Fall 2018, Math 320: Week 6 Problem Set
Due: Tuesday, October 9th, 2018
Isomorphisms and Homomorphisms

Discussion problems. The problems below should be completed in class.

(D1) *Homomorphisms.* The goal of this problem is to get practice with homomorphisms.

(a) Determine whether each of the following maps is a ring homomorphism.

(i) $\phi : \mathbb{Z} \to \mathbb{Z}$ given by $\phi(a) = a + 3$.

(ii) $\phi : \mathbb{Z} \to \mathbb{Z}$ given by $\phi(a) = 2a$.

(iii) $\phi : \mathbb{Z} \to M(\mathbb{R})$ given by $\phi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

(iv) $\phi : R \to S$ given by $\phi\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$, where $R, S \subset M(\mathbb{R})$ are the set of all upper triangular and lower triangular matrices, respectively.

(b) Consider a ring homomorphism $\phi : \mathbb{R}[x] \to \mathbb{R}[y]$ that sends $x \mapsto y + 1$.

(i) Find all possible values of $\phi(0)$ and $\phi(1)$. Hint: consider $\phi(1 \cdot 1)$.

(ii) Find all possible values of $\phi(x^3)$, $\phi(x + 1)$, and $\phi(x^2 + 2x + 1)$.

(iii) What possible values can $\phi(1/2)$ be? What about $\phi(\sqrt{2})$?

(iv) Conjecture how many possible maps ϕ exist (you do **not** need to prove it).

(c) Fix a ring homomorphism $\phi : R \to S$. Prove that the image of ϕ is a subring of S.

(D2) *Constructing isomorphisms.* The goal of this problem is to get practice with isomorphisms.

(a) Prove each of the following.

(i) $\mathbb{R} \times \mathbb{R} \cong R$, where

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\} \subset M(\mathbb{R})$$

(ii) $\mathbb{Z}_{12} \cong \mathbb{Z}_3 \times \mathbb{Z}_4$.

(iii) $\mathbb{Z} \cong (R, \oplus, \odot)$, where $R = \mathbb{Z}$,

$$a \oplus b = a + b + 1 \qquad \text{and} \qquad a \odot b = ab + a + b.$$

Hint: find 0_R and 1_R.

(iv) $\mathbb{Z} \cong R$, where $R = \{(a, a) : a \in \mathbb{Z}\} \subset \mathbb{Z} \times \mathbb{Z}$.

(v) $R \cong T$, where R and S are any rings and $T = \{(r, 0) : r \in R\} \subset R \times S$.

(b) Determine which of the following statements are true, and prove your assertions.

(i) $\mathbb{Z} \cong 2\mathbb{Z}$, where $2\mathbb{Z}$ denotes the set of even integers.

(ii) $\mathbb{Z}_{19} \cong \mathbb{Z}_{17}$.

(iii) $\mathbb{Z}_6$ is isomorphic to **some** subring of $M(\mathbb{R})$.

(iv) $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \cong M(\mathbb{R})$.

Required problems. As the name suggests, you must submit *all* required problem with this homework set in order to receive full credit.

(R1) Determine whether each of the following maps is a ring homomorphism.

(a) $\phi : \mathbb{Q} \to \mathbb{Q}$ given by $\phi(a) = \dfrac{1}{a^2+1}$.

(b) $\phi : \mathbb{Z} \to M(\mathbb{R})$ given by $\phi(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$.

(c) $\phi : \mathbb{Z} \to M(\mathbb{R})$ given by $\phi(a) = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$.

(d) $\phi : \mathbb{Z}_{24} \to \mathbb{Z}_6$ given by $\phi([a]_{24}) = [a]_6$. Hint: be sure to check ϕ is well-defined!

(e) $\phi : \mathbb{Z}_6 \to \mathbb{Z}_{12}$ given by $\phi([a]_6) = [a]_{12}$. Hint: be sure to check ϕ is well-defined!

(R2) Suppose $\phi : R \to S$ is a nonzero ring homomorphism (i.e. ϕ is not the map that sends every element of R to 0), R has a multiplicative identity, and S is an integral domain. Prove S has a multiplicative identity $1_S \in S$ and that $\phi(1_R) = 1_S$.

(R3) Determine whether each of the following statements is true or false. Prove your assertions.

(a) $\mathbb{Z}_9$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.

(b) The ring $(L, \oplus, \odot)$ from Problem (R2) on Homework 4 is isomorphic to $\mathbb{R}$.