Classifying numerical semigroups using polyhedral geometry

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$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

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Multiplicity: $\mathrm{m}(S) =$ smallest nonzero element

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Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

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For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
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The Apéry set is a "one stop shop" for computation.

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Theorem

If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup S with $\mathrm{Ap}(S) = A$ if and only if

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Big idea: the inequalities "$a_i + a_j \geq a_{i+j}$" to define a **cone** C_m.

Kunz cone

Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

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$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\ge 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

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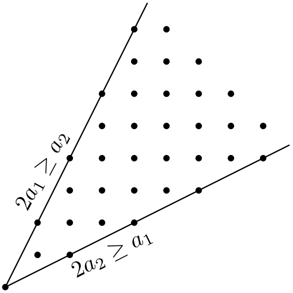
Example: C_3



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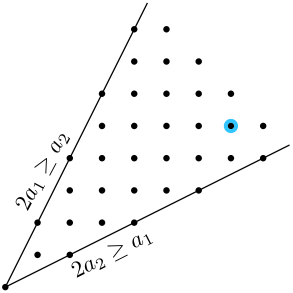

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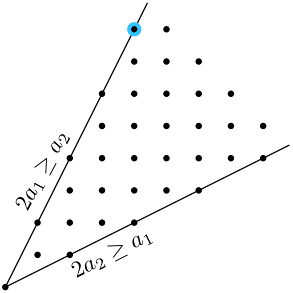

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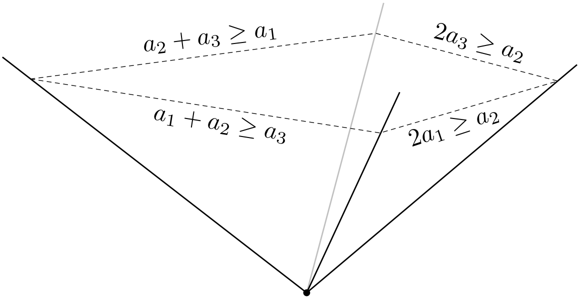

Faces of the Kunz cone

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When are numerical semigroups in (the relative interior of) the same face?

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What about the other faces?

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$$2a_1 > a_2 \qquad a_1 + a_2 = a_3$$
$$2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

Example: $S = \langle 4, 13 \rangle$

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$$2a_1 = a_2 \qquad a_1 + a_2 = a_3$$
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Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

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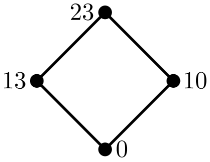
Definition

The *Apéry poset* of S: define $a \preceq a'$ whenever $a' - a \in S$.

$\mathrm{Ap}(S) = \{0, 13, 10, 23\}$



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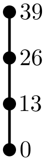

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$$S = \langle 6, 9, 20 \rangle$$
$$\mathrm{Ap}(S) = \{0, 49, 20, \ \ 9, 40, 29\}$$

$$S' = \langle 6, 26, 27 \rangle$$
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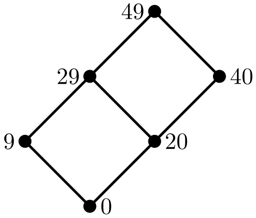

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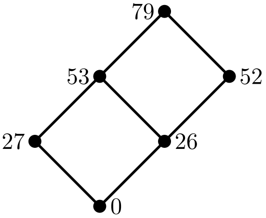

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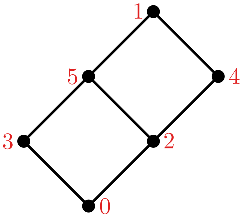

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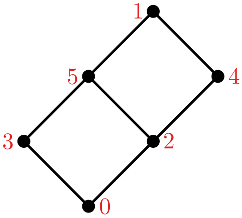

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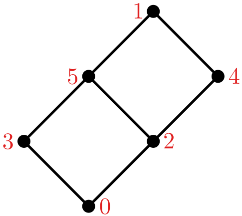

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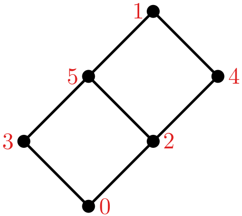

The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

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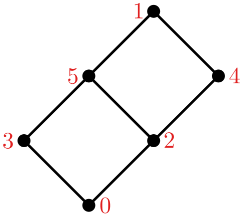

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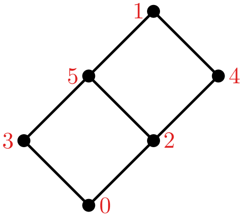

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Theorem (Bruns–García-Sánchez–O.–Wilburne)

Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

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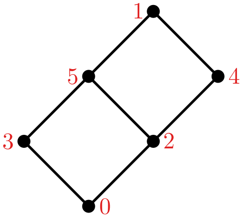

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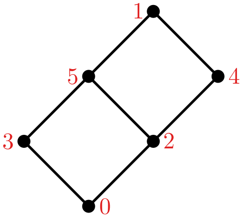

Defining facet equations:

$$2a_2 = a_4$$

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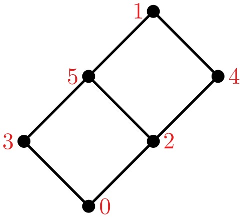

Defining facet equations:

$2a_2 = a_4$	$2 \preceq 4$
$a_2 + a_3 = a_5$	$2 \preceq 5$
	$3 \preceq 5$
$a_2 + a_5 = a_1$	$2 \preceq 1$
	$5 \preceq 1$
$a_3 + a_4 = a_1$	$3 \preceq 1$
	$4 \preceq 1$

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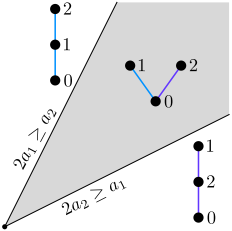
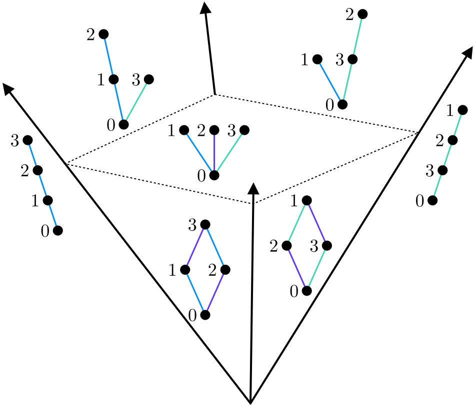
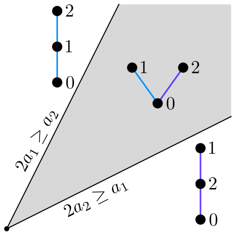
C_3 and C_4

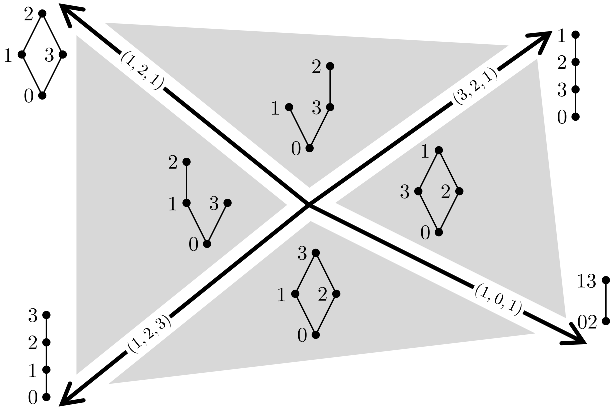




C_3 and C_4

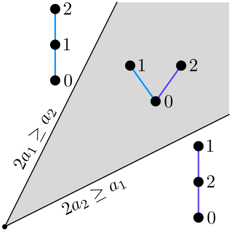


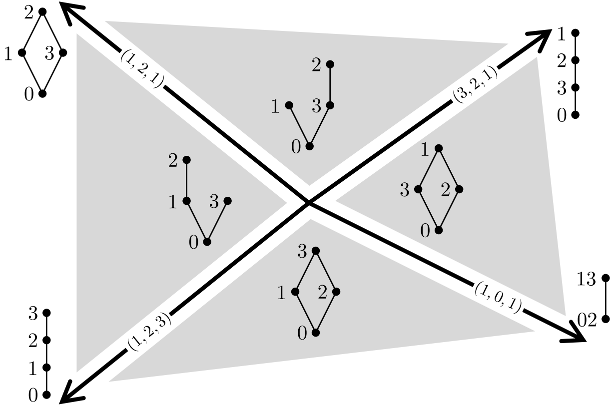
2
1
0
1
2
0
$2a_1 \geq a_2$
$2a_2 \geq a_1$
1
2
0



2
1
3
0
(1, 2, 1)
(3, 2, 1)
1
2
3
0
2
1
3
0
2
1
3
0
1
3
2
0
3
1
2
0
(1, 0, 1)
13
02
(1, 2, 3)
3
2
1
0

C_3 and C_4





Theorem (Kaplan–O.)

There is a natural labeling of the faces of C_m by finite posets.

Shared properties within a face

What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

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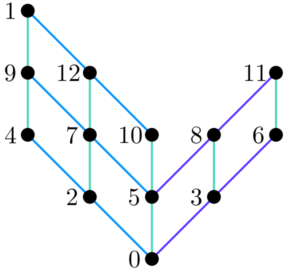
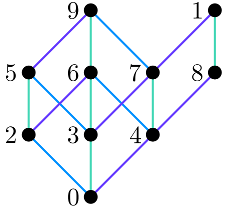
What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of P





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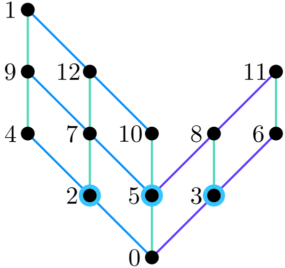

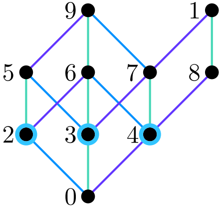

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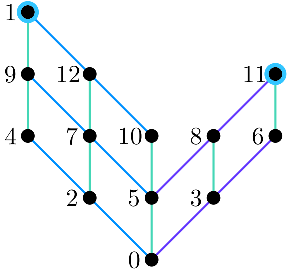
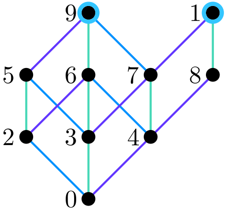
What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of P
- $\mathrm{t}(S) = \#$ maximal elements
  (Cohen-Macaulay type of S)





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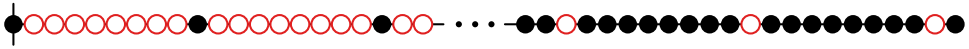
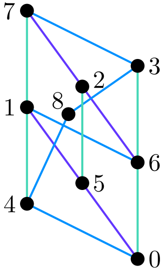
- Symmetric/Gorenstein?

$S = \langle 4, 7 \rangle$

$S = \langle 9, 40, 50, 60 \rangle$





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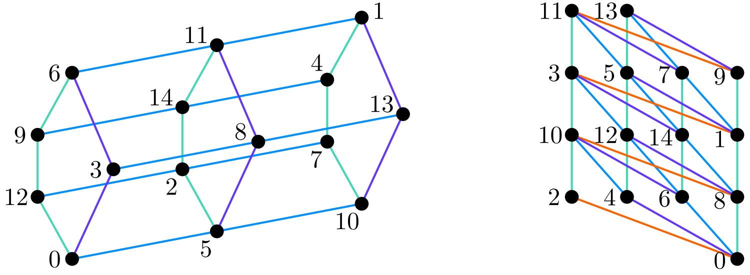

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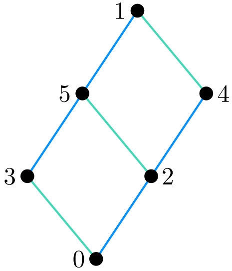
$$I_S = \ker\left(\Bbbk[\overline{x}] \to \Bbbk[t]\right) \qquad x_i \mapsto t^{n_i}$$

$$S = \langle 6, 9, 20 \rangle$$

$$I_S = \langle x^3 - y^2, x^4y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$$



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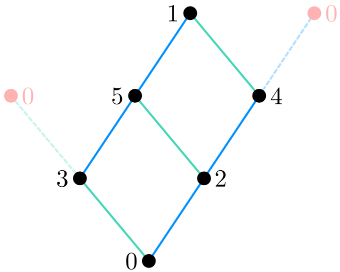

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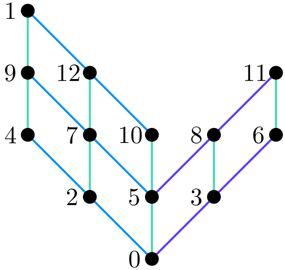

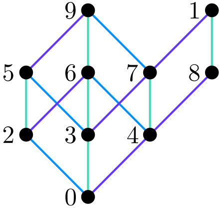

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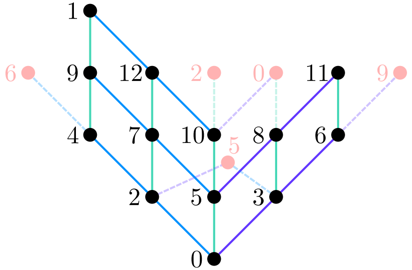

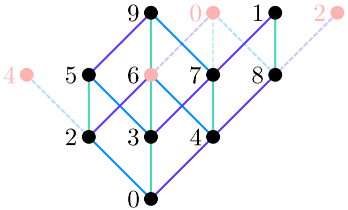

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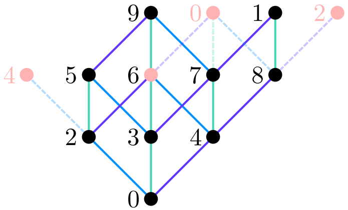
$S = \langle 10, a_2, a_3, a_4 \rangle$

$$I_S = \langle\, x_2^2 - y^* x_4,\ x_2 x_4 - x_3^2,\ x_3^2 x_4 - y^*,\ x_4^3 - y^* x_2 \,\rangle \subseteq \Bbbk[y, x_2, x_3, x_4]$$



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- Betti numbers of I_S over $\Bbbk[\overline{x}]$

$$
0 \leftarrow R \leftarrow R^6 \xleftarrow{\;\begin{array}{c|cccccccc}
 & 1,12 & 1,13 & 2,12 & 2,23 & 3,13 & 3,23 & 2,13 & 3,12 \\
\hline
1,1 & -x_2 & -x_3 & & & & & y^* & y^* \\
2,2 & -y^* & & x_1 & -x_3 & & y^* & & \\
3,3 & & & & & x_1 & x_2 & -y^* & \\
2,1 & x_1 & & -x_2 & y^* & y^* & & -x_3 & \\
3,1 & y^* & x_1 & & & -x_3 & -y^* & & -x_2 \\
3,2 & & -y^* & -y^* & x_2 & & -x_3 & x_1 & x_1
\end{array}\;} R^8 \xleftarrow{\;\begin{array}{c|ccc}
 & 1,[3] & 2,[3] & 3,[3] \\
\hline
1,12 & x_3 & -y^* & \\
1,13 & -x_2 & & y^* \\
2,12 & & x_3 & -y^* \\
2,23 & -y^* & x_1 & \\
3,13 & y^* & & -x_2 \\
3,23 & & -y^* & x_1 \\
2,13 & x_1 & -x_2 & \\
3,12 & -x_1 & & x_3
\end{array}\;} R^3 \leftarrow 0
$$

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- Betti numbers of I_S over $\Bbbk[\overline{x}]$
- Betti numbers of $\Bbbk$ over $\Bbbk[\overline{x}]/I_S$

$$0 \leftarrow R \xleftarrow{\begin{array}{cc} & \begin{array}{cccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \emptyset & \begin{bmatrix} y & x_1 & x_2 & x_3 \end{bmatrix} \end{array}} R^4 \xleftarrow{\begin{array}{cc} & \begin{array}{cccccccccccc} \mathbf{01} & \mathbf{02} & \mathbf{03} & \mathbf{11} & \mathbf{12} & \mathbf{13} & \mathbf{21} & \mathbf{22} & \mathbf{23} & \mathbf{31} & \mathbf{32} & \mathbf{33} \end{array} \\ \begin{array}{c} \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} & \begin{bmatrix} x_1 & x_2 & x_3 & & & -y^* & & -y^* & & -y^* & & \\ -y & & & x_1 & x_2 & x_3 & & & -y^* & & -y^* & \\ & -y & & -y^* & & & x_1 & x_2 & x_3 & & & -y^* \\ & & -y & & -y^* & & -y^* & & & x_1 & x_2 & x_3 \end{bmatrix} \end{array}} R^{12}$$

$$\longleftarrow R^{36} \longleftarrow R^{108} \longleftarrow R^{324} \longleftarrow R^{972} \longleftarrow R^{2916} \longleftarrow \cdots$$

Application 1: classifying minimal trades

Question

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$k = 3$ $k = 4$ $k = 5$ $k = 6$ $k = 7$

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Well known: $\beta_1(S) \le \binom{m}{2}$, with equality if and only if $k = m$

if $k = 3$, then $\beta_1(S) = 2, 3$

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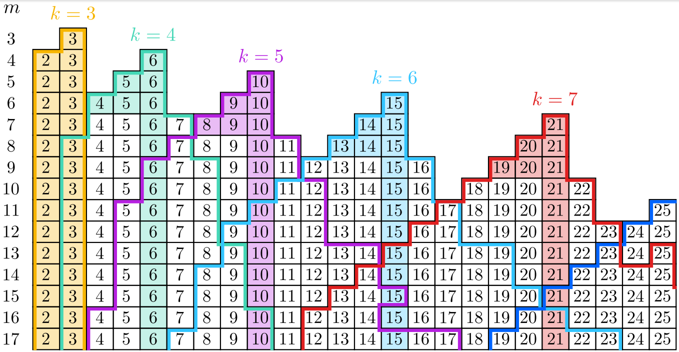

Prior work: a family has $\beta_1(S) = \binom{k}{2}$ for $3 \le k \le m$ (Rosales)

if $r = m - k \le 2$, then $\beta_1(S) \in [\binom{k}{2} - r, \binom{k}{2}]$ (GS-R)

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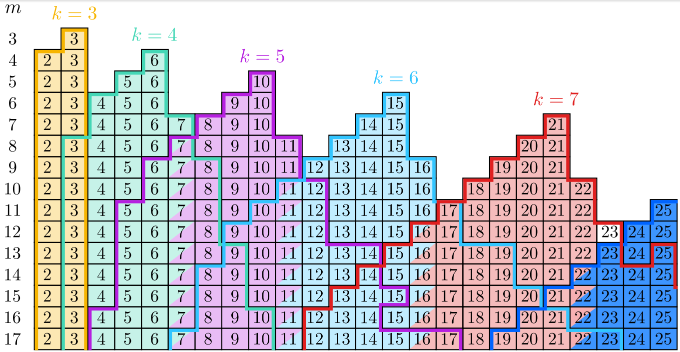

Using Kunz posets: a family hits each $\beta_1(S) \in [\binom{k}{2} - r, \binom{k}{2}]$

for $r = m - k \leq k - 2$

a family hits $\beta_1(S) = \binom{k}{2} + 1$ for each $m \geq k + 3$

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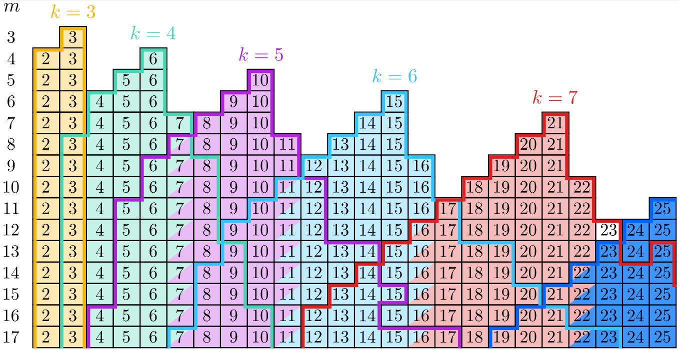

Bounds from Kunz posets: $\beta_1(S) \geq \binom{k}{2} - r$, where $r = m - k$

if $m - k = 3$, then $\beta_1(S) \in [\binom{k}{2} - 3, \binom{k}{2} + 1]$

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$k = 4$

m																								
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One more family: for $k = 4$, achieves each $\beta_1(S)$ with $(\beta_1(S) - 2)^2 \leq 4m$

conjectured to achieve every possible $\beta_1(S)$ for $k = 4$

Application 2: a long-standing (**hard**) conjecture

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For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathsf{F}(S) + 1 \le k(\mathsf{F}(S) + 1 - \mathsf{g}(S))$.

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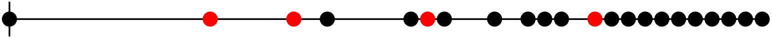
$$\frac{1}{k} \leq \underbrace{\frac{\mathrm{F}(S) + 1 - \mathrm{g}(S)}{\mathrm{F}(S) + 1}}_{\%\text{ of } [0, F(S)] \text{ in } S}$$



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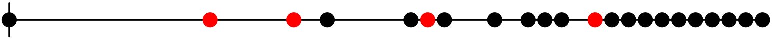
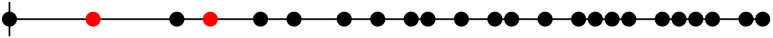


Equality holds when:

- $S = \langle a, b \rangle$



- $S = \langle m, m + 1, \ldots, 2m - 1 \rangle$

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If S corresponds to $x = (a_1, \ldots, a_{m-1}) \in C_m$,

$$\mathrm{g}(S) = \|x\|_1 - \tfrac{1}{2}m(m-1), \qquad \mathrm{F}(S) = \|x\|_\infty - m,$$

and $\#$ generators k is determined by the face $F \subseteq C_m$ containing x.

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Wilf's conjecture in fixed multiplicity
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