

Classifying numerical semigroups using polyhedral geometry

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Example:

$$McN = \langle 6, 9, 20 \rangle = \left\{ \begin{array}{l} 0, 6, 9, 12, 15, 18, 20, 21, 24, \dots \\ \dots, 36, 38, 39, 40, 41, 42, 44 \rightarrow \end{array} \right\}$$

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Fact

Every numerical semigroup has a unique minimal generating set.

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Every numerical semigroup has a unique minimal generating set.

Multiplicity: $m(S)$ = smallest nonzero element

Apéry sets

Fix a numerical semigroup S with $m(S) = m$.

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For 2 mod 6: $\{2, 8, 14, 20, 26, 32, \dots\} \cap S = \{20, 26, 32, \dots\}$

For 3 mod 6: $\{3, 9, 15, 21, \dots\} \cap S = \{9, 15, 21, \dots\}$

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- The elements of $\text{Ap}(S)$ are distinct modulo m
- $|\text{Ap}(S)| = m$

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$$n \in S \text{ if } n \geq a \text{ for } a \in \text{Ap}(S) \text{ with } a \equiv n \pmod{m}$$

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The Apéry set is a “one stop shop” for computation.

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Theorem

If $A = \{0, a_1, \dots, a_{m-1}\}$ with each $a_i > m$ and $a_i \equiv i \pmod{m}$, then there exists a numerical semigroup S with $\text{Ap}(S) = A$ if and only if

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Big idea: the inequalities “ $a_i + a_j \geq a_{i+j}$ ” to define a **cone** C_m .

Definition

The *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}$ is a pointed cone with defining inequalities

$$a_i + a_j \geq a_{i+j} \quad \text{whenever} \quad i + j \neq 0.$$

$$\begin{aligned} \{S \subseteq \mathbb{Z}_{\geq 0} : m(S) = m\} &\longrightarrow C_m \\ \text{Ap}(S) = \{0, a_1, \dots, a_{m-1}\} &\longmapsto (a_1, \dots, a_{m-1}) \end{aligned}$$

Kunz cone

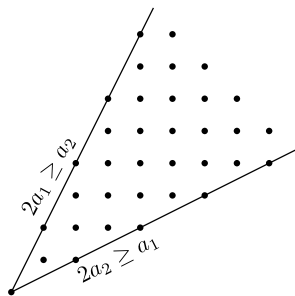
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Example: C_3



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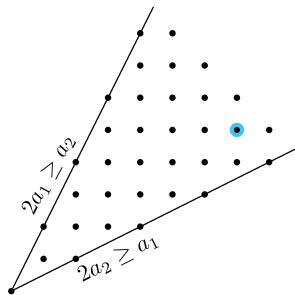
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$$\begin{aligned} S &= \langle 3, 5, 7 \rangle \\ \text{Ap}(S) &= \{0, 7, 5\} \end{aligned}$$

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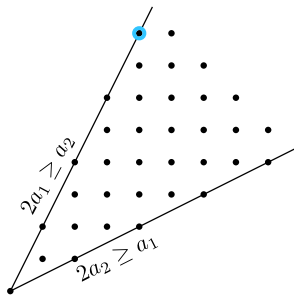
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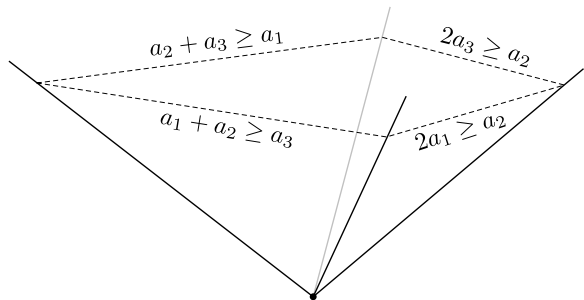
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Example: C_4



Faces of the Kunz cone

Question

When are numerical semigroups in (the relative interior of) the same face?

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Big picture: “parameter space” approach for studying XYZ 's

- Define a space with XYZ 's as points
Small changes to an $XYZ \rightsquigarrow$ small movements in space
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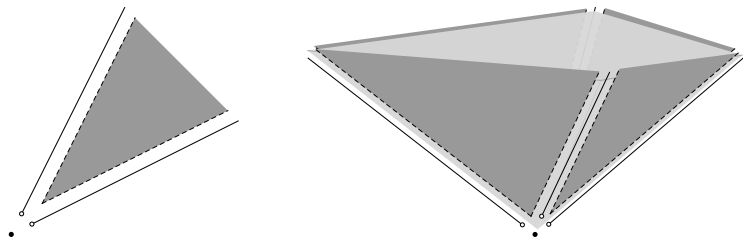
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More interesting example: C_m



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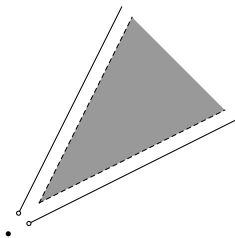
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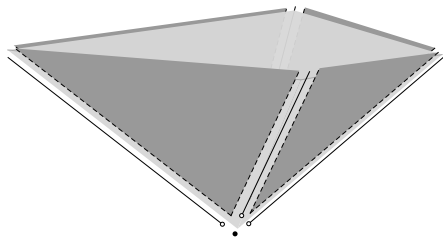
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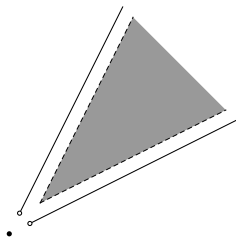


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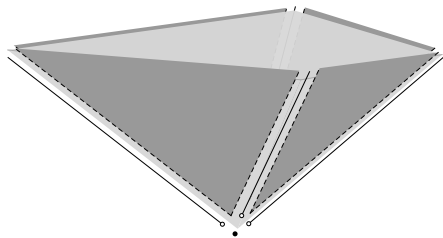
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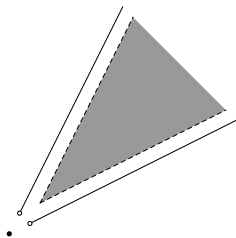
$$C_5 \subseteq \mathbb{R}^4?$$

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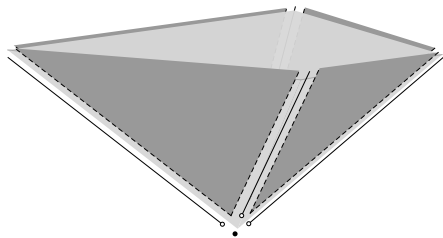
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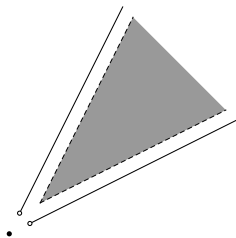
$$C_5 \subseteq \mathbb{R}^4? \quad \text{Cross section:}$$

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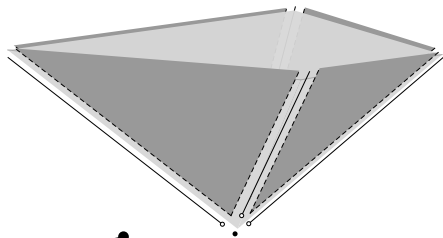
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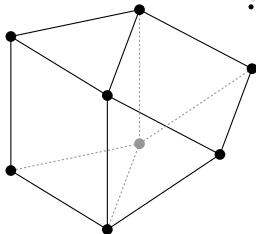
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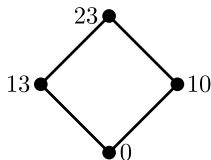
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The *Apéry poset* of S : define $a \preceq a'$ whenever $a' - a \in S$.

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$$\text{Ap}(S) = \{0, 13, 10, 23\}$$



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When are numerical semigroups in (the relative interior of) the same face?

$$S = \langle 6, 9, 20 \rangle$$

$$\text{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$$

$$S' = \langle 6, 26, 27 \rangle$$

$$\text{Ap}(S') = \{0, 79, 26, 27, 52, 53\}$$

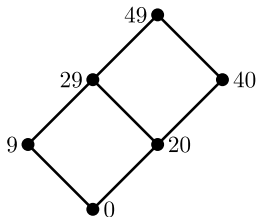
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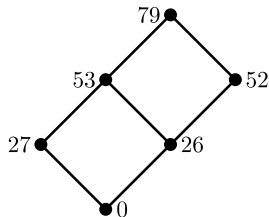
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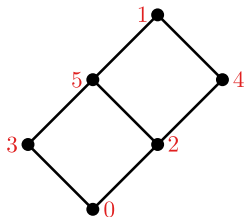
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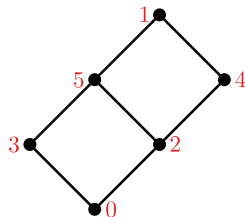
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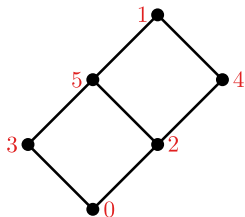
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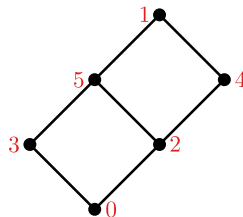
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The *Kunz poset* of S : use ground set $\mathbb{Z}_m$ instead of $\text{Ap}(S)$.

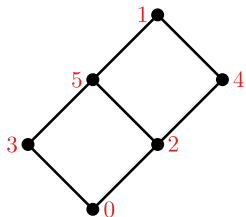
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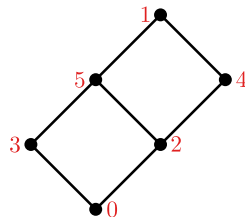
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Numerical semigroups lie in the relative interior of the same face of C_m if and only if their Kunz posets are identical.

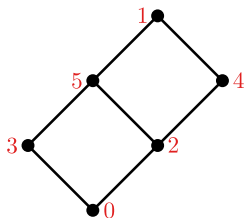
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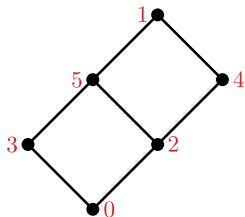
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Defining facet equations:

$$2a_2 = a_4$$

$$a_2 + a_3 = a_5$$

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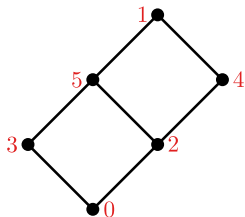
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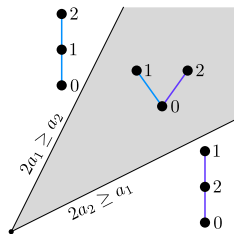
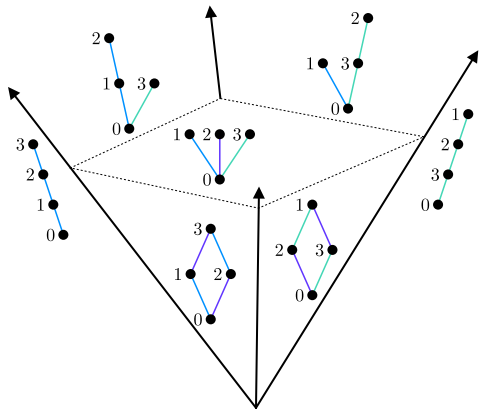
$2a_2 = a_4$	$2 \preceq 4$
$a_2 + a_3 = a_5$	$2 \preceq 5$
	$3 \preceq 5$
$a_2 + a_5 = a_1$	$2 \preceq 1$
	$5 \preceq 1$
$a_3 + a_4 = a_1$	$3 \preceq 1$
	$4 \preceq 1$

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C_3 and C_4



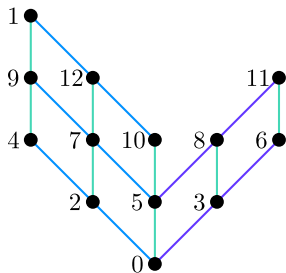
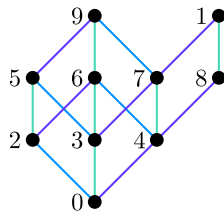
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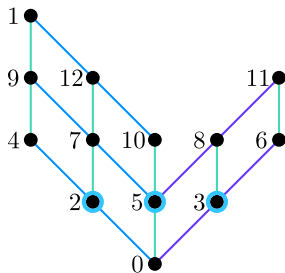
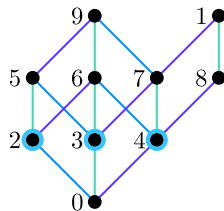
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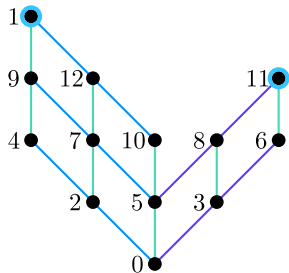
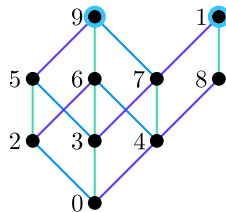
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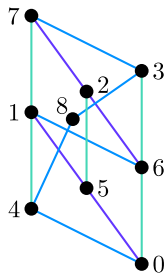
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$$S = \langle 4, 7 \rangle$$



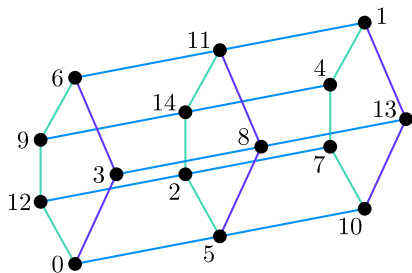
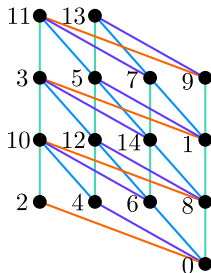
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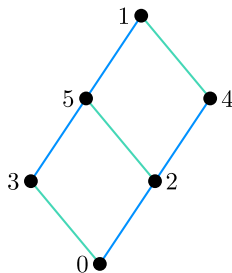
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$$I_S = \langle x^3 - y^2, x^4 y^4 - z^3 \rangle \\ \subseteq \mathbb{k}[x, y, z]$$



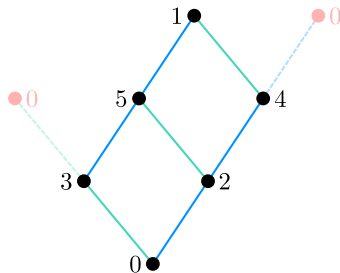
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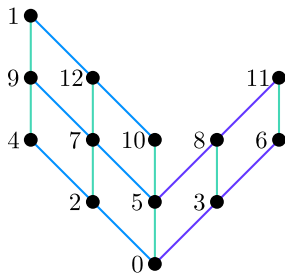
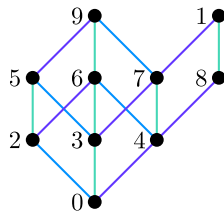
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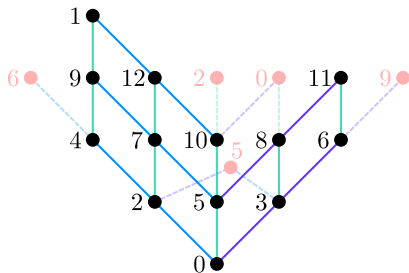
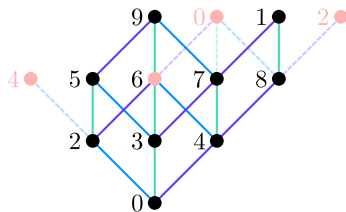
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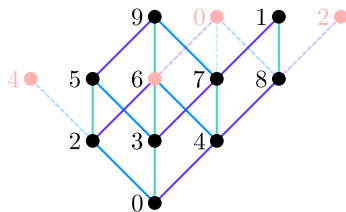
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$$S = \langle 10, a_2, a_3, a_4 \rangle$$

$$I_S = \langle x_2^2 - y^* x_4, x_2 x_4 - x_3^2, x_3^2 x_4 - y^*, x_4^3 - y^* x_2 \rangle$$

$$\subseteq \mathbb{k}[y, x_2, x_3, x_4]$$



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$$\begin{array}{cccccccc}
 & & & & & & & & 1, [3] & 2, [3] & 3, [3] \\
 & & & & & & & & 1, 12 & 1, 13 & 2, 12 & 2, 23 & 3, 13 & 3, 23 & 2, 13 & 3, 12 \\
 \begin{array}{l} 1, 1 \\ 2, 2 \\ 3, 3 \\ 2, 1 \\ 3, 1 \\ 3, 2 \end{array} & \left[\begin{array}{cccccccc}
 -x_2 & -x_3 & & & & & & \\
 -y^* & & x_1 & -x_3 & & & y^* & y^* \\
 & & & & x_1 & x_2 & -y^* & \\
 x_1 & & -x_2 & y^* & y^* & & -x_3 & \\
 y^* & x_1 & & & -x_3 & -y^* & & -x_2 \\
 & -y^* & -y^* & x_2 & & -x_3 & x_1 & x_1
 \end{array} \right] & \left[\begin{array}{ccc}
 x_3 & -y^* & \\
 -x_2 & & y^* \\
 & x_3 & -y^* \\
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$0 \leftarrow R \leftarrow R^6 \leftarrow \xrightarrow{\hspace{10em}} R^8 \leftarrow \xrightarrow{\hspace{10em}} R^3 \leftarrow 0$

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$$\begin{array}{c}
 \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \emptyset & [y & x_1 & x_2 & x_3] \end{array} \\
 0 \leftarrow R \leftarrow \leftarrow \leftarrow R^4 \leftarrow \leftarrow \leftarrow \leftarrow R^{12}
 \end{array}
 \begin{array}{cccccccccccc}
 & 01 & 02 & 03 & 11 & 12 & 13 & 21 & 22 & 23 & 31 & 32 & 33 \\
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 x_1 & x_2 & x_3 & & & -y^* & & -y^* & & -y^* & & \\
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 & & -y & & -y^* & & -y^* & & & & x_1 & x_2 & x_3
 \end{array} \right]
 \end{array}$$

$$\leftarrow \leftarrow R^{36} \leftarrow \leftarrow R^{108} \leftarrow \leftarrow R^{324} \leftarrow \leftarrow R^{972} \leftarrow \leftarrow R^{2916} \leftarrow \leftarrow \dots$$

Application 1: classifying minimal trades

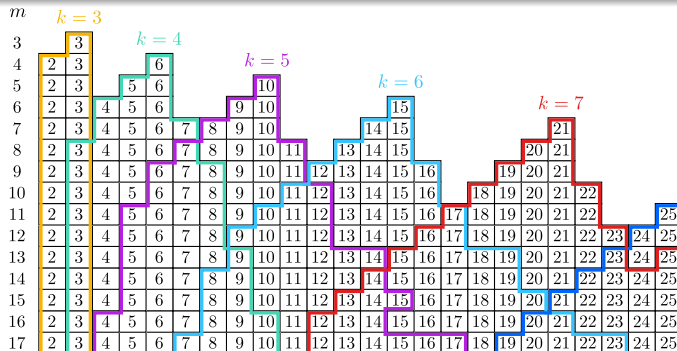
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Given the multiplicity $m = m(S)$ and $k = \#$ minimal generators of S , what can $\beta_1(I_S) = \#$ minimal generators of I_S be?

Application 1: classifying minimal trades

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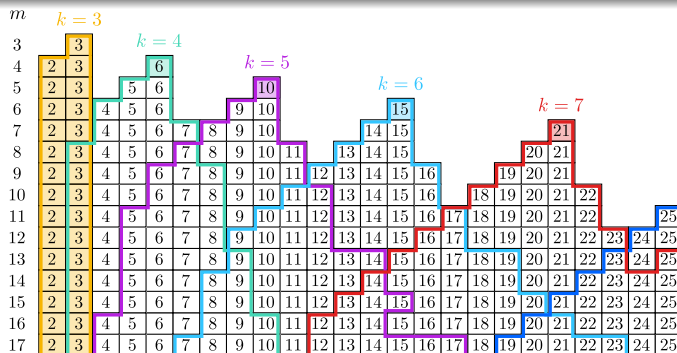
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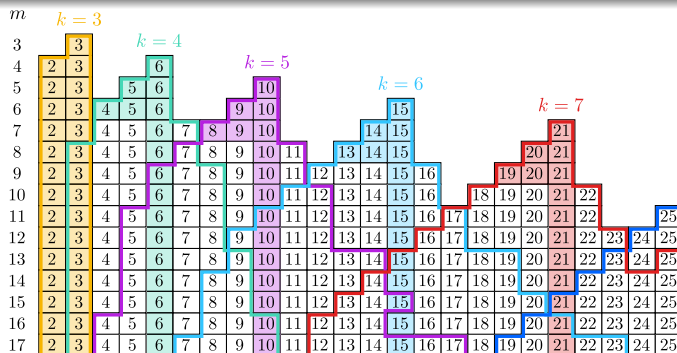


Well known: $\beta_1(S) \leq \binom{m}{2}$, with equality if and only if $k = m$
 if $k = 3$, then $\beta_1(S) = 2, 3$

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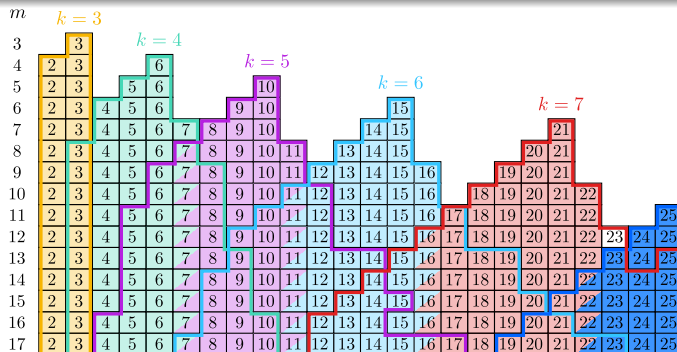


Prior work: a family has $\beta_1(S) = \binom{k}{2}$ for $3 \leq k \leq m$ (Rosales)
 if $r = m - k \leq 2$, then $\beta_1(S) \in [\binom{k}{2} - r, \binom{k}{2}]$ (GS-R)

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Using Kunz posets: a family hits each $\beta_1(S) \in [(\binom{k}{2} - r, \binom{k}{2})]$

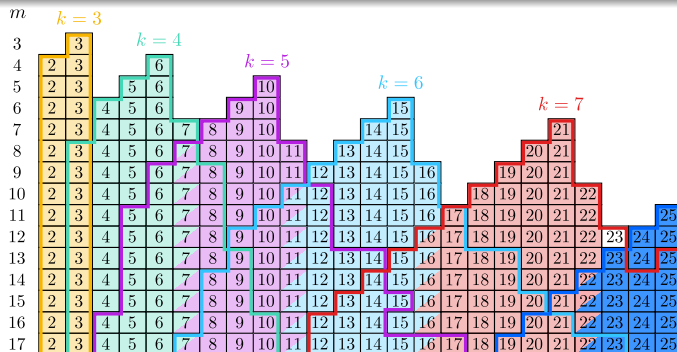
for $r = m - k \leq k - 2$

a family hits $\beta_1(S) = \binom{k}{2} + 1$ for each $m \geq k + 3$

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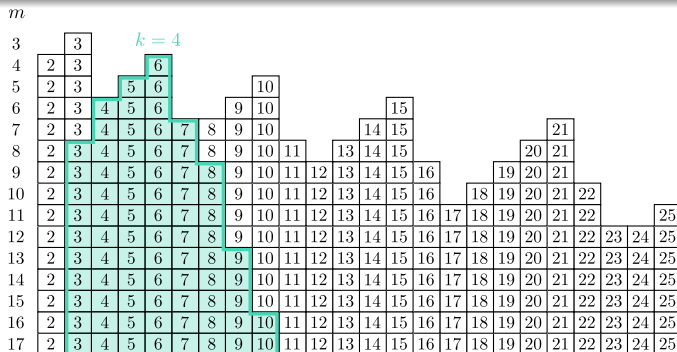


Bounds from Kunz posets: $\beta_1(S) \geq \binom{k}{2} - r$, where $r = m - k$
 if $m - k = 3$, then $\beta_1(S) \in [\binom{k}{2} - 3, \binom{k}{2} + 1]$

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One more family: for $k = 4$, achieves each $\beta_1(S)$ with $(\beta_1(S) - 2)^2 \leq 4m$ conjectured to achieve every possible $\beta_1(S)$ for $k = 4$

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Wilf's Conjecture

For any $S = \langle n_1, \dots, n_k \rangle$, we have $F(S) + 1 \leq k(F(S) + 1 - g(S))$.

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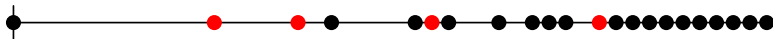
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Equivalently,

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Application 2: a long-standing (**hard**) conjecture

$$F(S) = \max(\mathbb{Z}_{\geq 0} \setminus S)$$

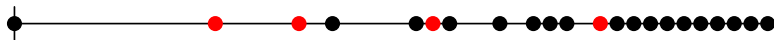
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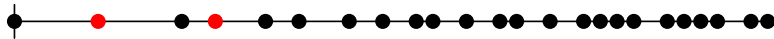
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Equality holds when:

- $S = \langle a, b \rangle$



- $S = \langle m, m + 1, \dots, 2m - 1 \rangle$



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If S corresponds to $x = (a_1, \dots, a_{m-1}) \in C_m$,

$$g(S) = \|x\|_1 - \frac{1}{2}m(m-1), \quad F(S) = \|x\|_\infty - m,$$

and # generators k is determined by the face $F \subseteq C_m$ containing x .

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Thanks!