# Classifying numerical semigroups using polyhedral geometry

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Slides available: `https://cdoneill.sdsu.edu/`

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*Multiplicity*: $\mathrm{m}(S) =$ smallest nonzero element

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The Apéry set is a "one stop shop" for computation.

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**Theorem**

*If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup $S$ with $\mathrm{Ap}(S) = A$ if and only if*

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Big idea: the inequalities "$a_i + a_j \geq a_{i+j}$" to define a **cone** $C_m$.

# Kunz cone

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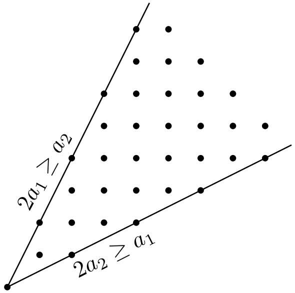

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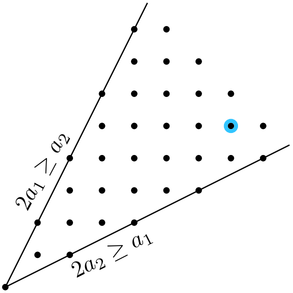


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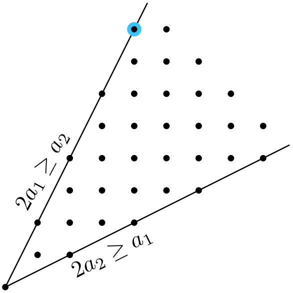


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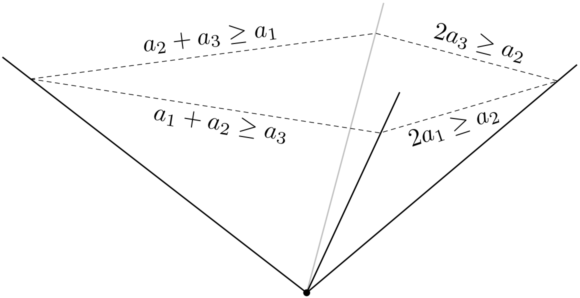

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Big picture: "parameter space" approach for studying $XYZ$'s

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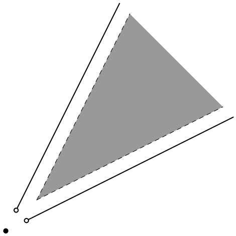
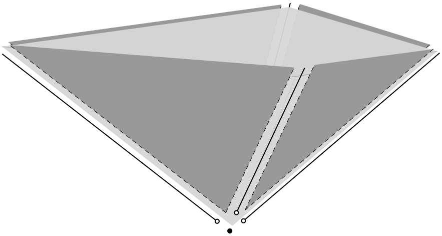
More interesting example: $C_m$





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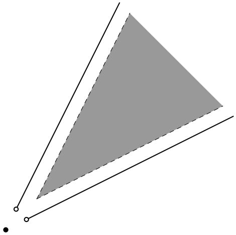

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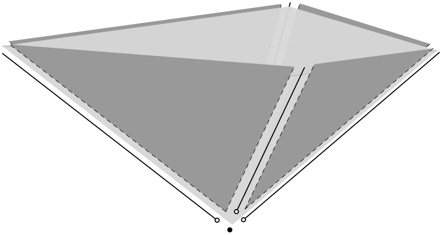

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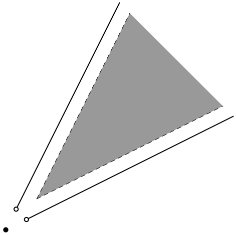

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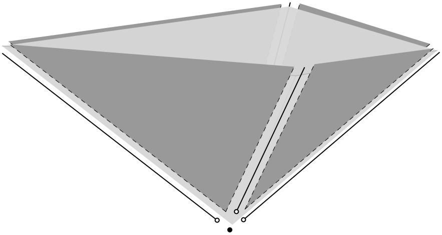

$C_5 \subseteq \mathbb{R}^4$?

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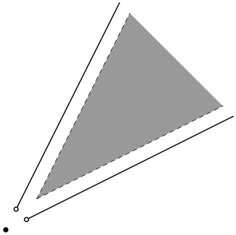

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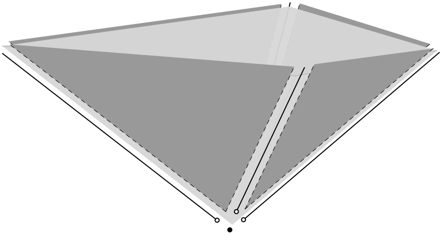


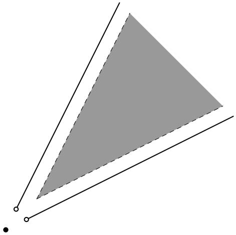
$C_5 \subseteq \mathbb{R}^4$? Cross section:

# Faces of the Kunz cone

## Question

When are numerical semigroups in (the relative interior of) the same face?

$C_3 \subseteq \mathbb{R}^2$



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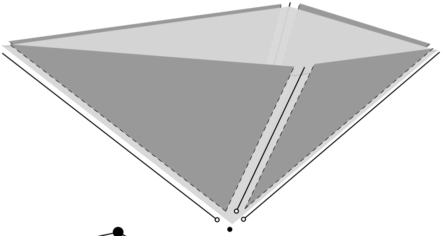

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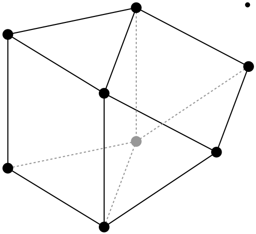


# Faces of the Kunz cone

**Question**

When are numerical semigroups in (the relative interior of) the same face?

# Faces of the Kunz cone

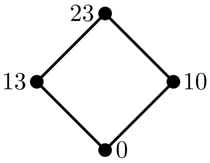
## Question

When are numerical semigroups in (the relative interior of) the same face?

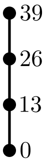
## Definition

The *Apéry poset* of $S$: define $a \preceq a'$ whenever $a' - a \in S$.

$$S = \langle 4, 10, 13 \rangle$$
$$\mathrm{Ap}(S) = \{0, 13, 10, 23\}$$




$$S = \langle 4, 13 \rangle$$
$$\mathrm{Ap}(S) = \{0, 13, 26, 39\}$$



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When are numerical semigroups in (the relative interior of) the same face?

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When are numerical semigroups in (the relative interior of) the same face?

$$S = \langle 6, 9, 20 \rangle \qquad S' = \langle 6, 26, 27 \rangle$$

$$\mathrm{Ap}(S) = \{0, 49, 20, \;\; 9, 40, 29\} \qquad \mathrm{Ap}(S') = \{0, 79, 26, 27, 52, 53\}$$

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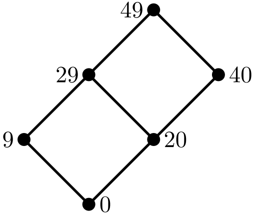


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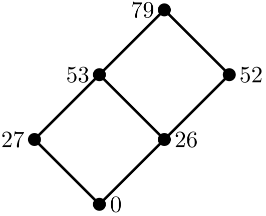

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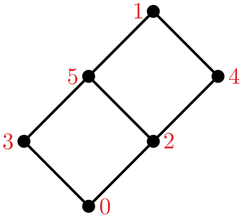


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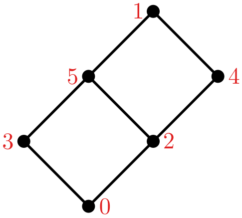

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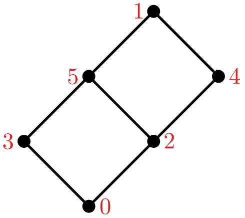


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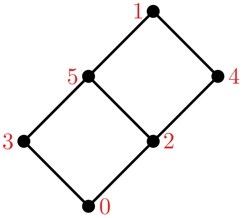


The *Kunz poset* of $S$: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

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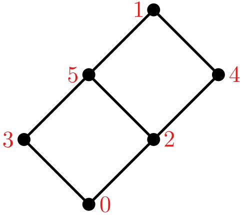


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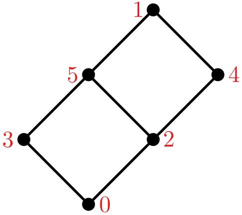


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## Theorem (Bruns–García-Sánchez–O.–Wilburne)

*Numerical semigroups lie in the relative interior of the same face of $C_m$ if and only if their Kunz posets are identical.*

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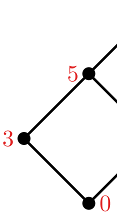


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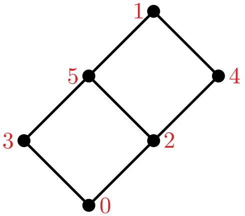


Defining facet equations:

$$2a_2 = a_4$$

$$a_2 + a_3 = a_5$$

$$a_2 + a_5 = a_1$$

$$a_3 + a_4 = a_1$$

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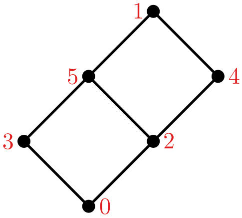


Defining facet equations:

| | |
|---|---|
| $2a_2 = a_4$ | $2 \preceq 4$ |
| $a_2 + a_3 = a_5$ | $2 \preceq 5$ |
| | $3 \preceq 5$ |
| $a_2 + a_5 = a_1$ | $2 \preceq 1$ |
| | $5 \preceq 1$ |
| $a_3 + a_4 = a_1$ | $3 \preceq 1$ |
| | $4 \preceq 1$ |

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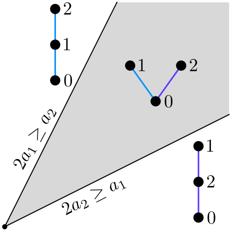
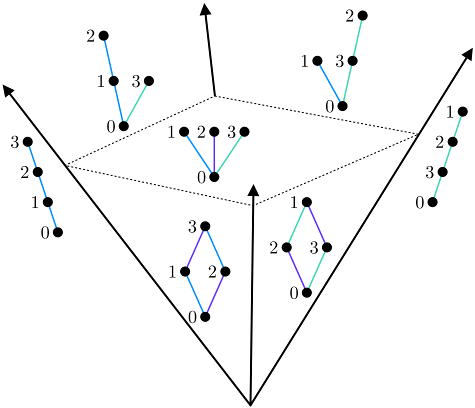
# $C_3$ and $C_4$



2
1
0
1
2
0
$2a_1 \geq a_2$
$2a_2 \geq a_1$
1
2
0

2
1
3
0
2
1
3
0
3
2
1
0
1
2
3
0
1
2
3
0
3
1
2
0
1
2
3
0

# Shared properties within a face

What properties are determined by the Kunz poset $P$ of $S = \langle n_1, \ldots, n_k \rangle$?

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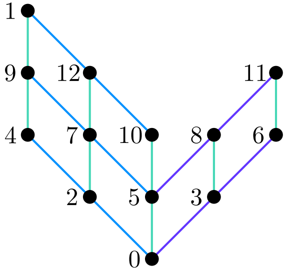


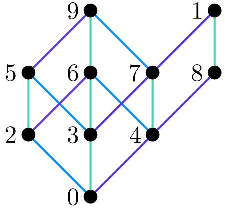

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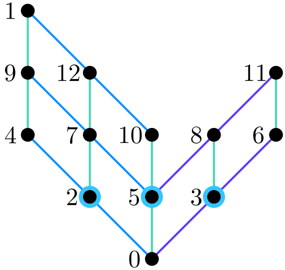


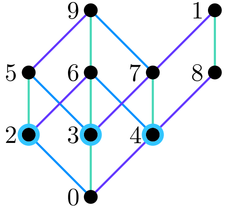

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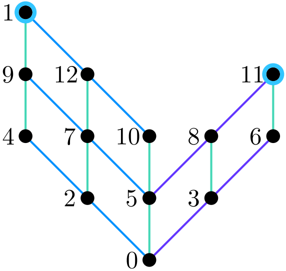


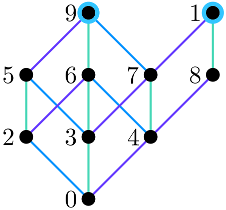

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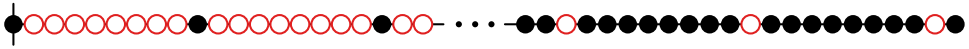
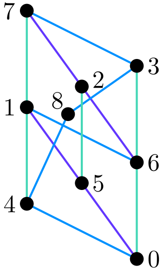
- Symmetric/Gorenstein?

---

$S = \langle 4, 7 \rangle$

$S = \langle 9, 40, 50, 60 \rangle$





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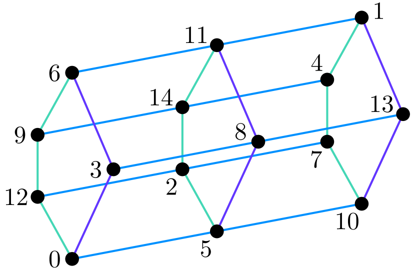


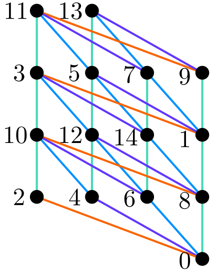

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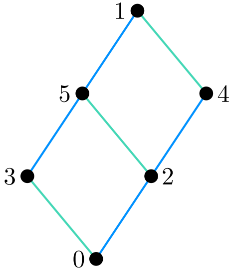
---

$$S = \langle 6, 9, 20 \rangle$$

$$I_S = \langle x^3 - y^2, x^4y^4 - z^3 \rangle \subseteq \Bbbk[x, y, z]$$



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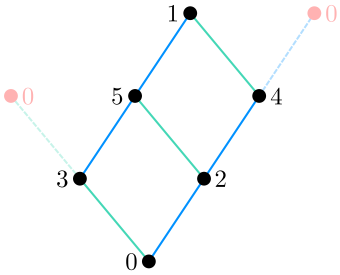

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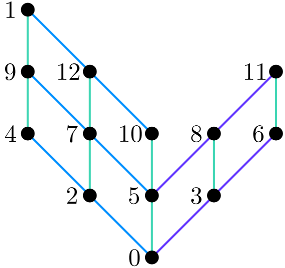


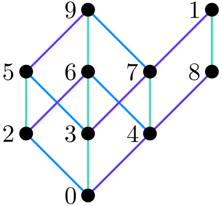

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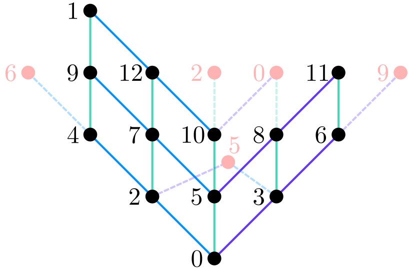

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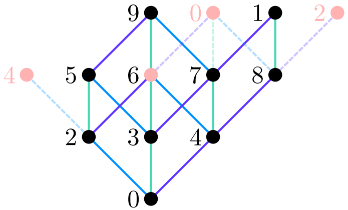
---

$S = \langle 10, a_2, a_3, a_4 \rangle$

$$I_S = \langle\, x_2^2 - y^* x_4,\ x_2x_4 - x_3^2,\ x_3^2 x_4 - y^*,\ x_4^3 - y^* x_2 \,\rangle \subseteq \Bbbk[y, x_2, x_3, x_4]$$



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---

$$0 \leftarrow R \leftarrow R^6 \xleftarrow{\begin{array}{c|cccccccc} & 1,12 & 1,13 & 2,12 & 2,23 & 3,13 & 3,23 & 2,13 & 3,12 \\ \hline 1,1 & -x_2 & -x_3 & & & & & y^* & y^* \\ 2,2 & -y^* & & x_1 & -x_3 & & y^* & & \\ 3,3 & & & & & x_1 & x_2 & -y^* & \\ 2,1 & x_1 & & -x_2 & y^* & y^* & & -x_3 & \\ 3,1 & y^* & x_1 & & & -x_3 & -y^* & & -x_2 \\ 3,2 & & -y^* & -y^* & x_2 & & -x_3 & x_1 & x_1 \end{array}} R^8 \xleftarrow{\begin{array}{c|ccc} & 1,[3] & 2,[3] & 3,[3] \\ \hline 1,12 & x_3 & -y^* & \\ 1,13 & -x_2 & & y^* \\ 2,12 & & x_3 & -y^* \\ 2,23 & -y^* & x_1 & \\ 3,13 & y^* & & -x_2 \\ 3,23 & & -y^* & x_1 \\ 2,13 & x_1 & -x_2 & \\ 3,12 & -x_1 & & x_3 \end{array}} R^3 \leftarrow 0$$

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- Betti numbers of $\Bbbk$ over $\Bbbk[\overline{x}]/I_S$

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$$0 \leftarrow R \xleftarrow{\begin{array}{c} \begin{array}{cccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \emptyset \begin{bmatrix} y & x_1 & x_2 & x_3 \end{bmatrix} \end{array}} R^4 \xleftarrow{\begin{array}{c|cccccccccccc} & \mathbf{01} & \mathbf{02} & \mathbf{03} & \mathbf{11} & \mathbf{12} & \mathbf{13} & \mathbf{21} & \mathbf{22} & \mathbf{23} & \mathbf{31} & \mathbf{32} & \mathbf{33} \\ \hline \mathbf{0} & x_1 & x_2 & x_3 & & & -y^* & & -y^* & & -y^* & & \\ \mathbf{1} & -y & & & x_1 & x_2 & x_3 & & & -y^* & & -y^* & \\ \mathbf{2} & & -y & & -y^* & & & x_1 & x_2 & x_3 & & & -y^* \\ \mathbf{3} & & & -y & & -y^* & & -y^* & & & x_1 & x_2 & x_3 \end{array}} R^{12}$$

$$\longleftarrow R^{36} \longleftarrow R^{108} \longleftarrow R^{324} \longleftarrow R^{972} \longleftarrow R^{2916} \longleftarrow \cdots$$

# Application 1: classifying minimal trades

**Question**

Given the multiplicity $m = \mathrm{m}(S)$ and $k = \#$ minimal generators of $S$, what can $\beta_1(I_S) = \#$ minimal generators of $I_S$ be?

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| $m$ | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | | 3 | | | | | | | | | | | | | | | | | | | | | | |
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$k = 3$ $\quad$ $k = 4$ $\quad$ $k = 5$ $\quad$ $k = 6$ $\quad$ $k = 7$

| $m$ | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
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Well known: $\beta_1(S) \le \binom{m}{2}$, with equality if and only if $k = m$

if $k = 3$, then $\beta_1(S) = 2, 3$

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$k = 3$ $k = 4$ $k = 5$ $k = 6$ $k = 7$

| $m$ | | | | | | | | | | | | | | | | | | | | | | | | |
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Prior work: a family has $\beta_1(S) = \binom{k}{2}$ for $3 \le k \le m$ (Rosales)

if $r = m - k \le 2$, then $\beta_1(S) \in [\binom{k}{2} - r, \binom{k}{2}]$ (GS-R)

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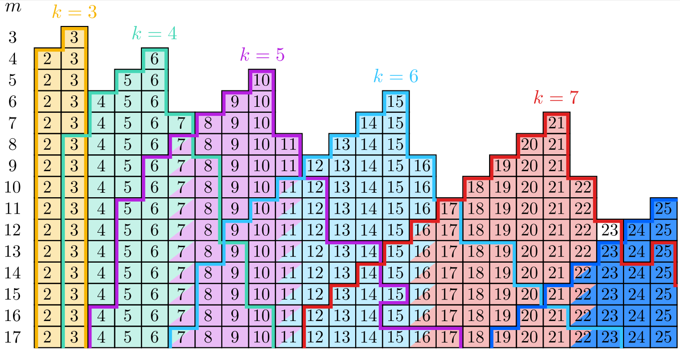
Given the multiplicity $m = \mathrm{m}(S)$ and $k = \#$ minimal generators of $S$, what can $\beta_1(I_S) = \#$ minimal generators of $I_S$ be?




Using Kunz posets: a family hits each $\beta_1(S) \in [\binom{k}{2} - r, \binom{k}{2}]$

for $r = m - k \leq k - 2$

a family hits $\beta_1(S) = \binom{k}{2} + 1$ for each $m \geq k + 3$

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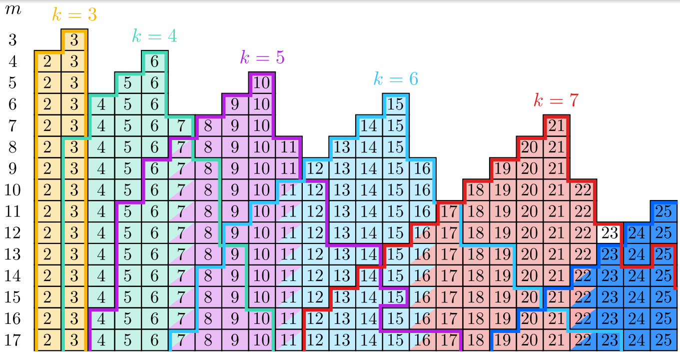


Bounds from Kunz posets: $\beta_1(S) \geq \binom{k}{2} - r$, where $r = m - k$

if $m - k = 3$, then $\beta_1(S) \in [\binom{k}{2} - 3, \binom{k}{2} + 1]$

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$k = 4$

| $m$ | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | | 3 | | | | | | | | | | | | | | | | | | | | | | |
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One more family: for $k = 4$, achieves each $\beta_1(S)$ with $(\beta_1(S) - 2)^2 \leq 4m$

conjectured to achieve every possible $\beta_1(S)$ for $k = 4$

# Application 2: a long-standing (**hard**) conjecture

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For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \leq k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

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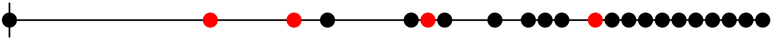

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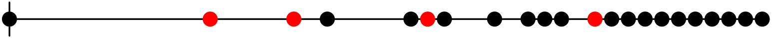
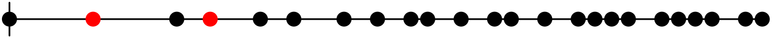


Equality holds when:

- $S = \langle a, b \rangle$



- $S = \langle m, m+1, \ldots, 2m-1 \rangle$

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If $S$ corresponds to $x = (a_1, \ldots, a_{m-1}) \in C_m$,

$$\mathrm{g}(S) = \|x\|_1 - \tfrac{1}{2}m(m-1), \qquad \mathrm{F}(S) = \|x\|_\infty - m,$$

and $\#$ generators $k$ is determined by the face $F \subseteq C_m$ containing $x$.

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