Fall 2015, Math 431: Week 7 Problem Set
Due: Thursday, October 29th, 2015
Matching and Bipartite Graphs

Discussion problems. The problems below should be completed in class.

(D1) *Chromatic polynomials.* Fix a graph G with $n = |V(G)|$. The *chromatic function* of G is

$$\chi_G(k) = \# \text{ proper colorings of } G \text{ with } k \text{ colors.}$$

(a) Find the chromatic function of each of the following graphs.

    (i) The competely disconnected graph D_n with n vertices and no edges.

    (ii) The complete graph K_n.

    (iii) The path graph P_n.

What do you notice about these functions?

(b) Let c_i denote the number of ways to properly color G using exactly i colors. Prove that for all $k \geq 1$,

$$\chi_G(k) = \sum_{i=1}^{n} \binom{k}{i} c_i.$$

(c) Deduce that $\chi_G(k)$ is a polynomial in k of degree n.

(d) Fix an edge $e \in E(G)$. Consider (without proof, for the moment) the equality

$$\chi_G(k) = \chi_{G \setminus e}(k) - \chi_{G/e}(k).$$

Using this equality and problem (a)(i), find the chromatic polynomial of $G = K_3$.

(e) Use the equality in part (d) to give an inductive proof for part (c).

(f) Give a (preferrably combinatorial) proof of the equality in part (d).

(D2) *Hall's Marriage Theorem.* Given a graph G and $T \subset V(G)$, let $N(T)$ denote the set of neighbors of vertices in T. In this problem, you will prove the following result.

Theorem. *Fix a bipartite graph $G = (X, Y)$. There exists a perfect matching of X into Y if and only if $|T| \leq |N(T)|$ for all $T \subset X$.*

(a) First, verify Hall's Marriage Theorem holds in the special cases when $|X| = 1$, $|X| = 2$, $|Y| = 1$ and $|Y| = 2$.

(b) Prove the forward direction of Hall's Marriage Theorem.

(c) We will prove the backward direction by induction on X. Prove the base case, and state the inductive hypothesis.

(d) Prove the inductive step under the stronger assumption that $|T| < |N(T)|$ for $T \subset X$.

(e) Next, suppose that $B \subset X$ satisfies $|B| = |N(B)|$. Let $B' = X \setminus B$, and let $A = N(B)$ and $A' = Y \setminus A$. Draw a diagram for G labeling these sets.

(f) Show that the restrictions of G to $B \cup A$ and $B' \cup A'$ (that is, the graphs obtained from each set by deleting the remaining vertices) satisfy the indutive hypothesis.

(g) Conclude that Hall's Marriage Theorem holds.

Required problems. As the name suggests, you must submit *all* required problem with this homework set in order to receive full credit.

(R1) Find the chromatic polynomial of $K_{3,3}$.

(R2) Prove that the leading coefficient of the chromatic polynomial of any graph is 1.

(R3) Which of the following can be an ordered degree sequence of a bipartite graph?

  (a) $3, 3, 3, 3, 3, 3$
  (b) $3, 3, 3, 3, 3, 3, 3, 3$ (8 vertices total)
  (c) $3, 3, 3, 3, 3, 5, 6, 6, 6$

(R4) Fix $n \geq 1$ and $k < n/2$. Let $G = (X, Y)$ denote a bipartite graph so that X consists of the k-element subsets of $[n]$, Y consists of the $(k+1)$-element subsets of $[n]$, and $(x, y) \in E(G)$ for $x \in X$ and $y \in Y$ precisely when $x \subset y$. Find a perfect matching for X into Y.

Selection problems. You are required to submit *one* selection problem with this problem set. You may also submit additional selection problems, but the total number of points awarded (excluding challenge problems) won't exceed the total possible score on this problem set.

(S1) A medium-sized city has 3 high schools, each with n students. Each student knows exactly $n + 1$ students at a high school different from his/her own. Prove that we can choose 3 students, one from each school, each of which knows the other two.

(S2) Fix a graph G with n vertices. Prove that G is a tree if and only if

$$\chi_G(k) = k(k-1)^{n-1}.$$

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded *on top of your score* for submitting a partial attempt or a complete solution.

(C1) Fix a graph G with n vertices and m edges. Prove that the coefficient of k^{n-1} in the chromatic polynomial $\chi_G(k)$ equals $-m$.