Fall 2015, Math 431: Week 1 Problem Set
Due: Tuesday, September 8th, 2015
Induction and the Pigeon-hole Principle

Discussion problems. The problems below should be completed in class.

(D1) *Strong induction.* Fix a statement $P(n)$ dependent on n, and suppose that:

- $P(1)$ holds (the *base case*); and
- if $P(k)$ holds for all $1 \le k \le n$, then $P(n+1)$ holds (the *inductive step*).

We can conclude that $P(n)$ holds for all n. This technique is called *strong induction* on n.

Prove the following results using induction. For each, indicate whether your proof uses strong induction or weak induction.

(a) Define $a_0 = 0$ and
$$a_n = a_0 + \cdots + a_{n-1} + n$$
for all $n \ge 1$. Prove that $a_n = 2^n - 1$ for $n \ge 0$.

(b) Prove that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all $n \ge 1$.

(c) Suppose $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(m+n) = f(m) + f(n)$ for all $m, n \ge 0$. Prove that there exists a constant c such that $f(n) = cn$ for all n. (For those who have taken abstract algebra, f is an *additive homomorphism*.)

(D2) *The generalized pigeon-hole principle.* Consider the generalization of the pigeon-hole principle given below.

Theorem. *Fix positive integers $n, m, r > 0$, and suppose $n > rm$. If n pigeons are placed into m boxes, then some box contains at least $r + 1$ pigeons.*

(a) Find a formula for the minimum value of r in terms of n and m. You may find the floor and ceiling functions helpful. You do *not* need to prove your formula.

(b) Prove the generalized pigeon-hole principle by induction on r. You may use the standard pigeon-hole principle in both the base case and inductive step.

(c) Does your argument for part (b) use strong induction?

(D3) *The pigeon-hole principle in geometry.* Solve each of the following problems using the pigeon-hole principle. Be sure to specify the version of the pigeon-hole principle used.

(a) If 10 points are chosen inside of a unit square, then there are two points with a distance at most 0.5 apart.

(b) If 10 points are chosen inside of a unit square, then three points can be covered by a disk of radius 0.5.

Required problems. As the name suggests, you must submit *all* required problem with this homework set in order to receive full credit.

(R1) Define the sequence f_n as follows: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$. Use induction to prove that
$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$
holds for all $n \ge 0$. Does your proof use strong induction?

(R2) Let $p : \mathbb{N} \to \mathbb{R}$ be a polynomial of degree d, and suppose $q(n) = \sum_{i=1}^{n} p(i)$.

(a) Prove that $q(n)$ is a polynomial of degree $d + 1$.

(b) Prove that $q(0) = 0$. (Note that both p and q are only defined on *positive* values.)

(R3) Suppose 200 balls are distributed into 100 boxes in such a way that no box contains more than 100 balls, and each box contains at least one ball. Then there are some boxes which together contain exactly 100 balls.

(R4) Prove that among 502 positive integers, there are always two integers whose sum or difference is a multiple of 1000.

Selection problems. You are required to submit all parts of *one* selection problem with this problem set. You may submit additional selection problems if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional selection problems.

(S1) Suppose every point in $\mathbb{N}^2$ is colored using one of six colors.

(a) Prove that there exists a rectangle whose vertices are monochromatic.

(b) Suppose $\mathbb{N}^2$ is colored using one of r colors, where $r > 0$. For which values of r does part (a) still hold?

(S2) For each of the following statements, find a sufficent condition on n and r that makes the statement true, and prove that it is sufficient. Is your condition "tight"?

(a) Given n consecutive integers, at least one is divisible by r.

(b) Fix a set M of n positive integers, none of which has a prime divisor larger than the r-th prime. Then there are two elements of M whose product is a perfect square.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Suppose every point in $\mathbb{N}^2$ is colored using one of three colors. Can we always find a square whose vertices are monochromatic?