Classifying numerical semigroups using polyhedral geometry

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Slides available: `https://cdoneill.sdsu.edu/`

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Example:

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \ldots \\ \ldots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

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Multiplicity: $\mathrm{m}(S) =$ smallest nonzero element

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Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

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For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
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- $|\,\mathrm{Ap}(S)| = m$

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The Apéry set is a "one stop shop" for computation.

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Theorem

If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup S with $\mathrm{Ap}(S) = A$ if and only if

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Big idea: the inequalities "$a_i + a_j \ge a_{i+j}$" to define a **cone** C_m.

Kunz cone

Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

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$$\begin{array}{rcl} \{S \subseteq \mathbb{Z}_{\geq 0} : \mathrm{m}(S) = m\} & \longrightarrow & C_m \\ \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\} & \longmapsto & (a_1, \ldots, a_{m-1}) \end{array}$$

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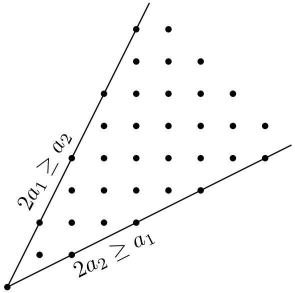

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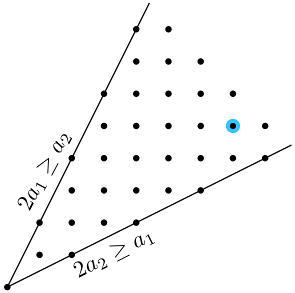

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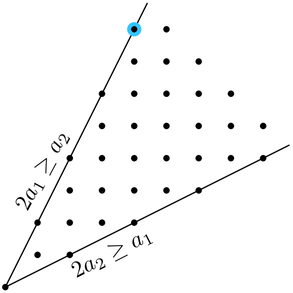

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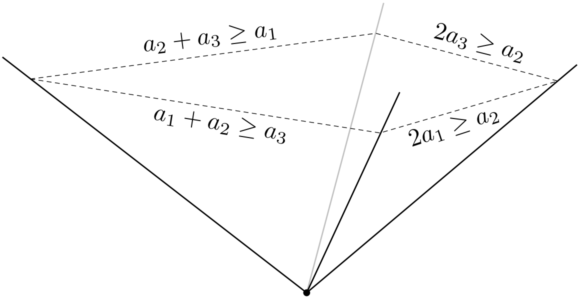

Faces of the Kunz cone

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When are numerical semigroups in (the relative interior of) the same face?

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Big picture: "moduli space" approach for studying XYZ's

- Define a space with XYZ's as points
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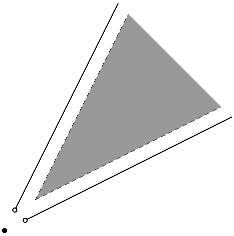
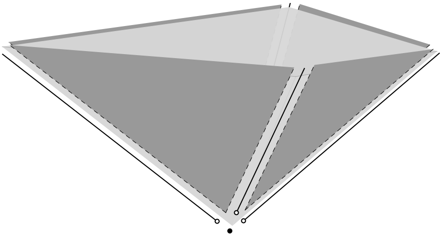
More interesting example: C_m





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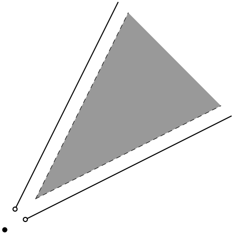

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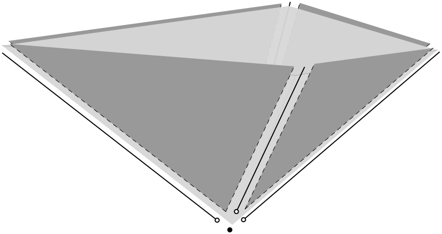

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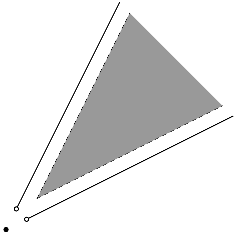

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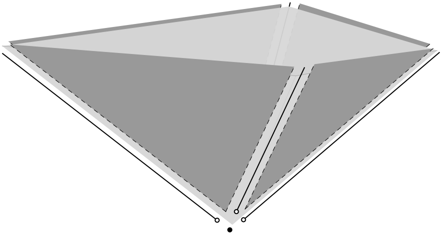

$C_5 \subseteq \mathbb{R}^4$?

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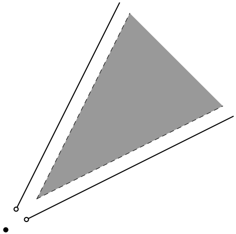

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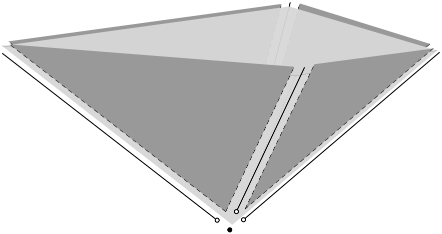

$C_5 \subseteq \mathbb{R}^4$? Cross section:

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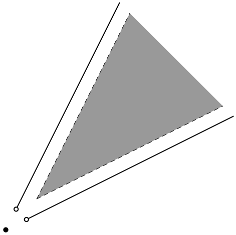

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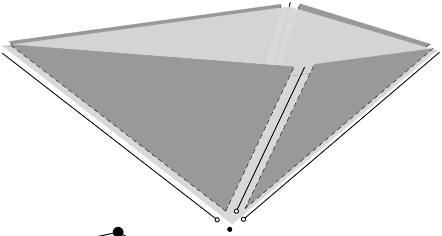
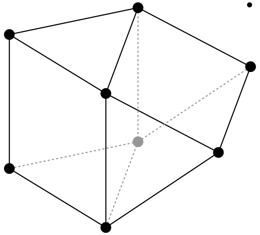


$C_5 \subseteq \mathbb{R}^4$? Cross section:



Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Faces of the Kunz cone

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First steps: $S \in \operatorname{Int}(C_m)$ if and only if S has *max embedding dimension*

Faces of the Kunz cone

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If $S = \langle n_1, \ldots, n_k \rangle$, then

$$n_i \not\equiv n_j \bmod n_1 \quad \Longrightarrow \quad k \le \mathrm{m}(S)$$

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$$S = \langle m, a_1, \ldots, a_{m-1} \rangle \quad \text{where} \quad \mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\}$$

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What about the other faces?

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Example: $S = \langle 4, 10, 11, 13 \rangle$

$$\mathrm{Ap}(S) = \{0, 13, 10, 11\}$$
$$a_1 = 13, \quad a_2 = 10, \quad a_3 = 11$$

$$2a_1 > a_2 \qquad a_1 + a_2 > a_3$$
$$2a_3 > a_2 \qquad a_2 + a_3 > a_1$$

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Example: $S = \langle 4, 10, 13 \rangle$

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Example: $S = \langle 4, 13 \rangle$

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Faces of the Kunz cone

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Faces of the Kunz cone

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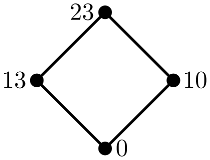
When are numerical semigroups in (the relative interior of) the same face?

Definition

The *Apéry poset* of S: define $a \preceq a'$ whenever $a' - a \in S$.

$\mathrm{Ap}(S) = \{0, 13, 10, 23\}$



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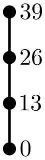

Faces of the Kunz cone

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When are numerical semigroups in (the relative interior of) the same face?

Faces of the Kunz cone

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When are numerical semigroups in (the relative interior of) the same face?

$$S = \langle 6, 9, 20 \rangle \qquad\qquad S' = \langle 6, 26, 27 \rangle$$

$$\mathrm{Ap}(S) = \{0, 49, 20, \; 9, 40, 29\} \qquad\qquad \mathrm{Ap}(S') = \{0, 79, 26, 27, 52, 53\}$$

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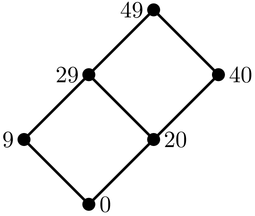

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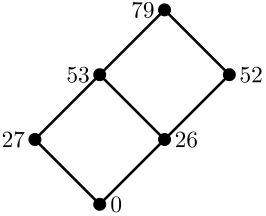

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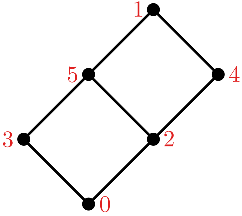

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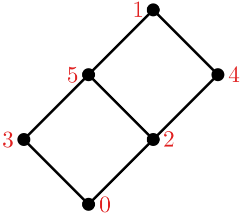

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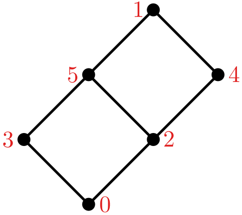
1 2 3 4 5



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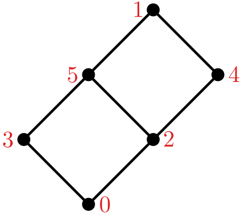

The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

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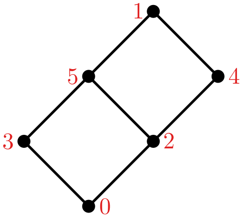

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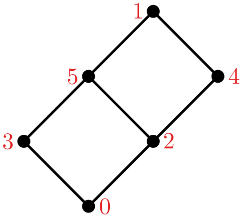

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Theorem (Bruns–García-Sánchez–O.–Wilburne)

Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

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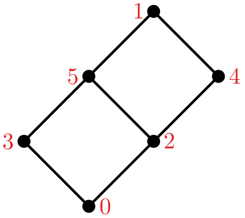

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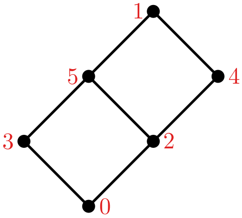

Defining facet equations:

$$2a_2 = a_4$$

$$a_2 + a_3 = a_5$$

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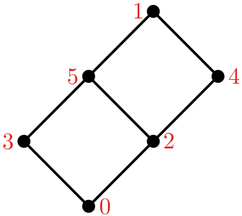
$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$ (indices 1, 2, 3, 4, 5)



Defining facet equations:

$2a_2 = a_4$	$2 \preceq 4$
$a_2 + a_3 = a_5$	$2 \preceq 5$
	$3 \preceq 5$
$a_2 + a_5 = a_1$	$2 \preceq 1$
	$5 \preceq 1$
$a_3 + a_4 = a_1$	$3 \preceq 1$
	$4 \preceq 1$

The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

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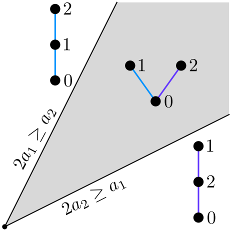
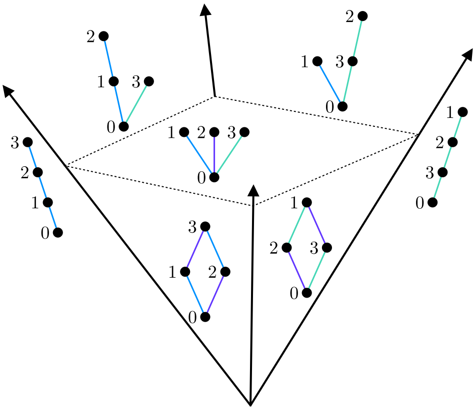
C_3 and C_4


2
1
0
1
2
0
$2a_1 \geq a_2$
$2a_2 \geq a_1$
1
2
0


2
1
3
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2
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2
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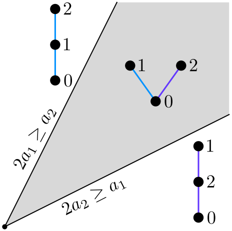
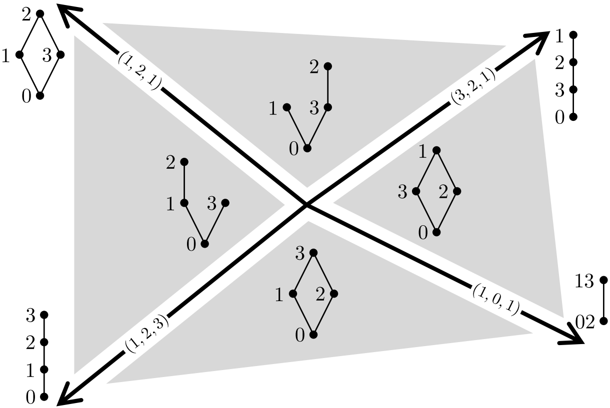
3
0

C_3 and C_4





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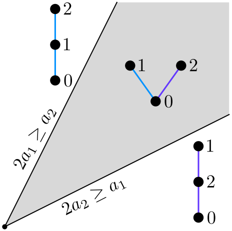

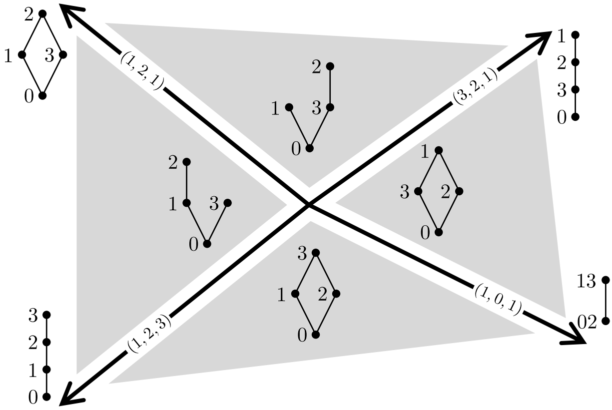

Theorem (Kaplan–O.)

There is a natural labeling of the faces of C_m by finite posets.

Shared properties within a face

What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

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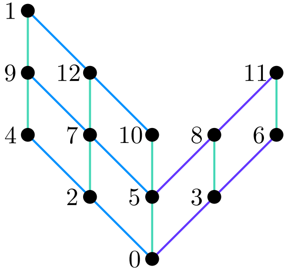

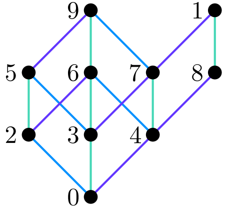

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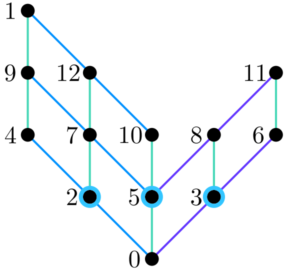

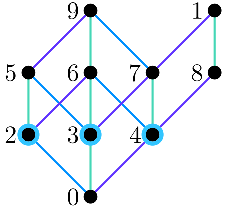

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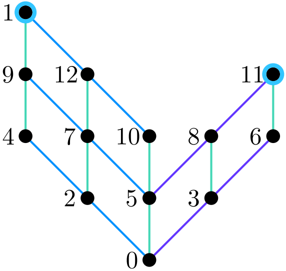
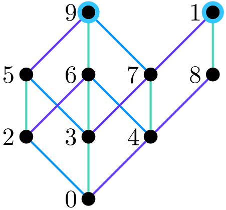
What properties are determined by the Kunz poset P of $S = \langle n_1, \ldots, n_k \rangle$?

- $k = 1 + \#$ atoms of P
- $\mathrm{t}(S) = \#$ maximal elements
  (Cohen-Macaulay type of S)





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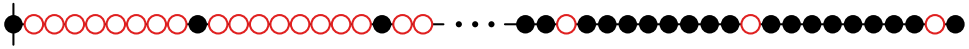
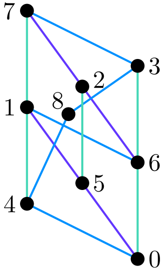
- Symmetric/Gorenstein?

$S = \langle 4, 7 \rangle$

$S = \langle 9, 40, 50, 60 \rangle$





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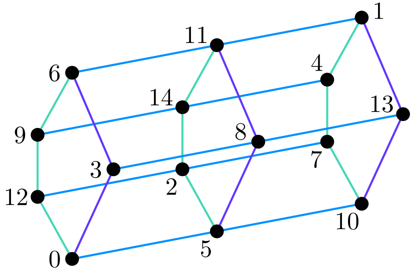
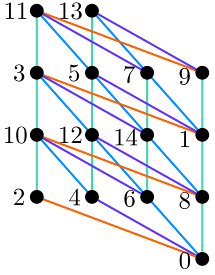
- Complete intersection?
- Generalized arithmetical?





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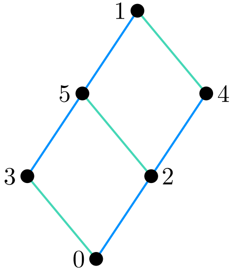
- Minimal binomial generators of the *defining toric ideal* of S:

$$I_S = \ker\left(\Bbbk[\overline{x}] \to \Bbbk[t]\right) \quad x_i \mapsto t^{n_i}$$

$S = \langle 6, 9, 20 \rangle$

$$I_S = \langle x^3 - y^2, x^4y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$$



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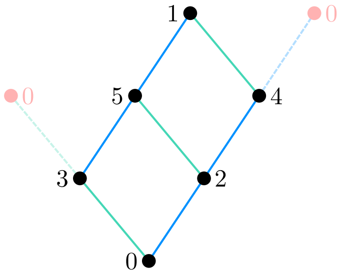

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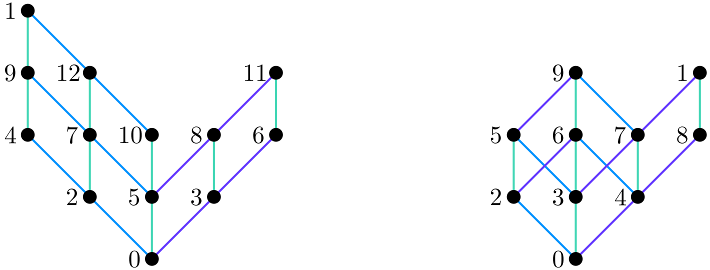

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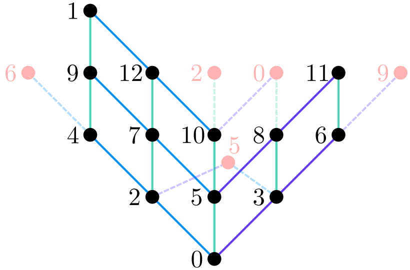

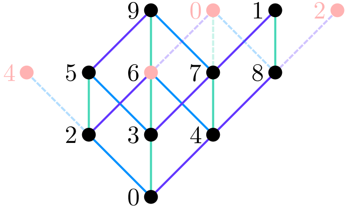

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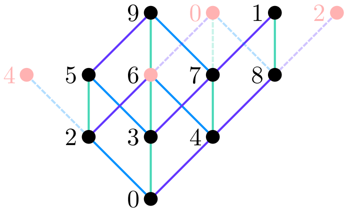
$S = \langle 10, a_2, a_3, a_4 \rangle$

$$I_S = \langle\, x_2^2 - y^* x_4,\ x_2x_4 - x_3^2,\ x_3^2x_4 - y^*,\ x_4^3 - y^* x_2 \,\rangle \subseteq \Bbbk[y, x_2, x_3, x_4]$$



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- Betti numbers of I_S over $\Bbbk[\overline{x}]$

$$0 \leftarrow R \leftarrow R^6 \longleftarrow R^8 \longleftarrow R^3 \leftarrow 0$$

Matrix for $R^6 \longleftarrow R^8$:

	1,12	**1,13**	**2,12**	**2,23**	**3,13**	**3,23**	**2,13**	**3,12**
1,1	$-x_2$	$-x_3$					y^*	y^*
2,2	$-y^*$		x_1	$-x_3$		y^*		
3,3					x_1	x_2	$-y^*$	
2,1	x_1		$-x_2$	y^*	y^*		$-x_3$	
3,1	y^*	x_1			$-x_3$	$-y^*$		$-x_2$
3,2		$-y^*$	$-y^*$	x_2		$-x_3$	x_1	x_1

Matrix for $R^8 \longleftarrow R^3$:

	1,[3]	**2,[3]**	**3,[3]**
1,12	x_3	$-y^*$	
1,13	$-x_2$		y^*
2,12		x_3	$-y^*$
2,23	$-y^*$	x_1	
3,13	y^*		$-x_2$
3,23		$-y^*$	x_1
2,13	x_1	$-x_2$	
3,12	$-x_1$		x_3

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$$0 \leftarrow R \xleftarrow{\begin{array}{c|cccc} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \emptyset & y & x_1 & x_2 & x_3 \end{array}} R^4 \xleftarrow{\begin{array}{c|cccccccccccc} & \mathbf{01} & \mathbf{02} & \mathbf{03} & \mathbf{11} & \mathbf{12} & \mathbf{13} & \mathbf{21} & \mathbf{22} & \mathbf{23} & \mathbf{31} & \mathbf{32} & \mathbf{33} \\ \mathbf{0} & x_1 & x_2 & x_3 & & & -y^* & & -y^* & & -y^* & & \\ \mathbf{1} & -y & & & x_1 & x_2 & x_3 & & & -y^* & & -y^* & \\ \mathbf{2} & & -y & & -y^* & & & x_1 & x_2 & x_3 & & & -y^* \\ \mathbf{3} & & & -y & & -y^* & & -y^* & & & x_1 & x_2 & x_3 \end{array}} R^{12}$$

$$\longleftarrow R^{36} \longleftarrow R^{108} \longleftarrow R^{324} \longleftarrow R^{972} \longleftarrow R^{2916} \longleftarrow \cdots$$

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For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \leq k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

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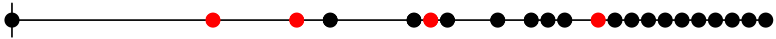
$$\frac{1}{k} \le \underbrace{\frac{\mathrm{F}(S) + 1 - \mathrm{g}(S)}{\mathrm{F}(S) + 1}}_{\%\text{ of } [0, F(S)] \text{ in } S}$$



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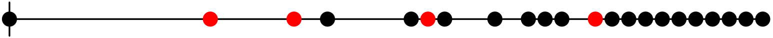
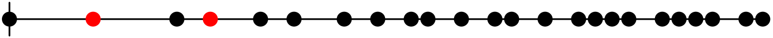


Equality holds when:

- $S = \langle a, b \rangle$



- $S = \langle m, m+1, \ldots, 2m-1 \rangle$

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Wilf's conjecture holds for all numerical semigroups S with $m \leq 18$.

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If S corresponds to $x = (a_1, \ldots, a_{m-1}) \in C_m$,

$$\mathrm{g}(S) = \|x\|_1 - \tfrac{1}{2}m(m-1), \qquad \mathrm{F}(S) = \|x\|_\infty - m,$$

and $\#$ generators k is determined by the face $F \subseteq C_m$ containing x.

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Example: $n_3 = 4$

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Not true for $n'_f = \#$ of numerical semigroups with Frobenius number f

$$n'_{11} = 51 \qquad n'_{12} = 40 \qquad n'_{13} = 106$$

References

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