Fall 2015, Math 431: Week 10 Problem Set
Due: Thursday, December 3rd, 2015
Combinatorics of Generating Functions

Discussion problems. The problems below should be completed in class.

(D1) *Combinatorial interpretation.* For each of the following, attempt to find each generating function using the combinatorial interpretation of multiplication and composition of formal power series, rather than by explicitly solving the given counting question.

(a) Suppose our course meets for n days. Let d_n denote the number of ways to split the available days into 3 units (counting methods, graph theory, and generating functions),

(i) select some collection of days from the first unit to give a pop quiz,
(ii) select an odd number of days from the second unit to hold discussions, and
(iii) select a single day from the third unit in which to give an exam.

Find the ordinary generating function for d_n.

(b) What if each unit should be at least 2 days long?

(c) What if exactly half of the days in the first unit must have a pop quiz, and they must alternate days (quiz, no quiz, quiz, no quiz, etc)?

(d) A permutation of $[n] = \{1, \ldots, n\}$ is called *indecomposable* if it cannot be split into a permutation on $\{1, \ldots, k\}$ and a permutation on $\{k+1, \ldots, n\}$ for $1 \leq k \leq n-1$. For example, 54321 is indecomposable, but $23154 = (231)(54)$ is not.

Let c_n denote the number of indecomposable permutations on $[n]$, and let $c_0 = 0$. Find an equation relating the ordinary generating function $C(x)$ for c_n and the ordinary generating function for the number of permutations of $[n]$, that is, $P(x) = \sum_{n=0}^{\infty}(n!)x^n$.

(D2) *Exponential generating functions.* Fix sequences $(f_0, f_1, \ldots)$ and $(g_0, g_1, \ldots)$, and consider the formal power series $F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}$ and $G(x) = \sum_{n=0}^{\infty} g_n \frac{x^n}{n!}$ (called *exponential* generating functions). Let e^x denote the exponential generating function $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(a) Verify (using algebra) that

$$F(x)G(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} f_k g_{n-k} \right) \frac{x^n}{n!}.$$

(b) How might we interpret the above coefficients combinatorially? Be sure to use the phrases "f-structure" and "g-structure" in your description!

(c) Assuming $g_0 = 0$, the n-th coefficient of $F(G(x))$ is the number of ways to partition the set $[n]$, place a g-structure on each block, and place an f-structure on the set of blocks. Fill in each combinatorial interpretation in the table below. What is the "main combinatorial difference" between ordinary and exponential generating functions?

(d) Let g_n denote the number of simple graphs on n vertices, let c_n denote the number of connected simple graphs on n vertices. Find an equation relating the exponential generating functions $G(x) = \sum_{n \geq 0} g_n \frac{x^n}{n!}$ and $C(x) = \sum_{n \geq 0} c_n \frac{x^n}{n!}$.

	Ordinary Generating Functions	Exponential Generating Functions
Multiplication		
Composition		

Required problems. As the name suggests, you must submit *all* required problem with this homework set in order to receive full credit.

(R1) Find an explicit formula for a_n if $a_0 = 0$ and $a_{n+1} = (n+1)a_n + n!$ for $n \geq 0$. Hint: use exponential generating functions.

(R2) Find the exponential generating function for the number t_n of ways to arrange n books on two bookshelves so that each shelf has at least one book. Then, find a closed form for t_n.

Selection problems. You are required to submit *two* selection problem with this problem set. You may also submit additional selection problems, but the total number of points awarded (excluding challenge problems) won't exceed the total possible score on this problem set.

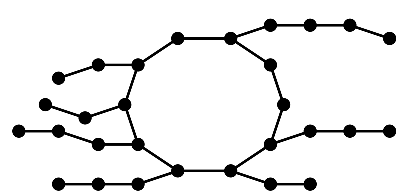
(S1) Suppose from our class of n students, we select an odd number of students to serve on a committee, and select an even number of committee members to serve on a subcommittee. Find the exponential generating function for the number c_n of ways to do this.

(S2) Suppose we have n cards. We want to split them into an even number of nonempty subsets, form a line within each subset, then arrange the subsets in a line. Use generating functions to determine the number of different ways to do this.

(S3) Let ℓ_n denote the number of linear orderings of $[n]$, and let $L(x) = \sum_{n=0}^{\infty} \ell_n \frac{x^n}{n!}$. Give a combinatorial proof that $L(x) = 1 + xL(x)$, and use this to derive the closed form $\ell_n = n!$.

(S4) A *rooted tree* is a tree with a disinguished vertex, called the *root*. Let t_n denote the number of rooted trees on $[n]$, and let $T(x) = \sum_{n=0}^{\infty} t_n \frac{x^n}{n!}$ denote its exponential generating function. Prove that $T(x) = xe^{T(x)}$. Hint: what is left if you remove the root of a rooted tree?

(S5) A *combinatorial octopus* is an undirected graph of the following form.



Let o_n denote the number of combinatorial octopii on n vertices. Additionally, let ℓ_n denote the number of ways to linearly order a set with n elements, and let c_n denote the number of ways to arrange n elements in a circle. Express the exponential generating function for o_n in terms of the exponential generating functions of ℓ_n and c_n. Use this to find o_n.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded *on top of your score* for submitting a partial attempt or a complete solution.

(C1) Let g_n denote the number of simple graphs on the vertex set $[n]$ in which every vertex has degree 2, and let $g_0 = 1$. Prove that

$$\sum_{n=0}^{\infty} g_n \frac{x^n}{n!} = \frac{e^{-\frac{1}{2}x - \frac{1}{4}x^2}}{\sqrt{1-x}}$$