Fall 2015, Math 431: Week 3 Problem Set
Due: Thursday, September 24th, 2015
Binomial Theorem and Inclusion-Exclusion

Discussion problems. The problems below should be completed in class.

(D1) *Binomial theorem.* Recall the binomial theorem from Tuesday:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

(a) Give a combinatorial proof of the following identity, valid whenever $k + m \leq n$:

$$\binom{n}{m}\binom{n-m}{k} = \binom{n}{k}\binom{n-k}{m}.$$

(b) Consider the following identity, valid whenever $n \geq 1$:

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

(i) Give a combinatorial proof of this identity.
(ii) Give an algebraic proof of this identity. Hint: apply $\frac{d}{dx}$ to the binomial theorem.

(D2) *Applications of Inclusion-Exclusion.* Recall the Sieve formula from Tuesday:

$$|A_1 \cup \cdots \cup A_n| = \textstyle\sum_{S \subset [n]} (-1)^{|S|} \left|\bigcap_{j \in S} A_j\right|.$$

(a) A *derangement* is a permutation that has no fixed points.

(i) Find the number D_n of derangements $[n] \to [n]$ using the Sieve formula.
(ii) Give a combinatorial proof that the sequence D_n satisfies

$$D_n = n! - \sum_{i=k}^{n} \binom{n}{k} D_{n-k}.$$

(iii) Give a combinatorial proof that the sequence D_n satisfies

$$D_{n+1} = n(D_n + D_{n-1}).$$

(b) For $n, m \geq 1$, let $O_{n,m}$ denote the total number of surjective functions $[n] \to [m]$.

(i) For $n \geq m$, use the Sieve formula to show that

$$O_{n,m} = m^n - \sum_{k=0}^{m-1} (-1)^{m-k} \binom{m}{k} k^n.$$

(ii) Show that $O_{n,m}$ satisfies the relation

$$O_{n,m} = m^n - \sum_{k=0}^{m-1} \binom{m}{k} O_{n,k}.$$

Hint: use the fact that any function is surjective onto its image.

Required problems. As the name suggests, you must submit *all* required problem with this homework set in order to receive full credit.

(R1) Find the coefficient of $x^{11}y^7$ in the expansion of $(x+y)^{18} + x^3(x-y)^{15}$.

(R2) How many ways can we arrange the characters $1, 1, 2, 2, 3, 4, 5$ so that no two consecutive digits are identical?

(R3) Use induction on n to prove that for all $n \geq 1$,

$$\sum_{i=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Hint: use the identity $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$ in your inductive step.

(R4) Find $\phi(210)$, where $\phi(n)$ denote the number of elements of $[n]$ relatively prime to n.

Selection problems. You are required to submit all parts of *one* selection problem with this problem set to be eligible for full credit. However, *on this assignment only*, you may also submit additional selection problems, and your scores *will* count towards your overall homework score.

(S1) Find 4 infinite subsets of $\mathbb{N}$ such that the intersection of any 3 of them is infinite, but the intersection of all 4 is empty.

(S2) Let $\phi(n)$ denote the number of elements of $[n]$ relatively prime to n. Find $\phi(p_1^{a_1} \cdots p_k^{a_k})$ for $a_1, \ldots, a_k \geq 1$ and distinct primes $p_1, \ldots, p_k$. Hint: first prove this for the case $k = 1$.

(S3) Prove that for all $n \geq 1$,

$$\sum_{a_1+a_2+a_3=n} \binom{n}{a_1, a_2, a_3} = 3^n.$$

(S4) Use the binomial theorem to prove that for all $n \geq 1$,

$$\sum_{k=0}^{n} \frac{(-1)^{k+1}}{k+1} \binom{n}{k} = 0.$$

Why might it be hard to find a combinatorial proof of this identity?

(S5) Consider the following identity, valid for all $n \geq 1$:

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

(a) Give an algebraic proof of this identity.

(b) Give a combinatorial proof of this identity.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Give a combinatorial proof of the following identity:

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n}{k-1}\binom{n-1}{k}\binom{n+1}{k+1}.$$