

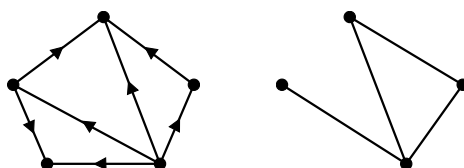
**Fall 2015, Math 431: Week 6 Problem Set**  
**Due: Thursday, October 22nd, 2015**  
**Trees**

**Discussion problems.** The problems below should be completed in class.

- (D1) *Counting spanning trees.* Fix a directed graph  $G = (V, E)$  with  $V = \{v_1, \dots, v_n\}$  and  $E = \{e_1, \dots, e_m\}$ . The *incidence matrix* of  $G$  is the  $n \times m$  matrix  $A$  defined by

$$A_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ is the head of } e_j \\ -1 & \text{if } v_i \text{ is the tail of } e_j \\ 0 & \text{otherwise} \end{cases}$$

- (a) Find the incidence matrix  $A$  of the directed graph depicted below.



- (b) Consider the matrix  $A_0$  obtained by omitting the last row of  $A$ . Compute the determinant of several  $(n-1) \times (n-1)$  submatrices of  $A_0$  (divide the work on this!).
- (c) Notice that the value of each determinant above is either 0 or 1. What do you notice about the edges corresponding to the columns when this value is nonzero?
- (d) Fix an arbitrary directed graph  $G$  with incidence matrix  $A$ , and let  $A_0$  denote the matrix obtained by removing the last row of  $A$ . Recall the Binet-Cauchy formula from linear algebra:

$$\det(A_0 A_0^T) = \sum_{\substack{(n-1) \times (n-1) \\ \text{submatrices} \\ B \text{ of } A_0}} (\det B)^2$$

Use this formula and your observation above to show that  $\det(A_0 A_0^T)$  equals the number of (undirected) spanning trees of  $G$ .

- (e) Fix an undirected graph  $G$  with vertices  $v_1, \dots, v_n$ , and consider the directed graph  $G'$  obtained from  $G$  by replacing each undirected edge of  $G$  with two directed edges, one in each direction. Let  $A$  denote the incidence matrix for  $G'$ . Compute  $AA^T$  for the undirected graph  $G$  depicted in part (a) above.
- (f) Now compute the adjacency matrix for  $G$ . What do you notice about these two matrices? Use this to prove the theorem stated in class on Tuesday to compute the number of spanning trees of an undirected graph  $G$ .

**Required problems.** As the name suggests, you must submit *all* required problem with this homework set in order to receive full credit.

- (R1) Find all non-isomorphic trees on 7 vertices.
- (R2) How many different (labeled) trees are there on  $[n]$  whose vertices have degree at most 2? How many unlabeled (that is, non-isomorphic) trees are there with this property?
- (R3) Find the number of spanning trees of the circle graph  $C_n$ . Verify your answer using the adjacency matrix of  $C_n$ .

- (R4) Prove that in any tree  $G$ , any two longest paths cross each other. Is the same true if  $G$  is connected but not necessarily a tree?

**Selection problems.** You are required to submit *one* selection problem with this problem set. You may also submit additional selection problems, but the total number of points awarded (excluding challenge problems) won't exceed the total possible score on this problem set.

- (S1) Suppose  $G$  is a tree, and no vertex of  $G$  has degree more than 3. Prove that number of vertices with degree 1 is two more than the number of vertices with degree 3.
- (S2) Find the number of spanning trees of the wheel graph  $W_n$ .

**Challenge problems.** Challenge problems are not required for submission, but bonus points will be awarded *on top of your score* for submitting a partial attempt or a complete solution.

- (C1) Suppose  $G$  is a tree with  $n$  vertices, and suppose each vertex of  $G$  has degree at most  $k$ . What is the maximum length path we can guarantee exists in  $G$ ?