Classifying numerical semigroups using polyhedral geometry

Christopher O'Neill

San Diego State University

cdoneill@sdsu.edu

Joint with (i) Winfred Bruns, Pedro García-Sánchez, Dane Wilbourne; (ii) Nathan Kaplan;
(iii) J. Autry, *A. Ezell, *T. Gomes, *C. Preuss, *T. Saluja, *E. Torres Dávila
(iv) B. Braun, T. Gomes, E. Miller, C. O'Neill, and A. Sobieska

* = undergraduate student

Slides available: `https://cdoneill.sdsu.edu/`

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Multiplicity: $\mathrm{m}(S) =$ smallest nonzero element

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Fix a numerical semigroup S with $\mathrm{m}(S) = m$.

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For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \ldots\} \cap S = \{20, 26, 32, \ldots\}$
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Observations:

- The elements of $\mathrm{Ap}(S)$ are distinct modulo m
- $|\mathrm{Ap}(S)| = m$

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The Apéry set is a "one stop shop" for computation.

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Theorem

If $A = \{0, a_1, \ldots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \bmod m$, then there exists a numerical semigroup S with $\mathrm{Ap}(S) = A$ if and only if

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Big idea: the inequalities "$a_i + a_j \geq a_{i+j}$" to define a **cone** C_m.

Kunz cone

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The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

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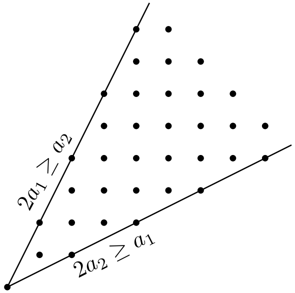

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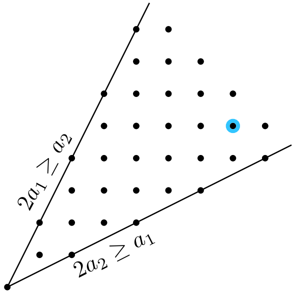

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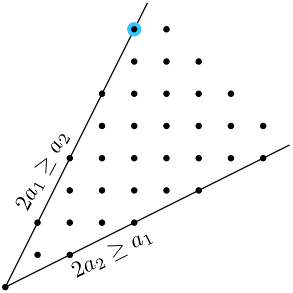

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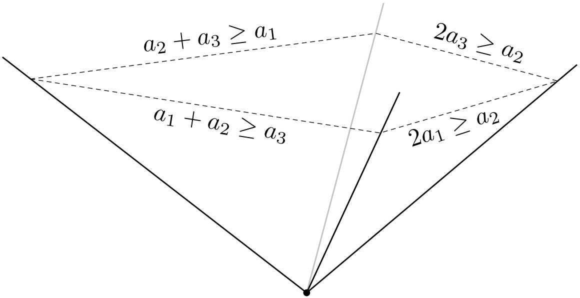

Faces of the Kunz cone

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When are numerical semigroups in (the relative interior of) the same face?

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Big picture: "parameter space" approach for studying XYZ's

- Define a space with XYZ's as points
  Small changes to an XYZ $\rightsquigarrow$ small movements in space
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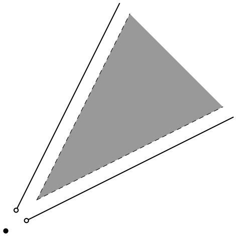
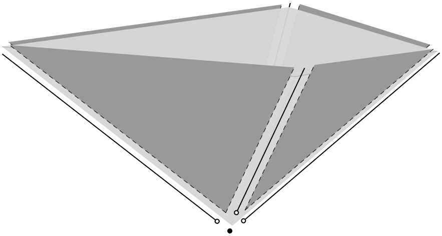
More interesting example: C_m





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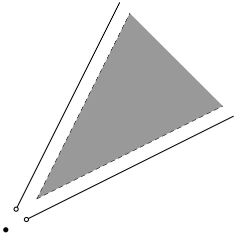

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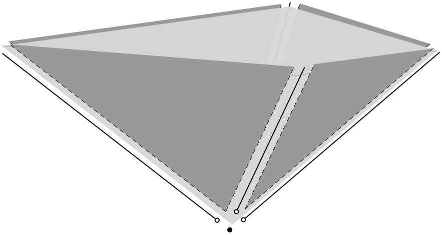

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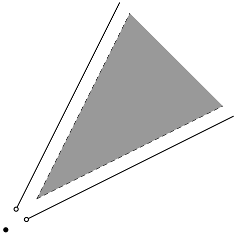

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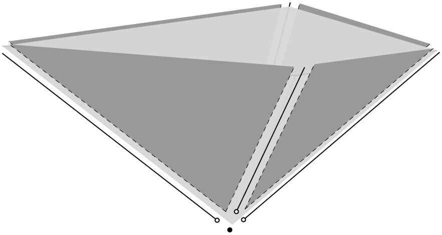

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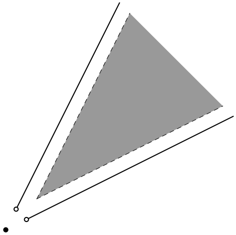

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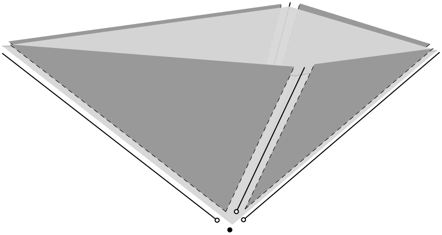

$C_5 \subseteq \mathbb{R}^4$? Cross section:

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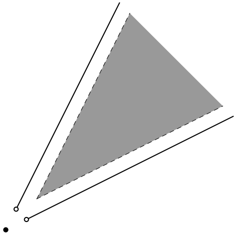

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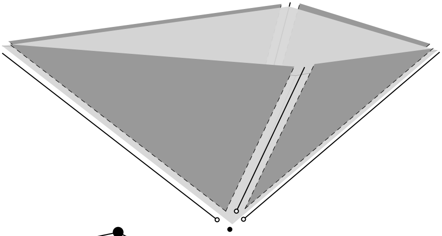

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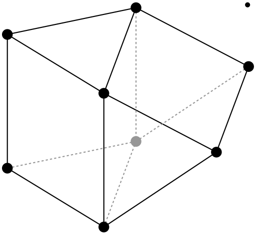

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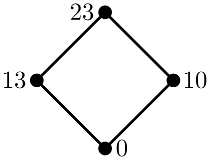
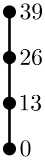
$\mathrm{Ap}(S) = \{0, 13, 10, 23\}$



$\mathrm{Ap}(S) = \{0, 13, 26, 39\}$



Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

Faces of the Kunz cone

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$$S = \langle 6, 9, 20 \rangle \qquad S' = \langle 6, 26, 27 \rangle$$

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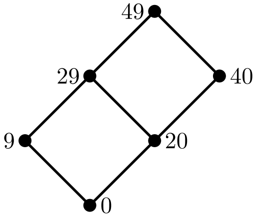

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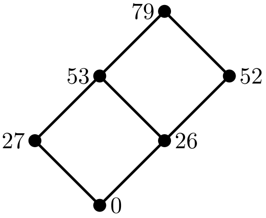

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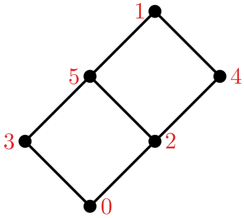

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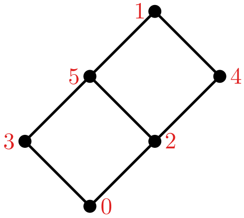

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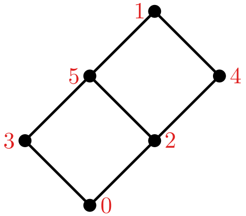
1 2 3 4 5



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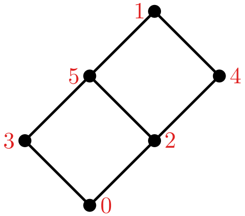

The *Kunz poset* of S: use ground set $\mathbb{Z}_m$ instead of $\mathrm{Ap}(S)$.

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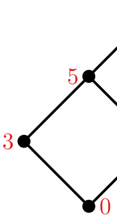

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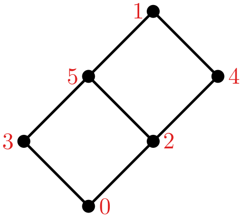

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Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

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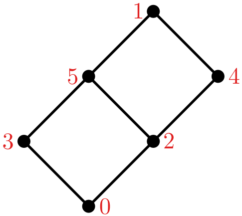

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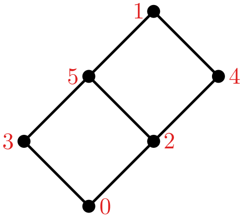

Defining facet equations:

$$2a_2 = a_4$$

$$a_2 + a_3 = a_5$$

$$a_2 + a_5 = a_1$$

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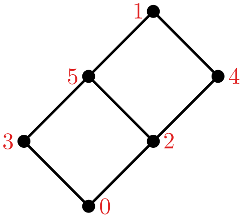

Defining facet equations:

$2a_2 = a_4$	$2 \preceq 4$
$a_2 + a_3 = a_5$	$2 \preceq 5$
	$3 \preceq 5$
$a_2 + a_5 = a_1$	$2 \preceq 1$
	$5 \preceq 1$
$a_3 + a_4 = a_1$	$3 \preceq 1$
	$4 \preceq 1$

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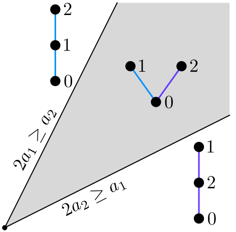

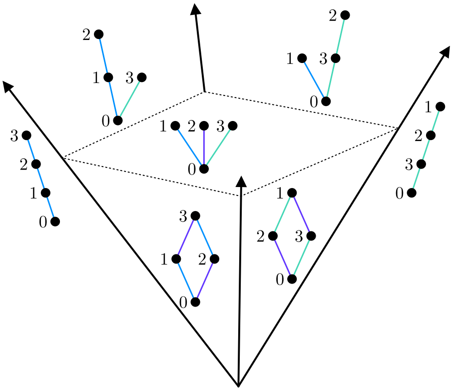

Connecting the algebra and the geometry

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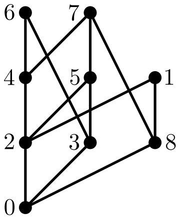

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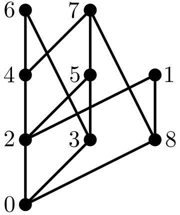
$$\mathrm{Ap}(S) = \{0, a_1, a_2, \ldots, a_8\}$$



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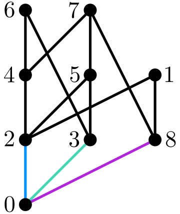

$$\mathrm{Ap}(S) = \{0, a_1, a_2, \ldots, a_8\}$$

$$S = \langle 9, a_2, a_3, a_8 \rangle$$

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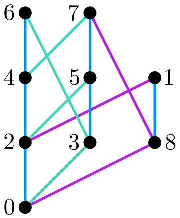

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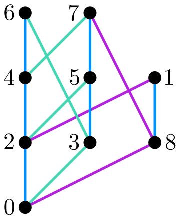

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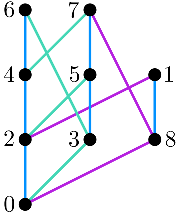

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$$a_7 = 2a_2 + a_3$$
$$= 2a_8$$

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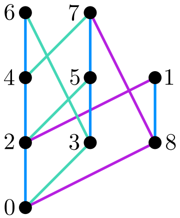

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Cover relations: add a generator

$a_5 = a_2 + a_3$

$$\begin{aligned} a_7 &= 2a_2 + a_3 \\ &= 2a_8 \end{aligned}$$

Moral: can recover expressions for $a \in \mathrm{Ap}(S)$ in terms of generators

Connecting the algebra and the geometry

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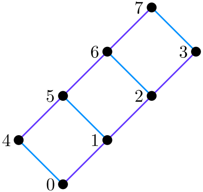
$S = \langle 8, a_1, a_4 \rangle$



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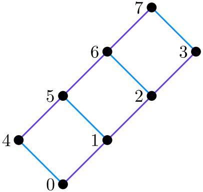

$S \rightsquigarrow (a_1, a_2, \ldots, a_7) \in F \subseteq C_8$

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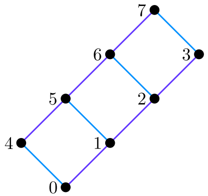

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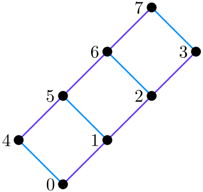

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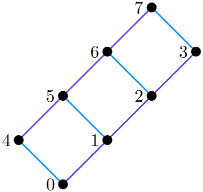

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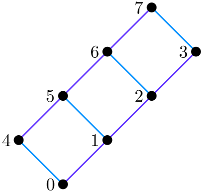

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$S = \langle 8, 9, 12 \rangle$? $\quad S = \langle 8, 9, 44 \rangle$?

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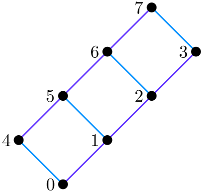

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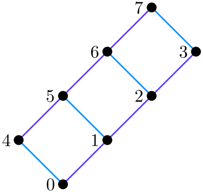

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$S = \langle 8, 9, 12 \rangle$? ~~$S = \langle 8, 9, 44 \rangle$?~~

Also need $4a_1 \geq a_4$, $a_4 \geq 0$

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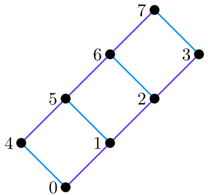
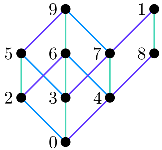


$S = \langle 10, a_2, a_3, a_4 \rangle$



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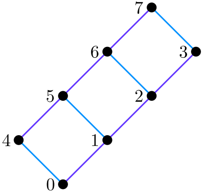

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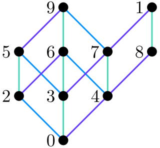

a_2, a_3, and a_4 “span” F

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Connecting the algebra and the geometry

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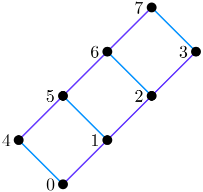

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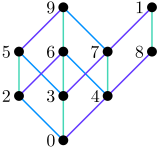

a_2, a_3, and a_4 "span" F
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$a_6 = 2a_3$
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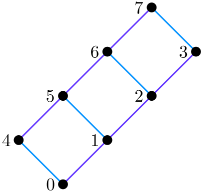

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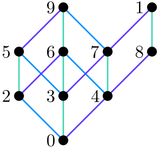


a_2, a_3, and a_4 "span" F

$\Rightarrow \dim F \leq 3$

$a_6 = 2a_3$

$= a_2 + a_4$

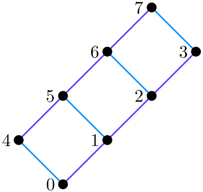
"trades" $\rightsquigarrow$ linear dependencies

Connecting the algebra and the geometry

Question

How is "face dimension" encoded in the Kunz poset?

$S = \langle 8, a_1, a_4 \rangle$



$S \rightsquigarrow (a_1, a_2, \ldots, a_7) \in F \subseteq C_8$

$a_2 = 2a_1, \quad a_6 = 2a_1 + a_4, \quad \ldots$

Idea: a_1 and a_4 "span" F
$\Rightarrow \dim F \le 2$

$S = \langle 8, 9, 12 \rangle$? ~~$S = \langle 8, 9, 44 \rangle$?~~

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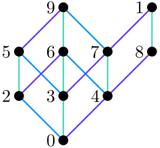

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Connecting the algebra and the geometry

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Arithmetical numerical semigroups:

$S = \langle m, m + d, m + 2d, \ldots, m + kd \rangle$

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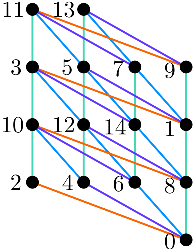

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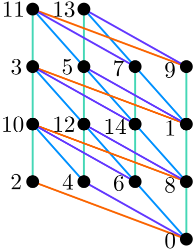

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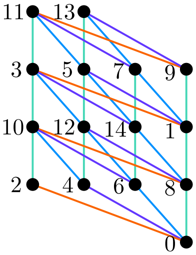

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Other families with descriptions: symmetric, complete intersection, ...

Connecting the algebra and the geometry

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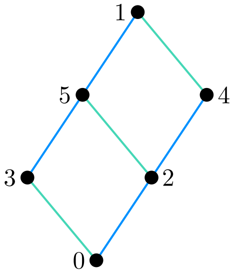

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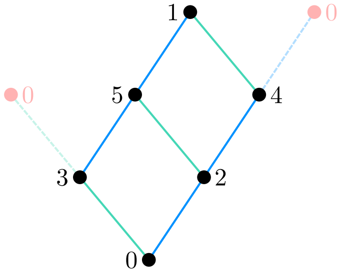

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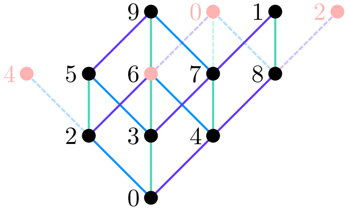
$$I_S = \langle x_2^2 - y^* x_4,\ x_2 x_4 - x_3^2,\ x_3^2 x_4 - y^*,\ x_4^3 - y^* x_2 \rangle \subseteq \Bbbk[y, x_2, x_3, x_4]$$



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For any $S = \langle n_1, \ldots, n_k \rangle$, we have $\mathrm{F}(S) + 1 \leq k(\mathrm{F}(S) + 1 - \mathrm{g}(S))$.

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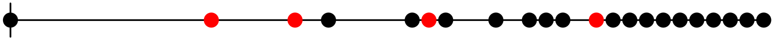

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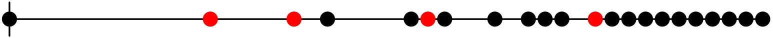
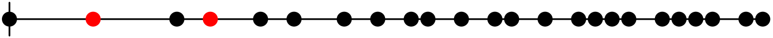


Equality holds when:

- $S = \langle a, b \rangle$



- $S = \langle m, m+1, \ldots, 2m-1 \rangle$

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Wilf's conjecture holds for all numerical semigroups S with $m \le 18$.

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If S corresponds to $x = (a_1, \ldots, a_{m-1}) \in C_m$,

$$\mathrm{g}(S) = \|x\|_1 - \tfrac{1}{2}m(m-1), \qquad \mathrm{F}(S) = \|x\|_\infty - m,$$

and # generators k is determined by the face $F \subseteq C_m$ containing x.

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Example: $n_3 = 4$

$$\begin{aligned}
\langle 2, 7 \rangle &= \{0, \quad 2, \quad 4, \quad 6, 7, 8, \ldots\} \\
\langle 3, 4 \rangle &= \{0, \qquad 3, 4, \quad 6, 7, 8, \ldots\} \\
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Not true for $n'_f = \#$ of numerical semigroups with Frobenius number f

$$n'_{11} = 51 \qquad n'_{12} = 40 \qquad n'_{13} = 106$$

References

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