Classifying numerical semigroups using polyhedral geometry

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$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

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Fact

Every numerical semigroup has a unique minimal generating set.

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Example: if $S = \langle 6, 9, 20 \rangle$, then $F(S) = 43$ and $g(S) = 22$ since

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Computing the genus is equally hard

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Goal (as mathematicians)

Understand the structure of numerical semigroups

A couple of long-standing (**hard**) conjectures

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Example: $n_3 = 4$

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\langle 2, 7 \rangle &= \{0, \quad 2, \quad 4, \quad 6, 7, 8, \ldots\} \\
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Conjecture (Bras-Amoros, 2008)

For all g, we have $n_g \geq n_{g-1}$.

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For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathsf{F}(S) + 1 \leq k(\mathsf{F}(S) + 1 - \mathsf{g}(S))$.

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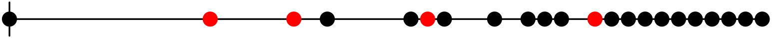

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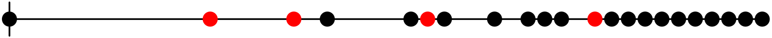
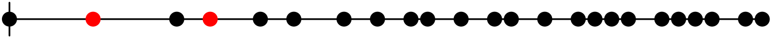


Equality holds when:

- $S = \langle a, b \rangle$



- $S = \langle m, m + 1, \ldots, 2m - 1 \rangle$

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$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
For 3 mod 6: $\{3, 9, 15, 21, \ldots\} \cap S = \{9, 15, 21, \ldots\}$
For 4 mod 6: $\{4, 10, 16, 22, \ldots\} \cap S = \{40, 46, 52, \ldots\}$

Facts: $|\,\mathrm{Ap}(S)| = m$, and the elements of $\mathrm{Ap}(S)$ are distinct modulo m

$$\mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} \qquad \text{where each} \qquad a_i \equiv i \bmod m$$

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The Apéry set is a "one stop shop" for computation.

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If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup S with $\mathrm{Ap}(S) = A$ if and only if

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Big idea: the inequalities "$a_i + a_j \geq a_{i+j}$" to define a **cone** C_m.

Kunz cone

Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

$$a_i + a_j \geq a_{i+j} \qquad \text{whenever} \qquad i + j \neq 0.$$

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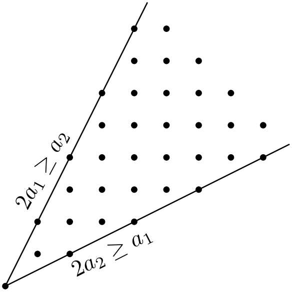
Example: C_3



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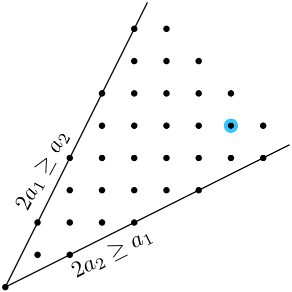

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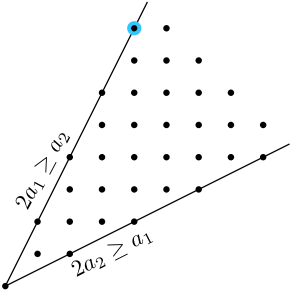

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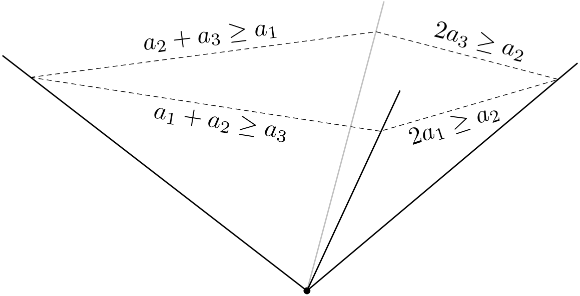
Example: C_4



Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

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Big picture: "moduli space" approach for studying XYZ's

- Define a space with XYZ's as points
  Small changes to an XYZ $\rightsquigarrow$ small movements in space
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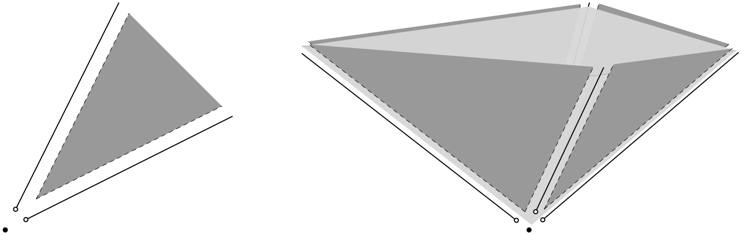
More interesting example: C_m



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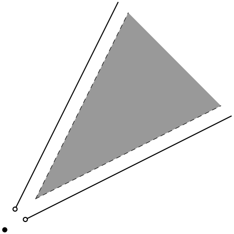

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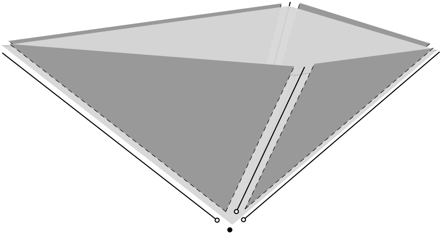

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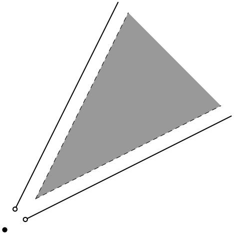

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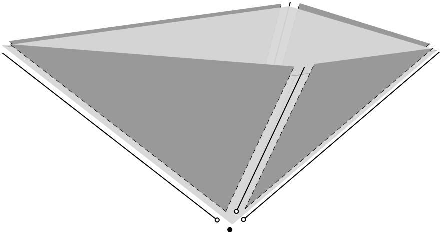

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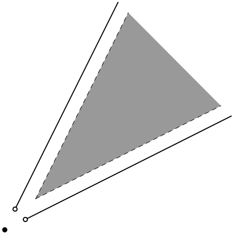

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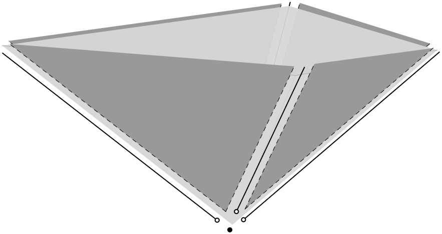

$C_5 \subseteq \mathbb{R}^4$? Cross section:

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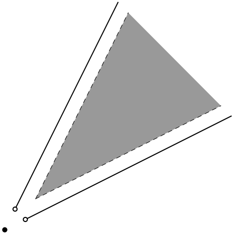

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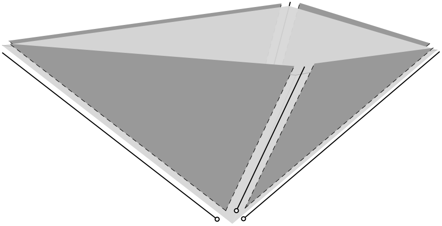

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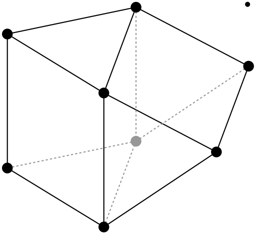

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What about the other faces?

Faces of the Kunz cone

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Example: $S = \langle 4, 10, 11, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 11\}$$
$$a_1 = 13, \quad a_2 = 10, \quad a_3 = 11$$

$$2a_1 > a_2 \qquad a_1 + a_2 > a_3$$
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The *Apéry poset* of S: define $a \preceq a'$ whenever $a' - a \in S$.

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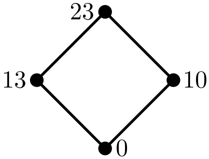

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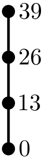

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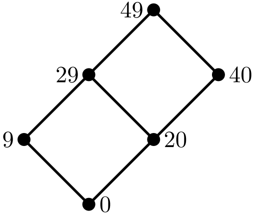

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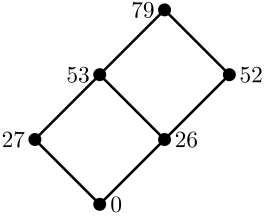
$\mathrm{Ap}(S') = \{0, 79, 26, 27, 52, 53\}$



Faces of the Kunz cone

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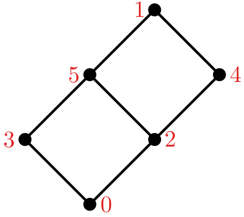

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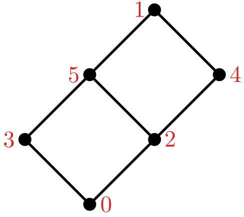

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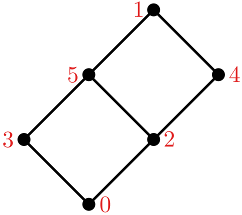

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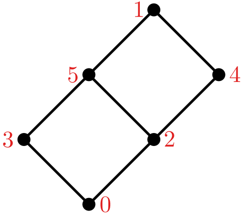

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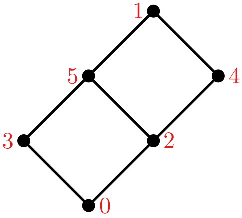

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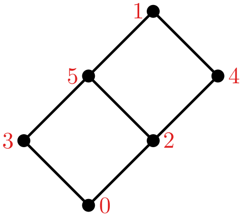

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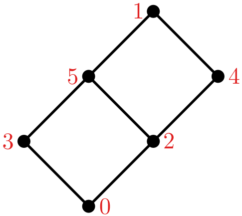

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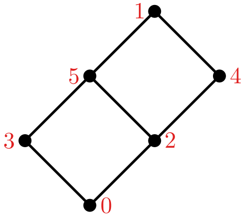

Defining facet equations:

$$2a_2 = a_4$$

$$a_2 + a_3 = a_5$$

$$a_2 + a_5 = a_1$$

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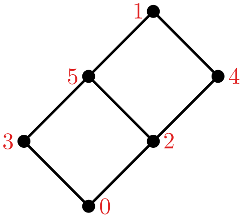

Defining facet equations:

$2a_2 = a_4$	$2 \preceq 4$
$a_2 + a_3 = a_5$	$2 \preceq 5$
	$3 \preceq 5$
$a_2 + a_5 = a_1$	$2 \preceq 1$
	$5 \preceq 1$
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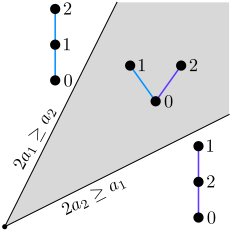
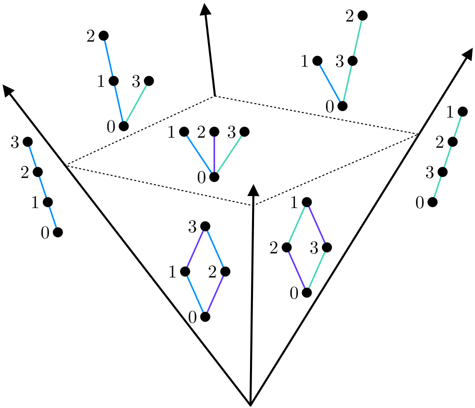
C_3 and C_4


2
1
0
1
2
0
1
2
0
$2a_1 \geq a_2$
$2a_2 \geq a_1$


2
1
3
0
2
1
3
0
3
2
1
0
1
2
3
0
1
2
3
0
3
1
2
0
1
2
3
0

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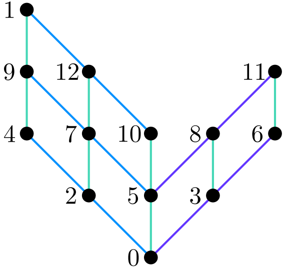

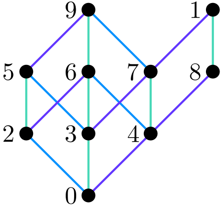

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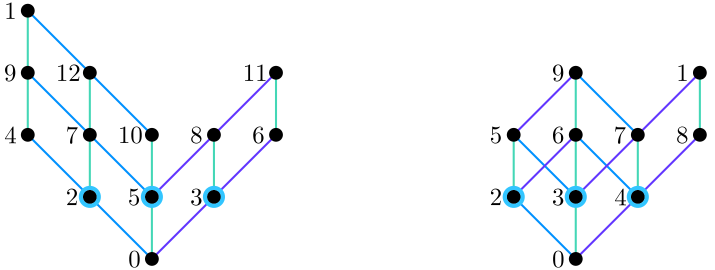

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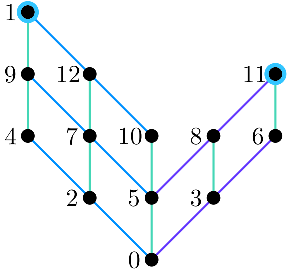

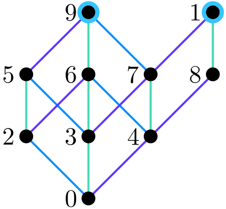

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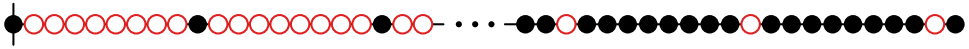
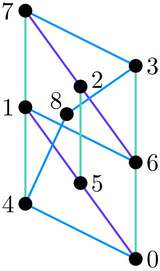
$S = \langle 4, 7 \rangle$

$S = \langle 9, 40, 50, 60 \rangle$





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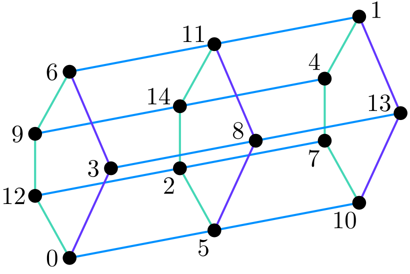

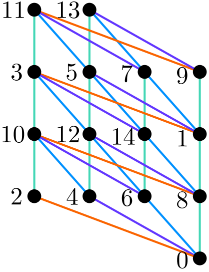

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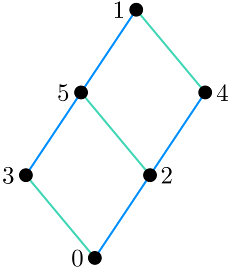

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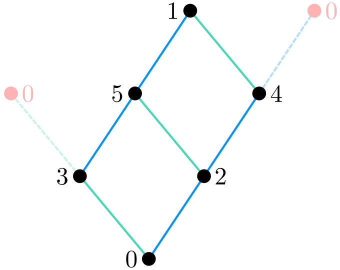

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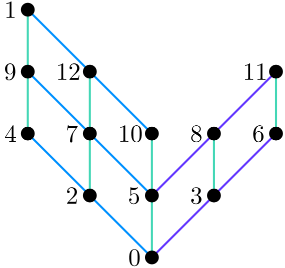

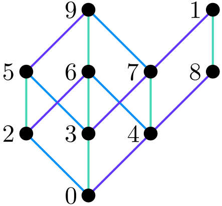

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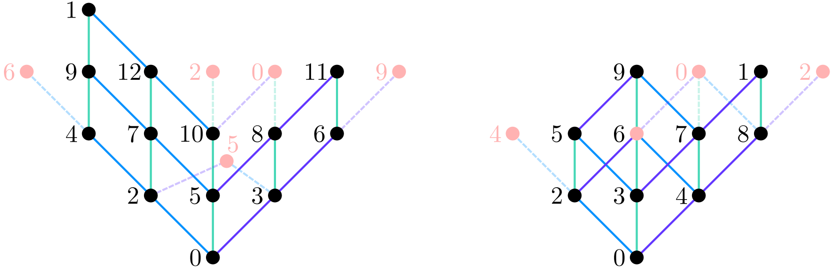

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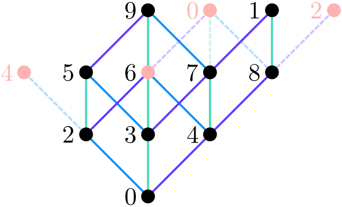
$$S = \langle 10, a_2, a_3, a_4 \rangle$$

$$\begin{aligned} I_S = \langle\, & x_2^2 - \textcolor{red}{y^*} x_4,\ \ x_2 x_4 - x_3^2, \\ & x_3^2 x_4 - \textcolor{red}{y^*},\ \ \ x_4^3 - \textcolor{red}{y^*} x_2 \,\rangle \\ \subseteq\ & \Bbbk[y, x_2, x_3, x_4] \end{aligned}$$



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$$0 \leftarrow R \leftarrow R^6 \xleftarrow{\begin{array}{c|cccccccc} & \mathbf{1,12} & \mathbf{1,13} & \mathbf{2,12} & \mathbf{2,23} & \mathbf{3,13} & \mathbf{3,23} & \mathbf{2,13} & \mathbf{3,12} \\ \hline \mathbf{1,1} & -x_2 & -x_3 & & & & & y^* & y^* \\ \mathbf{2,2} & -y^* & & x_1 & -x_3 & & y^* & & \\ \mathbf{3,3} & & & & & x_1 & x_2 & -y^* & \\ \mathbf{2,1} & x_1 & & -x_2 & y^* & y^* & & -x_3 & \\ \mathbf{3,1} & y^* & x_1 & & & -x_3 & -y^* & & -x_2 \\ \mathbf{3,2} & & -y^* & -y^* & x_2 & & -x_3 & x_1 & x_1 \end{array}} R^8 \xleftarrow{\begin{array}{c|ccc} & \mathbf{1,[3]} & \mathbf{2,[3]} & \mathbf{3,[3]} \\ \hline \mathbf{1,12} & x_3 & -y^* & \\ \mathbf{1,13} & -x_2 & & y^* \\ \mathbf{2,12} & & x_3 & -y^* \\ \mathbf{2,23} & -y^* & x_1 & \\ \mathbf{3,13} & y^* & & -x_2 \\ \mathbf{3,23} & & -y^* & x_1 \\ \mathbf{2,13} & x_1 & -x_2 & \\ \mathbf{3,12} & -x_1 & & x_3 \end{array}} R^3 \leftarrow 0$$

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- Betti numbers of $\Bbbk$ over $\Bbbk[\overline{x}]/I_S$

$$0 \leftarrow R \xleftarrow{\begin{array}{cccccc} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \\ \emptyset & [y & x_1 & x_2 & x_3] & \end{array}} R^4 \xleftarrow{\begin{array}{c} \begin{array}{ccccccccccccc} & \mathbf{01} & \mathbf{02} & \mathbf{03} & \mathbf{11} & \mathbf{12} & \mathbf{13} & \mathbf{21} & \mathbf{22} & \mathbf{23} & \mathbf{31} & \mathbf{32} & \mathbf{33} \\ \mathbf{0} & x_1 & x_2 & x_3 & & & -y^* & & -y^* & & -y^* & & \\ \mathbf{1} & -y & & & x_1 & x_2 & x_3 & & & -y^* & & -y^* & \\ \mathbf{2} & & -y & & -y^* & & & x_1 & x_2 & x_3 & & & -y^* \\ \mathbf{3} & & & -y & & -y^* & & -y^* & & & x_1 & x_2 & x_3 \end{array} \end{array}} R^{12}$$

$$\longleftarrow R^{36} \longleftarrow R^{108} \longleftarrow R^{324} \longleftarrow R^{972} \longleftarrow R^{2916} \longleftarrow \cdots$$

References

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