Fall 2015, Math 431: Week 4 Problem Set
Due: Thursday, October 8th, 2015
Partitions

Discussion problems. The problems below should be completed in class.

(D1) This problem involves Ferres diagrams for integer partitions, seen at the end of class on Tuesday. Recall that the *conjugate* of a partition is obtained by reflecting the Ferres diagram about the diagonal, and a partition is *self-conjugate* if it equals its conjugate.

   (i) Prove that for $n \geq 2$, the number of partitions of n in which the two largest parts are equal is given by $p(n) - p(n-1)$.

   (ii) Prove that the number of partitions of n into at most k parts equals $p_k(n+k)$.

   (iii) Prove that the number partitions of n into distinct odd parts equals the number of self-conjugate partitions.

(D2) Recall that $S(n,k)$ denotes the number of partitions of the set $[n]$ into exactly k blocks. Define $B(n) = \sum_{k=1}^{n} S(n,k)$, the n-th *Bell number*.

   (i) Give a combinatorial proof that

$$B(n+1) = \sum_{k=1}^{n} \binom{n}{k} B(k).$$

   (ii) Use the identity $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$ from class to prove that

$$B(n+1) - B(n) = \sum_{k=1}^{n} kS(n,k).$$

   (iii) Give a combinatorial proof for part (ii).

   (iv) Give a combinatorial proof that

$$B(n+m) = \sum_{k=1}^{n} \sum_{j=1}^{m} S(m,j) \binom{n}{k} j^{n-k} B(k).$$

(D3) For $n, m \geq 1$, let $O_{n,m}$ denote the total number of surjective functions $[n] \to [m]$.

   (i) For $n \geq m$, use the Sieve formula to show that

$$O_{n,m} = m^n - \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} (m-k)^n.$$

   (ii) Prove that $S(n,m) = m! \cdot O_{n,m}$.

Required problems. As the name suggests, you must submit *all* required problem with this homework set in order to receive full credit.

(R1) Find the number of compositions of 10 into even parts. Find an expression for the number of compositions of n into even parts.

(R2) Find a closed formula for $S(n, n-2)$ for all $n \geq 2$.

(R3) In class, we proved that the number of partitions of n into at most k parts equals the number of partitions of n into parts of size at most k. Use this to find an expression for the number of partitions of n into precisely k parts.

(R4) Complete problem (D2)(ii) above.

Selection problems. You are required to submit *one* selection problem with this problem set. You may also submit additional selection problems, but the total number of points awarded (excluding challenge problems) won't exceed the total possible score on this problem set.

(S1) Let $F(n)$ denote the number of all partitions of $[n]$ with no singleton blocks. Prove that $B(n) = F(n) + F(n+1)$.

(S2) Prove that the number of self-conjugate partitions of n has the same parity as $p(n)$.

(S3) Find an expression for the number of partitions of n in which each part appears an even number of times. For example, this includes $(4, 4, 3, 3, 3, 3, 1, 1)$, but not $(8, 3, 3, 1, 1)$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded *on top of your score* for submitting a partial attempt or a complete solution.

(C1) Let $F(n)$ denote the number of all partitions of $[n]$ with no singleton blocks. Find a recursive formula for $F(n)$ in terms of $F(k)$ for $k \leq n - 1$.