Fall 2015, Math 431: Week 9 Problem Set
Due: Thursday, November 19th, 2015
Ordinary Generating Functions

Discussion problems. The problems below should be completed in class.

(D1) *Practice with formal power series.* Recall that in lecture, we saw $\sum_{n=0}^{\infty} p(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$, where $p(n)$ denotes the number of partitions of n.

(a) Find the coefficients of the generating function $D(x) = 1/(1-x)^2$. What do you notice about both sides? Hint: think back to your days in calculus.

(b) Find the ordinary generating functions for the number $p_{\text{odd}}(n)$ of partitions of n into odd parts and the number $p_{\text{d}}(n)$ of partitions of n into distinct parts. Prove that these two formal power series are equal.

(D2) *Multiplying ordinary generating functions.* Recall that for any two generating functions $F(x) = \sum_{n=0}^{\infty} f_n x^n$ and $G(x) = \sum_{n=0}^{\infty} g_n x^n$, we have

$$F(x)G(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} f_k g_{n-k} \right) x^n$$

The goal of this problem is to interpret the above coefficients combinatorially.

(a) Let c_n denote the number of non-isomorphic, connected, simple graphs on n vertices, and write $C(x) = \sum_{n=1}^{\infty} c_n x^n$. Write out the coefficients of $F(x) = C(x)C(x)$ in terms of c_n. Find a combinatorial interpretation for these values.

(b) Is the same true if c_n denotes the number of distinct (but potentially isomorphic) connected simple graphs on n vertices?

(c) Fix two generating functions $F(x) = \sum_{n=0}^{\infty} f_n x^n$ and $G(x) = \sum_{n=0}^{\infty} g_n x^n$, where f_n and g_n each denote the number of ways to put some particular structure on a set with n elements (for instance, a graph, a partition, etc). Using part (a) and our examples from Tuesday, give a combinatorial interpretation of the coefficients of $F(x)G(x)$.

Hint: you may find it helpful to use the statement of the Theorem in (D3) as a model.

(D3) *Composing generating functions.* The goal of this problem is to make sense of the following.

Theorem. *Suppose f_n (resp. g_n) denotes the number of ways to put an f-stucture (resp. g-structure) on the set $[n] = \{1, 2, \ldots, n\}$, and suppose $g_0 = 0$. Let $F(x) = \sum_{n=0}^{\infty} f_n x^n$ and $G(x) = \sum_{n=0}^{\infty} g_n x^n$. The coefficient of x^n in the composition*

$$F(G(x)) = \sum_{n=0}^{\infty} f_n (G(x))^n$$

equals the number of ways to split the set $[n]$ into k nonempty subintervals, place a g-structure on each interval, and place an f-structure on the set of intervals.

(a) Look back at the preliminary problem where you found the coefficients of $A(A(x))$ for $A(x) = \sum_{n=1}^{\infty} x^n$. Interpret the coefficients of $A(x)$ and $A(A(x))$ combinatorially.

(b) Why do we require that $g_0 = 0$ when defining composition?

(c) Let $c_n = n$ denote the number of ways to pick a person from a set of n people, and let $C(x) = \sum_{n=1}^{\infty} c_n x^n$ denote its generating function. Find $C(A(x))$ using rational functions, and describe what the coefficients represent.

(d) Find $A(C(x))$ using rational functions, and describe what the coefficients represent.

Required problems. As the name suggests, you must submit *all* required problem with this homework set in order to receive full credit.

(R1) Use generating functions to find an explicit formula for a_n if $a_0 = 1$ and $a_n = 3a_{n-1} + 2^n$.

(R2) Use generating functions to find L_n if $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$.

(R3) Prove that $F(x) = \sum_{n=0}^{\infty} f_n x^n$ has a multiplicative inverse if and only if $f_0 \neq 0$. If each $f_n \in \mathbb{Z}$, under what conditions are the coefficients of $1/F(x)$ necessarily integers?

Selection problems. You are required to submit *one* selection problem with this problem set. You may also submit additional selection problems, but the total number of points awarded (excluding challenge problems) won't exceed the total possible score on this problem set.

(S1) Let $m(n)$ denote the number of ways to express $n \geq 0$ as a sum the values 6, 9 and 20. For instance, $m(18) = 2$, since $18 = 6 + 6 + 6 = 9 + 9$, and $18 < 20$. Express the generating function $M(x) = \sum_{n \geq 0} m(n) x^n$ as a rational function of x with denominator $(1 - x^{60})^3$. How would the denominator change if n_1, n_2, and n_3 were used in place of 6, 9 and 20?

(S2) Define $c(n) = 1$ if the set $\{(a, b) \in \mathbb{Z}_{\geq 0}^2 : 7a + 11b = n\}$ is nonempty (that is, there exists a way to write n as a sum of the values 7 and 11), and $c(n) = 0$ otherwise. For instance, $c(77) = 1$ since $(a, b) = (11, 0)$ and $(a, b) = (0, 7)$ both lie in this set, and $c(10) = 0$ since there is no choice of a and b to produce 10. Prove that

$$\sum_{n=0}^{\infty} c(n) x^n = \frac{1 - x^{77}}{(1 - x^7)(1 - x^{11})}.$$

Conjecture and prove an analogous result if positive n_1 and n_2 replace 7 and 11.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded *on top of your score* for submitting a partial attempt or a complete solution.

(C1) Fix a function $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$. Prove that we can express the generating function

$$F(x) = \sum_{n=0}^{\infty} f(n) x^n = \frac{Q(x)}{(1 - x)^{d+1}}$$

for some polynomial $Q(x)$ of degree at most d if and only if $f(n) = a_d n^d + \cdots + a_1 n + a_0$ is a polynomial of degree exactly d. Hint: start with $Q(x) = 1$.