Fall 2018, Math 320: Week 4 Problem Set
Due: Tuesday, September 25th, 2018
Introduction To Rings

Discussion problems. The problems below should be completed in class.

(D1) *Checking ring axioms.* Determine which of the following sets $(R, +, \cdot)$ forms a ring under the given addition and multiplication. For each R that is indeed a ring, determine whether R is (i) commutative, (ii) an integral domain, and (iii) a field.

(a) The set R of all 2×2 real matrices (under matrix addition/multiplication) given by

$$R = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\} \subset M(\mathbb{R}).$$

(b) The set $R = \{r_5 x^5 + \cdots + r_1 x + r_0 : r_i \in \mathbb{R}\}$ of polynomials in a variable x with real coefficients and **degree at most** 5, under the usual addition and multiplication.

(c) The set $R = \mathbb{R} \cup \{\infty\}$ of real numbers together with infinity, and addition and multiplication operations $a \oplus b = \min(a, b)$ and $a \odot b = a + b$, respectively.

(d) The set $R = \mathbb{Z}$ with operations $\oplus$ and $\odot$ given by $a \oplus b = a + b$ and $a \odot b = a + b$ (in particular, **both** addition and multiplication in R correspond to integer addition).

(e) The set $R = \{p(x) \in \mathbb{R}[x] : p(0) \in \mathbb{Z}\}$ of polynomials in a variable x with real coefficients and **integer constant term**, under the usual addition and multiplication. For example, $2x^2 + \frac{1}{2}x + 5 \in R$ and $\frac{6}{5}x \in R$, but $5x + \frac{1}{3} \notin R$.

(D2) *Cartesian products.* Recall that the Cartesian product of two rings R_1 and R_2 is the set

$$R_1 \times R_2 = \{(a, b) : a \in R_1, b \in R_2\}$$

with addition $(a, b) + (a', b') = (a + a', b + b')$ and multiplication $(a, b) \cdot (a', b') = (a \cdot a', b \cdot b')$.

(a) Find the addivite inverses of $([1]_6, [0]_3)$, $([3]_6, [2]_3)$, and $([5]_6, [1]_3) \in \mathbb{Z}_6 \times \mathbb{Z}_3$.

(b) What is the multiplicative identity of $\mathbb{Z}_6 \times \mathbb{Z}_3$? Which elements listed in part (a) have a multiplicative inverse?

(c) Justify each "=" in the following proof that addition is commutative in $R_1 \times R_2$ for any rings R_1 and R_2.

Proof. Given $(a, b), (c, d) \in R_1 \times R_2$, we have

$$(a, b) + (c, d) = (a + c, b + d) = (c + a, b + d) = (c + a, d + b) = (c, d) + (a, b),$$

which completes the proof. □

(d) Prove that every element of $R_1 \times R_2$ has an additive inverse.

(D3) *Arithmetic properties.* In this section, you will prove several of the basic properties of the addition and multiplication operations on $\mathbb{Z}_n$.

(a) Given below is a proof that addition in $\mathbb{Z}_n$ is associative. Modify this to obtain a proof that $\mathbb{Z}_n$ satisfies the distributivity axiom.

Proof. For any $[a]_n, [b]_n, [c]_n \in \mathbb{Z}_n$, we have

$$\begin{aligned} [a]_n + ([b]_n + [c]_n) &= [a]_n + [b + c]_n = [a + (b + c)]_n = [(a + b) + c]_n \\ &= [a + b]_n + [c]_n = ([a]_n + [b]_n) + [c]_n, \end{aligned}$$

which verifies associativity of addition. □

(b) Is $\mathbb{Z}_6$ an integral domain? What about $\mathbb{Z}_5$? What about $\mathbb{Z}_{mn}$ for some $m, n \in \mathbb{Z}_{\geq 2}$?

Required problems. As the name suggests, you must submit *all* required problem with this homework set in order to receive full credit.

(R1) Prove that

$$R = \{[3k]_{18} : k \in \mathbb{Z}\} = \{[0]_{18}, [3]_{18}, [6]_{18}, [9]_{18}, [12]_{18}, [15]_{18}\}$$

is a subring of $\mathbb{Z}_{18}$. Does R have a multiplicative identity?

(R2) Let L denote the set of positive real numbers. Define the operations

$$a \oplus b = ab \qquad \text{and} \qquad a \odot b = a^{\ln(b)}$$

for all $a, b \in L$. Is $(L, \oplus, \odot)$ a ring? If so, is it commutative? Is it a field?

(R3) Let

$$R = \left\{ \begin{pmatrix} a & a \\ b & b \end{pmatrix} : a, b, \in \mathbb{R} \right\} \subset M(\mathbb{R}).$$

(a) Prove that R is a subring of $M(\mathbb{R})$.

(b) Prove the matrix

$$J = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

is a *right identity* in R (that is, $AJ = A$ for every $A \in R$).

(c) Demonstrate the matrix J above is **not** a *left identity* in R by finding a matrix $A \in R$ so that $JA \neq A$.

(R4) Determine whether each of the following statements is true or false. Prove each true statement, and give a counterexample for each false statement.

(a) If R is a ring and $S \subset R$, then S is a subring of R.

(b) If R and S are both integral domains, then $R \times S$ is an integral domain.

Selection problems. You are required to submit all parts of *one* selection problem with this problem set. You may submit additional selection problems if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional selection problems.

(S1) Prove that

$$\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\},$$

under the usual addition and multiplication of real numbers, is a field.

(S2) Consider $(C, +, \odot)$, where $C = \mathbb{R}\times\mathbb{R}$, "$+$" is the standard componentwise addition on $\mathbb{R}\times\mathbb{R}$, and "$\odot$" is given by

$$(a, b) \odot (c, d) = (ac - bd, ad + bc)$$

for all $(a, b), (c, d) \in C$. Prove that C is a field.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Prove there is only one way to fill the addition and multiplication tables for a 3-element field $F = \{0, 1, a\}$, and give the operation tables. What (more familiar) ring is this?

In this problem, you may use the following result (which we will prove next week): if R is a ring and $r \in R$, then $0_R \cdot r = r \cdot 0_R = 0_R$.