Fall 2015, Math 431: Week 5 Problem Set
Due: Thursday, October 15th, 2015
Introduction to Graphs

Discussion problems. The problems below should be completed in class.

(D1) *Graph automorphisms.* An *automorphism* of a graph G is an unlabeled graph isomorphism from G to itself.

(a) Find all automorphisms of the complete graph K_n for $n \geq 2$.

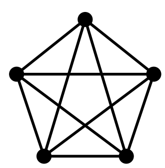
(b) Find all automorphisms of the circle graph C_n for $n \geq 3$.

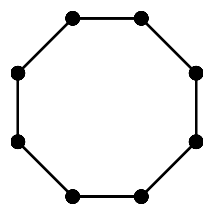
(c) Find all automorphisms of the star graph S_n for $n \geq 2$.

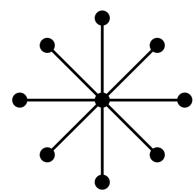
(d) Find all automorphisms of the wheel graph W_n for $n \geq 3$.

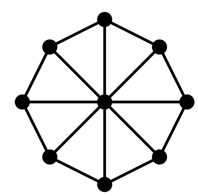
(e) Are any of the above graphs isomorphic?

Use the following graphs as a guide for the above problem.


Complete graph K_5

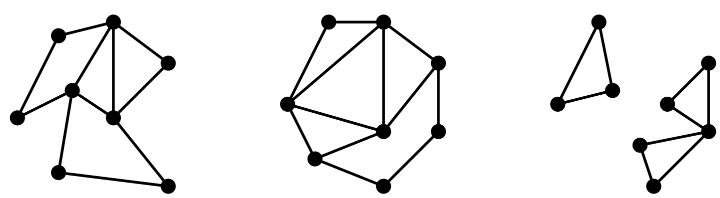

Circle graph C_8


Star graph S_8


Wheel graph W_8

(D2) *Eulerian walks.* Fix a graph $G = (V, E)$. A walk on G is *Eulerian* if every edge in G is used exactly once.

(a) Which of the following graphs have *closed* Eulerian walk? Justify your answers.



(b) Can you find a criterion to ensure a given simple graph G has a closed Eulerian walk?

(c) Which of the above graphs has a (not necessarily closed) Eulerian walk?

(d) Under what conditions on G can we find an Eulerian walk?

(e) Make each of the above graphs *directed* by manually adding a direction to each edge. Now, which graphs have an Eulerian walk?

(f) Suppose G is a directed graph. What conditions must G satisfy in order to apply your proof from part (b)?

Required problems. As the name suggests, you must submit *all* required problem with this homework set in order to receive full credit.

(R1) Prove that if there is a walk from a vertex v_1 to a vertex v_2 in a simple graph G, then there is a path from v_1 to v_2.

(R2) Is there a non-connected graph on 7 vertices such that every vertex has degree at least 3?

(R3) How many distinct Hamiltonian cycles does the complete graph K_n have?

(R4) Suppose G is a k-regular graph (that is, a simple graph in which every vertex has degree exactly k). Prove that G has a cycle of length at least $k + 1$.

(R5) Find all non-isomorphic simple graphs on four vertices. Be sure to give an argument that you found all of them!

Selection problems. You are required to submit *one* selection problem with this problem set. You may also submit additional selection problems, but the total number of points awarded (excluding challenge problems) won't exceed the total possible score on this problem set.

(S1) Find a simple graph G that has no nontrivial automorphisms. What is the minimum number $n > 1$ of vertices a simple graph with this property can have?

(S2) Suppose G is a simple graph with n vertices and no Hamiltonian cycles. What is the maximum number of edges G can have?

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded *on top of your score* for submitting a partial attempt or a complete solution.

(C1) Up to isomorphism, how many 4-regular graphs are there with exactly 10 edges?