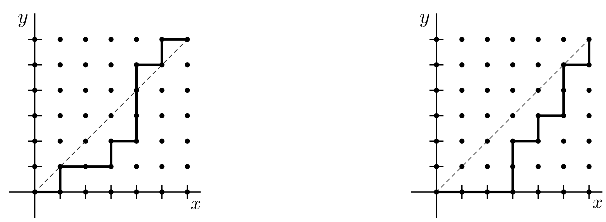
**Fall 2015, Math 431: Review Problems**
**Due: Friday, December 11th, 2015**
**Exam 3 Review**

**Exam review problems.** As the name suggests, these problems are intended to help you prepare for the upcoming exam.

(ER1) A *lattice path* is a path consisting only of unit moves up and right. For example:




Find the number of lattice paths between $(0,0)$ and $(m,n)$ for $m,n \in \mathbb{Z}_{\geq 0}$.

(ER2) Give a combinatorial proof that for all $n \geq 2$,

$$\sum_{k=0}^{n} k(k-1)\binom{n}{k} = n(n-1)2^{n-2}.$$

(ER3) Give another proof of the above identity using the binomial theorem.

(ER4) Recall that $S(n,k)$ denotes the number of partitions of the set $[n] = \{1,2,\ldots,n\}$ into exactly $k$ blocks, and $B(n) = \sum_{k=0}^{n} S(n,k)$ denotes the number of partitions of $[n]$ into any number of blocks. Give a combinatorial proof of the identity

$$B(n+1) - B(n) = \sum_{k=1}^{n} kS(n,k).$$

(ER5) Find all automorphisms of the complete bipartite graph $K_{m,n}$.

(ER6) Suppose a simple, connected graph $G$ with $n$ vertices has a unique cycle, the length of which is 3. Find the chromatic polynomial of $G$.

(ER7) Determine the minimum number of vertices that must be removed from a complete bipartite graph $K_{m,n}$ in order to yield a planar graph.

(ER8) Determine when two trees $T_1$ and $T_2$ have isomorphic dual graphs.

(ER9) In each theorem involving a composition $F(G(x))$ of formal power series, we have required that $G(x)$ have constant term 0. Why is this?

(ER10) Find a simple expression for the ordinary generating function of the sequence $a_n = n^2$. Do the same for its exponential generating function.

(ER11) Use ordinary generating functions to find a closed form for the recurrence relation given by $b_0 = 1$ and $b_n = 2b_{n-1} + n^2$.

(ER12) Fix power series $F(x) = \sum_{n=0}^{\infty} f_n x^n$ and $G(x) = \sum_{n=0}^{\infty} g_n x^n$, and let $\frac{d}{dx}$ denote term-by-term differentation. For instance, $\frac{d}{dx}F(x) = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n$.

Verify the product rule for formal power series:

$$\frac{d}{dx}\left(F(x)G(x)\right) = \left(\frac{d}{dx}F(x)\right)G(x) + F(x)\left(\frac{d}{dx}G(x)\right).$$

If you are feeling adventurous, verify the quotient rule for formal power series:

$$\frac{d}{dx}\left(\frac{F(x)}{G(x)}\right) = \frac{\left(\frac{d}{dx}F(x)\right)G(x) - F(x)\left(\frac{d}{dx}G(x)\right)}{(G(x))^2}$$

Where in your proof of the quotient rule did you use that $g_0 \neq 0$?

(ER13) Use induction and the product rule for formal power series (given above) to prove

$$\frac{d}{dx}(F(x))^k = k(F(x))^{k-1}\left(\frac{d}{dx}F(x)\right)$$

for all $k \geq 1$. Hint: this can be done without writing any infinite sums.

(ER14) Recall that $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$, and consider the formal power series $\ln(x)$ defined so that

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}x^{n+1}.$$

(a) Justify the above definition by differentiating both sides (using calculus).

(b) Prove that $(e^x)^m = e^{mx}$ for all $m \geq 0$. Hint: induct on $m$.

(c) Prove that $\ln(e^x) = x$.

(d) Pick your favorite algebraic property involving $\ln(x)$ and/or $e^x$, and prove that it holds in formal power series-land. Alternatively, look up the power series expansions for $\sin(x)$ and $\cos(x)$ in your favorite Calculus textbook, and prove (using formal power series) that $\sin(2x) = 2\sin(x)\cos(x)$, or that $(\sin(x))^2 + (\cos(x))^2 = 1$.

(ER15) For $k \geq 1$, find an expression for the exponential generating function

$$S_k(x) = \sum_{n=0}^{\infty} S(n,k)\frac{x^n}{n!}$$

in terms of familiar exponential generating functions ($e^x$, $\ln(x)$, etc.). Use this to find a closed form when $k = 1$, $k = 2$ and $k = 3$. Note: there is no known closed form for general $k$, so do not attempt to solve for the coefficients in general!

(ER16) A permutation of $[n] = \{1, \ldots, n\}$ is called *indecomposable* if it cannot be split into a permutation on $\{1, \ldots, k\}$ and a permutation on $\{k+1, \ldots, n\}$ for $1 \leq k \leq n-1$. For example, 54321 is indecomposable, but $23154 = (231)(54)$ is not.

Let $c_n$ denote the number of indecomposable permutations on $[n]$, and let $c_0 = 0$. Find an equation relating the ordinary generating function $C(x)$ for $c_n$ and the ordinary generating function for the number of permutations of $[n]$, that is, $P(x) = \sum_{n=0}^{\infty}(n!)x^n$.