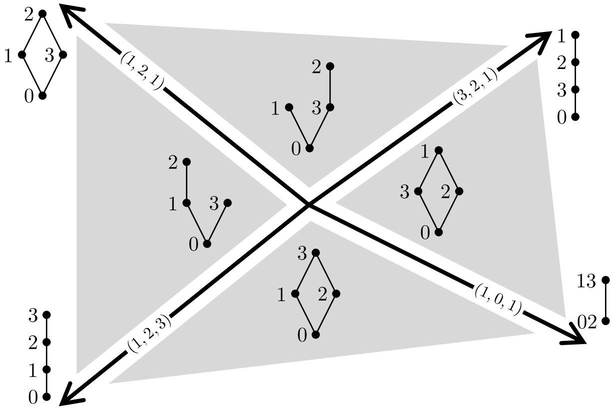
# $C_3$ and $C_4$



# $C_3$ and $C_4$




## Theorem (Kaplan–O.)

*There is a natural labeling of the faces of $C_m$ by finite posets.*

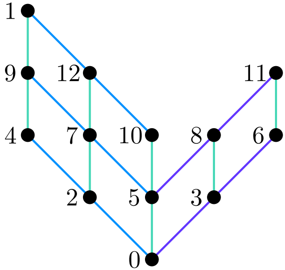
# Shared properties within a face

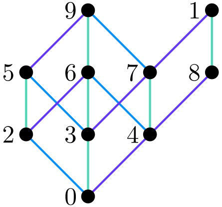
What properties are determined by the Kunz poset $P$ of $S = \langle n_1, \ldots, n_k \rangle$?

# Shared properties within a face

What properties are determined by the Kunz poset $P$ of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of $P$

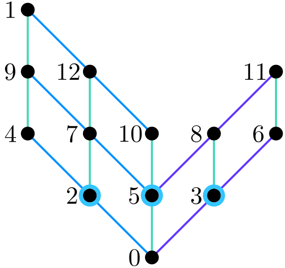


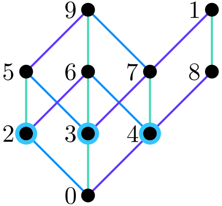



# Shared properties within a face

What properties are determined by the Kunz poset $P$ of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of $P$



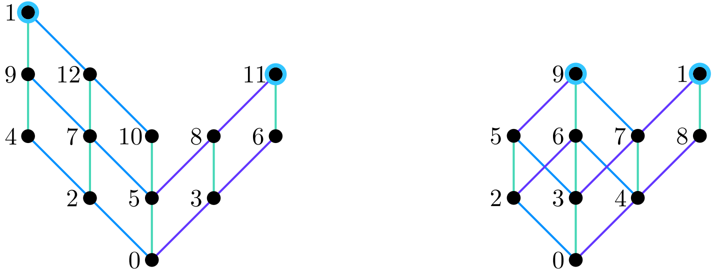



# Shared properties within a face

What properties are determined by the Kunz poset $P$ of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of $P$
- $\mathsf{t}(S) = \#$ maximal elements
  (Cohen-Macaulay type of $S$)

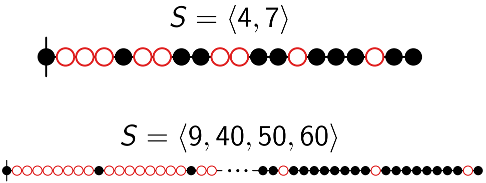
---

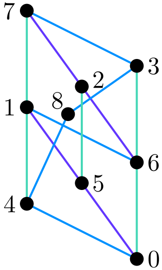


# Shared properties within a face

What properties are determined by the Kunz poset $P$ of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of $P$
- $\mathrm{t}(S) = \#$ maximal elements (Cohen-Macaulay type of $S$)
- Symmetric/Gorenstein?

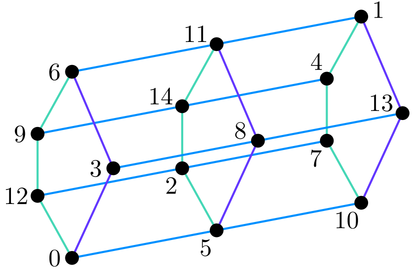


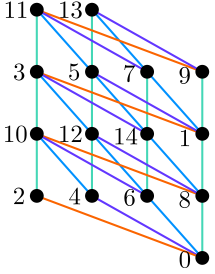



# Shared properties within a face

What properties are determined by the Kunz poset $P$ of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of $P$
- $\mathsf{t}(S) = \#$ maximal elements (Cohen-Macaulay type of $S$)
- Symmetric/Gorenstein?
- Complete intersection?
- Generalized arithmetical?



# Shared properties within a face

What properties are determined by the Kunz poset $P$ of $S = \langle n_1, \ldots, n_k \rangle$?

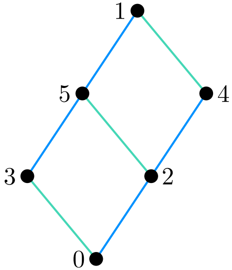
- $k = 1 + \#$ atoms of $P$
- $\mathrm{t}(S) = \#$ maximal elements (Cohen-Macaulay type of $S$)
- Symmetric/Gorenstein?
- Complete intersection?
- Generalized arithmetical?
- Minimal binomial generators of the *defining toric ideal* of $S$:

$$I_S = \ker\left(\Bbbk[\overline{x}] \to \Bbbk[t]\right)$$
$$x_i \mapsto t^{n_i}$$

---

$S = \langle 6, 9, 20 \rangle$

$$I_S = \langle x^3 - y^2, x^4y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$$



# Shared properties within a face

What properties are determined by the Kunz poset $P$ of $S = \langle n_1, \ldots, n_k \rangle$?

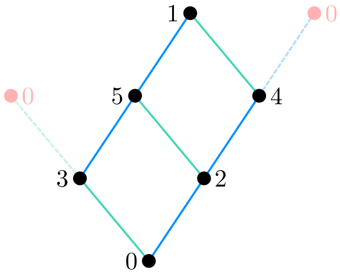
- $k = 1 + \#$ atoms of $P$
- $\mathrm{t}(S) = \#$ maximal elements (Cohen-Macaulay type of $S$)
- Symmetric/Gorenstein?
- Complete intersection?
- Generalized arithmetical?
- Minimal binomial generators of the *defining toric ideal* of $S$:

$$I_S = \ker\left(\Bbbk[\overline{x}] \to \Bbbk[t]\right)$$
$$x_i \mapsto t^{n_i}$$

---

$$S = \langle 6, 9, 20 \rangle$$

$$I_S = \langle x^3 - y^2, x^4y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$$



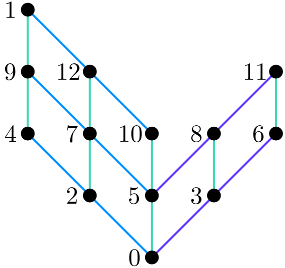
# Shared properties within a face

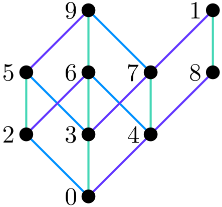
What properties are determined by the Kunz poset $P$ of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of $P$
- $\mathrm{t}(S) = \#$ maximal elements (Cohen-Macaulay type of $S$)
- Symmetric/Gorenstein?
- Complete intersection?
- Generalized arithmetical?
- Minimal binomial generators of the *defining toric ideal* of $S$:

$$I_S = \ker\left(\mathbb{k}[\overline{x}] \to \mathbb{k}[t]\right) \qquad x_i \mapsto t^{n_i}$$

---



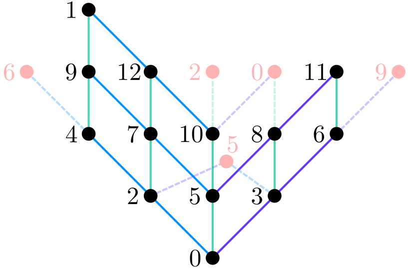
# Shared properties within a face

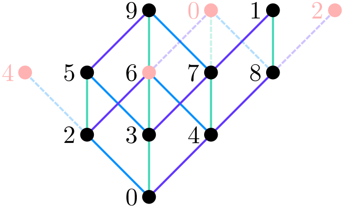
What properties are determined by the Kunz poset $P$ of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of $P$
- $\mathrm{t}(S) = \#$ maximal elements (Cohen-Macaulay type of $S$)
- Symmetric/Gorenstein?
- Complete intersection?
- Generalized arithmetical?
- Minimal binomial generators of the *defining toric ideal* of $S$:

$$I_S = \ker\left(\Bbbk[\overline{x}] \to \Bbbk[t]\right)$$
$$x_i \mapsto t^{n_i}$$

---



# Shared properties within a face

What properties are determined by the Kunz poset $P$ of $S = \langle n_1, \ldots, n_k \rangle$?

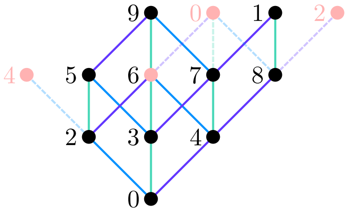
- $k = 1 + \#$ atoms of $P$
- $\mathrm{t}(S) = \#$ maximal elements (Cohen-Macaulay type of $S$)
- Symmetric/Gorenstein?
- Complete intersection?
- Generalized arithmetical?
- Minimal binomial generators of the *defining toric ideal* of $S$:

$$I_S = \ker \left( \mathbb{k}[\overline{x}] \to \mathbb{k}[t] \right) \qquad x_i \mapsto t^{n_i}$$

---

$S = \langle 10, a_2, a_3, a_4 \rangle$

$$I_S = \langle\, x_2^2 - y^* x_4,\ \ x_2 x_4 - x_3^2,\ \ x_3^2 x_4 - y^*,\ \ x_4^3 - y^* x_2 \,\rangle \subseteq \mathbb{k}[y, x_2, x_3, x_4]$$



# Shared properties within a face

What properties are determined by the Kunz poset $P$ of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of $P$
- $\mathrm{t}(S) = \#$ maximal elements (Cohen-Macaulay type of $S$)
- Symmetric/Gorenstein?
- Complete intersection?
- Generalized arithmetical?
- Minimal binomial generators of the *defining toric ideal* of $S$:
$$I_S = \ker\left(\Bbbk[\overline{x}] \to \Bbbk[t]\right) \quad x_i \mapsto t^{n_i}$$
- Betti numbers of $I_S$ over $\Bbbk[\overline{x}]$

---

| | 1,12 | 1,13 | 2,12 | 2,23 | 3,13 | 3,23 | 2,13 | 3,12 |
|---|---|---|---|---|---|---|---|---|
| 1,1 | $-x_2$ | $-x_3$ | | | | | $y^*$ | $y^*$ |
| 2,2 | $-y^*$ | | $x_1$ | $-x_3$ | | $y^*$ | | |
| 3,3 | | | | | $x_1$ | $x_2$ | $-y^*$ | |
| 2,1 | $x_1$ | | $-x_2$ | $y^*$ | $y^*$ | | $-x_3$ | |
| 3,1 | $y^*$ | $x_1$ | | | $-x_3$ | $-y^*$ | | $-x_2$ |
| 3,2 | | $-y^*$ | $-y^*$ | $x_2$ | | $-x_3$ | $x_1$ | $x_1$ |

| | 1,[3] | 2,[3] | 3,[3] |
|---|---|---|---|
| 1,12 | $x_3$ | $-y^*$ | |
| 1,13 | $-x_2$ | | $y^*$ |
| 2,12 | | $x_3$ | $-y^*$ |
| 2,23 | $-y^*$ | $x_1$ | |
| 3,13 | $y^*$ | | $-x_2$ |
| 3,23 | | $-y^*$ | $x_1$ |
| 2,13 | $x_1$ | $-x_2$ | |
| 3,12 | $-x_1$ | | $x_3$ |

$$0 \leftarrow R \leftarrow R^6 \longleftarrow R^8 \longleftarrow R^3 \leftarrow 0$$

# Shared properties within a face

What properties are determined by the Kunz poset $P$ of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of $P$
- $\mathrm{t}(S) = \#$ maximal elements (Cohen-Macaulay type of $S$)
- Symmetric/Gorenstein?
- Complete intersection?
- Generalized arithmetical?
- Minimal binomial generators of the *defining toric ideal* of $S$:
$$I_S = \ker\left(\Bbbk[\overline{x}] \to \Bbbk[t]\right) \qquad x_i \mapsto t^{n_i}$$
- Betti numbers of $I_S$ over $\Bbbk[\overline{x}]$
- Betti numbers of $\Bbbk$ over $\Bbbk[\overline{x}]/I_S$

---

$$0 \leftarrow R \xleftarrow{\begin{array}{c} \quad 0 \quad 1 \quad 2 \quad 3 \\ \emptyset \begin{bmatrix} y & x_1 & x_2 & x_3 \end{bmatrix} \end{array}} R^4 \xleftarrow{\begin{array}{c|cccccccccccc} & 01 & 02 & 03 & 11 & 12 & 13 & 21 & 22 & 23 & 31 & 32 & 33 \\ \hline 0 & x_1 & x_2 & x_3 & & & -y^* & & -y^* & & -y^* & & \\ 1 & -y & & & x_1 & x_2 & x_3 & & & -y^* & & -y^* & \\ 2 & & -y & & -y^* & & & x_1 & x_2 & x_3 & & & -y^* \\ 3 & & & -y & & -y^* & & -y^* & & & x_1 & x_2 & x_3 \end{array}} R^{12}$$

$$\longleftarrow R^{36} \longleftarrow R^{108} \longleftarrow R^{324} \longleftarrow R^{972} \longleftarrow R^{2916} \longleftarrow \cdots$$

# Application 1: classifying minimal trades

## Question

Given the multiplicity $m = \mathrm{m}(S)$ and $k = \#$ minimal generators of $S$, what can $\beta_1(I_S) = \#$ minimal trades be?

# Application 1: classifying minimal trades

## Question

Given the multiplicity $m = \mathrm{m}(S)$ and $k = \#$ minimal generators of $S$, what can $\beta_1(I_S) = \#$ minimal trades be?

| $m$ | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | | 3 | | | | | | | | | | | | | | | | | | | | | | |
| 4 | 2 | 3 | | | 6 | | | | | | | | | | | | | | | | | | | |
| 5 | 2 | 3 | | 5 | 6 | | | | 10 | | | | | | | | | | | | | | | |
| 6 | 2 | 3 | 4 | 5 | 6 | | | 9 | 10 | | | | | 15 | | | | | | | | | | |
| 7 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | | | | 14 | 15 | | | | | | 21 | | | | |
| 8 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | | 13 | 14 | 15 | | | | | 20 | 21 | | | | |
| 9 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | | | 19 | 20 | 21 | | | | |
| 10 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | | 18 | 19 | 20 | 21 | 22 | | | |
| 11 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | | | 25 |
| 12 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 13 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 14 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 15 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 16 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 17 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |

# Application 1: classifying minimal trades

## Question

Given the multiplicity $m = \mathrm{m}(S)$ and $k = \#$ minimal generators of $S$, what can $\beta_1(I_S) = \#$ minimal trades be?

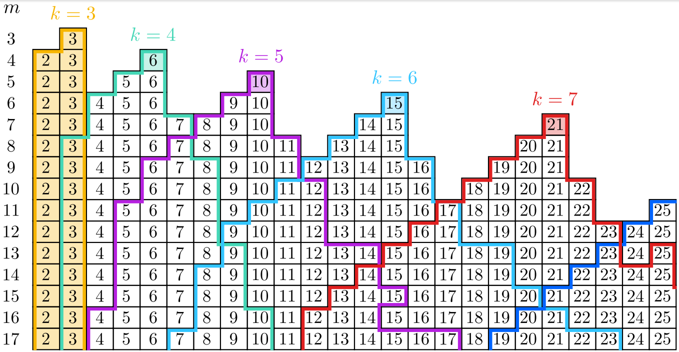
$k = 3$ $k = 4$ $k = 5$ $k = 6$ $k = 7$

| $m$ | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | | 3 | | | | | | | | | | | | | | | | | | | | | | |
| 4 | 2 | 3 | | | 6 | | | | | | | | | | | | | | | | | | | |
| 5 | 2 | 3 | | 5 | 6 | | | | 10 | | | | | | | | | | | | | | | |
| 6 | 2 | 3 | 4 | 5 | 6 | | | 9 | 10 | | | | | 15 | | | | | | | | | | |
| 7 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | | | | 14 | 15 | | | | | | 21 | | | | |
| 8 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | | 13 | 14 | 15 | | | | | 20 | 21 | | | | |
| 9 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | | | 19 | 20 | 21 | | | | |
| 10 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | | 18 | 19 | 20 | 21 | 22 | | | |
| 11 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | | | 25 |
| 12 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 13 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 14 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 15 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 16 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 17 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |

# Application 1: classifying minimal trades

## Question

Given the multiplicity $m = \mathrm{m}(S)$ and $k = \#$ minimal generators of $S$, what can $\beta_1(I_S) = \#$ minimal trades be?




Well known: $\beta_1(S) \le \binom{m}{2}$, with equality if and only if $k = m$

if $k = 3$, then $\beta_1(S) = 2, 3$

# Application 1: classifying minimal trades

## Question

Given the multiplicity $m = \mathrm{m}(S)$ and $k = \#$ minimal generators of $S$, what can $\beta_1(I_S) = \#$ minimal trades be?

$k=3$, $k=4$, $k=5$, $k=6$, $k=7$

| $m$ | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | | 3 | | | | | | | | | | | | | | | | | | | | | | |
| 4 | 2 | 3 | | | 6 | | | | | | | | | | | | | | | | | | | |
| 5 | 2 | 3 | | 5 | 6 | | | | 10 | | | | | | | | | | | | | | | |
| 6 | 2 | 3 | 4 | 5 | 6 | | | 9 | 10 | | | | | 15 | | | | | | | | | | |
| 7 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | | | | 14 | 15 | | | | | | 21 | | | | |
| 8 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | | 13 | 14 | 15 | | | | | 20 | 21 | | | | |
| 9 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | | | 19 | 20 | 21 | | | | |
| 10 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | | 18 | 19 | 20 | 21 | 22 | | | |
| 11 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | | | 25 |
| 12 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 13 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 14 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 15 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 16 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 17 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |

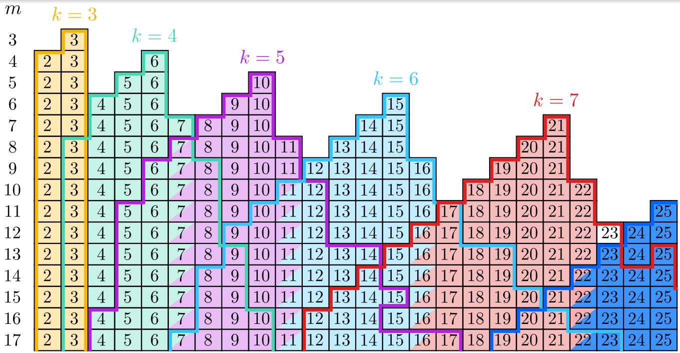
Prior work: a family has $\beta_1(S) = \binom{k}{2}$ for $3 \le k \le m$ (Rosales)

if $r = m - k \le 2$, then $\beta_1(S) \in [\binom{k}{2} - r, \binom{k}{2}]$ (GS-R)

# Application 1: classifying minimal trades

## Question

Given the multiplicity $m = \mathrm{m}(S)$ and $k = \#$ minimal generators of $S$, what can $\beta_1(I_S) = \#$ minimal trades be?




Using Kunz posets: a family hits each $\beta_1(S) \in [\binom{k}{2} - r, \binom{k}{2}]$

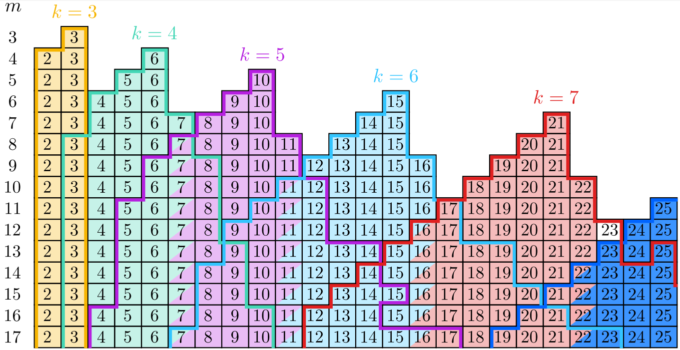
for $r = m - k \leq k - 2$

a family hits $\beta_1(S) = \binom{k}{2} + 1$ for each $m \geq k + 3$

# Application 1: classifying minimal trades

## Question

Given the multiplicity $m = \mathrm{m}(S)$ and $k = \#$ minimal generators of $S$, what can $\beta_1(I_S) = \#$ minimal trades be?




Bounds from Kunz posets: $\beta_1(S) \geq \binom{k}{2} - r$, where $r = m - k$

if $m - k = 3$, then $\beta_1(S) \in [\binom{k}{2} - 3, \binom{k}{2} + 1]$

# Application 1: classifying minimal trades

## Question

Given the multiplicity $m = \mathrm{m}(S)$ and $k = \#$ minimal generators of $S$, what can $\beta_1(I_S) = \#$ minimal trades be?

$k = 4$

| $m$ | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | | 3 | | | | | | | | | | | | | | | | | | | | | | |
| 4 | 2 | 3 | | | 6 | | | | | | | | | | | | | | | | | | | |
| 5 | 2 | 3 | | 5 | 6 | | | | 10 | | | | | | | | | | | | | | | |
| 6 | 2 | 3 | 4 | 5 | 6 | | | 9 | 10 | | | | | 15 | | | | | | | | | | |
| 7 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | | | | 14 | 15 | | | | | | 21 | | | | |
| 8 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | | 13 | 14 | 15 | | | | | 20 | 21 | | | | |
| 9 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | | | 19 | 20 | 21 | | | | |
| 10 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | | 18 | 19 | 20 | 21 | 22 | | | |
| 11 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | | | 25 |
| 12 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 13 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 14 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 15 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 16 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 17 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |

One more family: for $k = 4$, achieves each $\beta_1(S)$ with $(\beta_1(S) - 2)^2 \leq 4m$

conjectured to achieve every possible $\beta_1(S)$ for $k = 4$

# Application 2: a long-standing (**hard**) conjecture

# Application 2: a long-standing (**hard**) conjecture

$\mathsf{F}(S) = \max(\mathbb{Z}_{\geq 0} \setminus S)$ $\qquad$ $\mathsf{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$

# Application 2: a long-standing (**hard**) conjecture

$$\mathrm{F}(S) = \max(\mathbb{Z}_{\geq 0} \setminus S) \qquad \mathrm{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$$

**Wilf's Conjecture**

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \leq k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

# Application 2: a long-standing (**hard**) conjecture

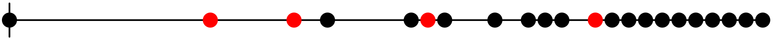
$$\mathrm{F}(S) = \max(\mathbb{Z}_{\geq 0} \setminus S) \qquad \mathrm{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$$

## Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \leq k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

Equivalently,

$$\frac{1}{k} \leq \underbrace{\frac{\mathrm{F}(S) + 1 - \mathrm{g}(S)}{\mathrm{F}(S) + 1}}_{\%\text{ of } [0, F(S)] \text{ in } S}$$



# Application 2: a long-standing (**hard**) conjecture

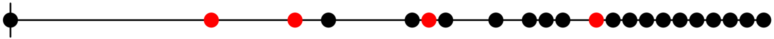
$$\mathrm{F}(S) = \max(\mathbb{Z}_{\geq 0} \setminus S) \qquad\qquad \mathrm{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$$

## Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \leq k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

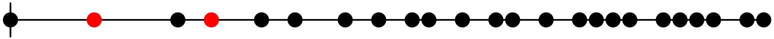
Equivalently,

$$\frac{1}{k} \leq \underbrace{\frac{\mathrm{F}(S) + 1 - \mathrm{g}(S)}{\mathrm{F}(S) + 1}}_{\%\text{ of } [0, F(S)] \text{ in } S}$$



Equality holds when:

- $S = \langle a, b \rangle$



- $S = \langle m, m+1, \ldots, 2m-1 \rangle$

# Application 2: a long-standing (**hard**) conjecture

$$\mathrm{F}(S) = \max(\mathbb{Z}_{\geq 0} \setminus S) \qquad \mathrm{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$$

## Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \leq k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

# Application 2: a long-standing (**hard**) conjecture

$$\mathrm{F}(S) = \max(\mathbb{Z}_{\geq 0} \setminus S) \qquad\qquad \mathrm{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$$

**Wilf's Conjecture**

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \leq k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

Proved in many special cases, including $\mathrm{g}(S) \leq 66$

# Application 2: a long-standing (**hard**) conjecture

$$\mathrm{F}(S) = \max(\mathbb{Z}_{\geq 0} \setminus S) \qquad \mathrm{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$$

**Wilf's Conjecture**

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \leq k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

Proved in many special cases, including $\mathrm{g}(S) \leq$ ~~66~~ 100

# Application 2: a long-standing (**hard**) conjecture

$$\mathsf{F}(S) = \max(\mathbb{Z}_{\geq 0} \setminus S) \qquad \mathsf{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$$

**Wilf's Conjecture**

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathsf{F}(S) + 1 \leq k(\mathsf{F}(S) + 1 - \mathsf{g}(S))$.

Proved in many special cases, including $\mathsf{g}(S) \leq$ ~~66~~ 100 ($\sim 10^{21}$ sgps)

# Application 2: a long-standing (**hard**) conjecture

$$\mathsf{F}(S) = \max(\mathbb{Z}_{\geq 0} \setminus S) \qquad \mathsf{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$$

## Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathsf{F}(S) + 1 \leq k(\mathsf{F}(S) + 1 - \mathsf{g}(S))$.

Proved in many special cases, including $\mathsf{g}(S) \leq$ ~~66~~ 100 ($\sim 10^{21}$ sgps)

## Theorem (Bruns-García-Sánchez-O.-Wilburne, 2020)

*Wilf's conjecture holds for all numerical semigroups $S$ with $m \leq 18$.*

# Application 2: a long-standing (**hard**) conjecture

$$\mathrm{F}(S) = \max(\mathbb{Z}_{\geq 0} \setminus S) \qquad \mathrm{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$$

**Wilf's Conjecture**

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \leq k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

Proved in many special cases, including $\mathrm{g}(S) \leq$ ~~66~~ 100 ($\sim 10^{21}$ sgps)

**Theorem (Bruns-García-Sánchez-O.-Wilburne, 2020)**

*Wilf's conjecture holds for all numerical semigroups $S$ with $m \leq 18$.*

Proved *computationally*!!?! But that's infinitely many semigroups!

# Application 2: a long-standing (**hard**) conjecture

$$\mathrm{F}(S) = \max(\mathbb{Z}_{\geq 0} \setminus S) \qquad \mathrm{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$$

**Wilf's Conjecture**

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \leq k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

Proved in many special cases, including $\mathrm{g}(S) \leq$ ~~66~~ 100 ($\sim 10^{21}$ sgps)

**Theorem (Bruns-García-Sánchez-O.-Wilburne, 2020)**

*Wilf's conjecture holds for all numerical semigroups $S$ with $m \leq 18$.*

Proved *computationally*!!?! But that's infinitely many semigroups!
The key: discrete optimization (integer solutions to linear inequalities)

# Application 2: a long-standing (**hard**) conjecture

$$\mathrm{F}(S) = \max(\mathbb{Z}_{\geq 0} \setminus S) \qquad \mathrm{g}(S) = |\mathbb{Z}_{\geq 0} \setminus S|$$

## Wilf's Conjecture

For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \leq k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

Proved in many special cases, including $\mathrm{g}(S) \leq$ ~~66~~ 100 ($\sim 10^{21}$ sgps)

## Theorem (Bruns-García-Sánchez-O.-Wilburne, 2020)

*Wilf's conjecture holds for all numerical semigroups $S$ with $m \leq 18$.*

Proved *computationally*!!?! But that's infinitely many semigroups!
The key: discrete optimization (integer solutions to linear inequalities)

If $S$ corresponds to $x = (a_1, \ldots, a_{m-1}) \in C_m$,

$$\mathrm{g}(S) = \|x\|_1 - \tfrac{1}{2}m(m-1), \qquad \mathrm{F}(S) = \|x\|_\infty - m,$$

and # generators $k$ is determined by the face $F \subseteq C_m$ containing $x$.

# References


W. Bruns, P. García-Sánchez, C. O'Neill, D. Wilburne (2020)
Wilf's conjecture in fixed multiplicity
International Journal of Algebra and Computation **30** (2020), no. 4, 861–882. (arXiv:1903.04342)

N. Kaplan, C. O'Neill, (2021)
Numerical semigroups, polyhedra, and posets I: the group cone
Combinatorial Theory **1** (2021), #19. (arXiv:1912.03741)

J. Autry, A. Ezell, T. Gomes, C. O'Neill, C. Preuss, T. Saluja, E. Torres Davila (2022)
Numerical semigroups, polyhedra, and posets II: locating certain families of semigroups.
Advances in Geometry **22** (2022), no. 1, 33–48. (arXiv:1912.04460)

T. Gomes, C. O'Neill, E. Torres Davila (2023)
Numerical semigroups, polyhedra, and posets III: minimal presentations and face dimension.
Electronic Journal of Combinatorics **30** (2023), no. 2, #P2.5. (arXiv:2009.05921)

B. Braun, T. Gomes, E. Miller, C. O'Neill, and A. Sobieska (2023)
Minimal free resolutions of numerical semigroup algebras via Apéry specialization
under review. (arXiv:2310.03612)

# References


W. Bruns, P. García-Sánchez, C. O'Neill, D. Wilburne (2020)
Wilf's conjecture in fixed multiplicity
International Journal of Algebra and Computation **30** (2020), no. 4, 861–882. (arXiv:1903.04342)

N. Kaplan, C. O'Neill, (2021)
Numerical semigroups, polyhedra, and posets I: the group cone
Combinatorial Theory **1** (2021), #19. (arXiv:1912.03741)

J. Autry, A. Ezell, T. Gomes, C. O'Neill, C. Preuss, T. Saluja, E. Torres Davila (2022)
Numerical semigroups, polyhedra, and posets II: locating certain families of semigroups.
Advances in Geometry **22** (2022), no. 1, 33–48. (arXiv:1912.04460)

T. Gomes, C. O'Neill, E. Torres Davila (2023)
Numerical semigroups, polyhedra, and posets III: minimal presentations and face dimension.
Electronic Journal of Combinatorics **30** (2023), no. 2, #P2.5. (arXiv:2009.05921)

B. Braun, T. Gomes, E. Miller, C. O'Neill, and A. Sobieska (2023)
Minimal free resolutions of numerical semigroup algebras via Apéry specialization
under review. (arXiv:2310.03612)


Thanks!